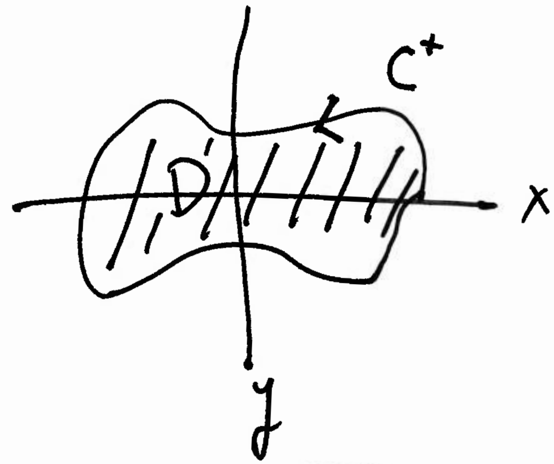
Green's Theorem

D - elementary region, $P, Q : D \to \mathbb{R}$ class C^1

C^+ = boundary of D with counterclockwise orientation

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \int_{C^+} P\,dx + Q\,dy$$



$F = (P, Q)$ is vector field

$$\iint_D \text{"derivative" of } F =$$

$$= \int_{\text{boundary of } D} F \cdot d\vec{s}$$

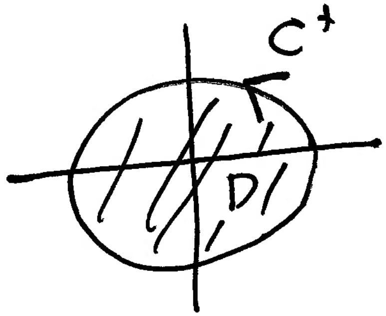
$\Phi(u,v) = (u,v,0)$

(u,v) in D

$T_u \times T_v = (0,0,1)$

Examples

① Verify Green's Theorem for $P(x,y) = x$, $Q(x,y) = xy$ where D is the unit disk $x^2+y^2 \le 1$.



$D: x^2+y^2 \le 1$

$\partial D = C: x^2+y^2 = 1$

parametrize C^+: $c(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$

$c'(t) = (-\sin t, \cos t)$

$x = \cos t$, $y = \sin t$

$dx = -\sin t \quad dy = \cos t$

line integral

$$\int_{C^+} P\,dx + Q\,dy =$$

$$= \int_0^{2\pi} P(\cos t, \sin t) \cdot -\sin t + Q(\cos t, \sin t) \cdot \cos t \, dt$$

$$= \int_0^{2\pi} -\cos t \sin t + \cos^2 t \sin t \, dt$$

$u = \cos t$

$du = -\sin t\,dt$

$$= \int_1^1 \text{blah} = 0$$

$t=0, \quad u=\cos(0)=1$

$t=2\pi, \quad u=\cos(2\pi)=1$

<u>double integral</u>

$D: x^2+y^2 \le 1, \quad P(x,y)=x, \quad Q(x,y)=xy$

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA, \qquad \frac{\partial Q}{\partial x} = y, \quad \frac{\partial P}{\partial y} = 0$$

$$\| $$

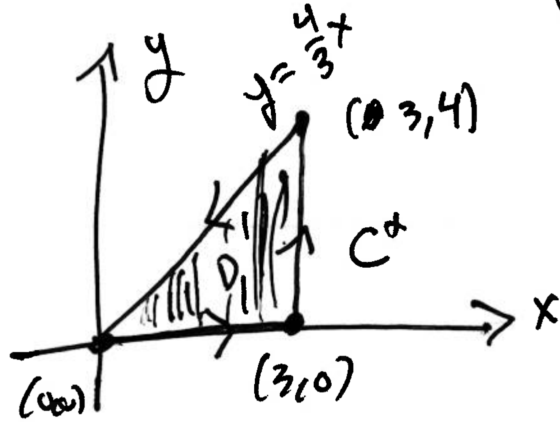
$$\iint_{x^2+y^2\le 1} y\, dx\,dy \underset{\text{Polar}}{=} \int_0^{2\pi}\int_0^1 r\sin\theta \; r\,dr\,d\theta$$

$D: 0 \le r \le 1$
$\quad\; 0 \le \theta \le 2\pi$, $\quad y = r\sin\theta$

$$\int_0^{2\pi} \sin\theta\, d\theta \cdot \int_0^1 r^2\,dr$$

$$-\cos\theta\Big|_0^{2\pi} \cdot \int_0^1 r^2\,dr = 0$$

② Particle travels the following path:



A force field $F(x,y) = (3x+4y^2, 10xy)$ acts on the particle. What is the work of F done on the particle?

Use Green's Thm: $P = 3x+4y^2$, $Q = 10xy$

$\frac{\partial P}{\partial y} = 8y$, $\frac{\partial Q}{\partial x} = 10y$

Answer $\int_{C^+} F \cdot d\vec{s}$

D: $0 \leq x \leq 3$

$0 \leq y \leq \frac{4}{3}x$

Greens Thm

$$\int_{C^+} \overset{\overset{Pdx+Qdy}{\shortparallel}}{F \cdot d\vec{s}} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

$$= \int_0^3 \int_0^{\frac{4}{3}x} 10y - 8y \, dy\, dx$$

$$= \int_0^3 \int_0^{\frac{4}{3}x} 2y\, dy\, dx$$

$$= \int_0^3 y^2 \Big|_0^{\frac{4}{3}x} dx$$

$$= \int_0^3 \cancel{16} \frac{16x^2}{9} dx$$

$$= \frac{16x^3}{27} \Big|_0^3 = \boxed{16}$$

$\vec{v} = (1,2,3)$

Vector field $\quad F = (F_1, F_2) \qquad F = (P, Q)$

F - vector field

F_1, F_2, F_3, P, Q, R

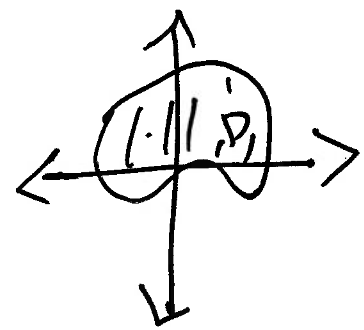
scalar valued functions $\quad F = (F_1, F_2, F_3) \qquad F = (P, Q, R)$

$$F(x,y) = (\underbrace{3x + 4y^2}_{P(x,y)}, \underbrace{10xy}_{Q(x,y)})$$

Theorem (Area of Region) If C is a simple closed curve that bounds a region D. Then the area of D is

$$\text{Area of } D = \frac{1}{2}\int_{C^+} x\,dy - y\,dx$$

proof: $\text{Area of } D = \iint_D 1\,dA = \int_{\partial D^+} P\,dx + Q\,dy$ where $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 1$



Green's Thm

$$\frac{1}{2}\int_{C^+} x\,dy - y\,dx = \frac{1}{2}\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = \frac{1}{2}\iint_D 2\,dA = \iint_D 1\,dA \qquad \square$$

$P = -y \qquad \dfrac{\partial Q}{\partial x} = 1, \quad \dfrac{\partial P}{\partial y} = -1$

$Q = x$

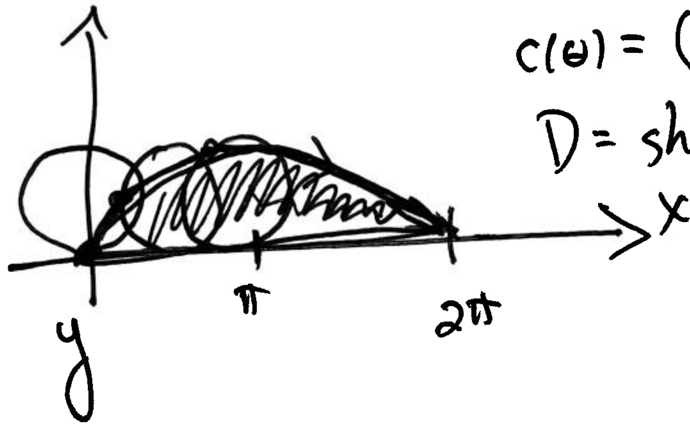
Example

Find area bounded by one arc of the cycloid

$$x = a(\theta - \sin\theta), \quad y = a(1-\cos\theta), \quad 0 \le \theta \le 2\pi$$

where $a > 0$ is fixed and x-axis.

Picture



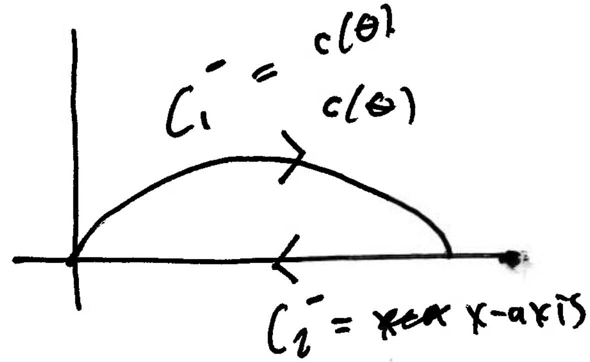
$c(\theta) = (a(\theta - \sin\theta), a(1-\cos\theta)), \ 0 \le \theta \le 2\pi$

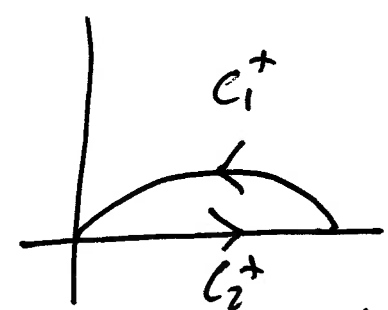
D = shaded region

How to express D as an elementary region? Not clear

$$\iint_D 1 \, dA$$

Use theorem





$C^+ = C_1^+ \cup C_2^+$

$$\text{Area of } D = \frac{1}{2}\int_{C^+} x\,dy - y\,dx = \frac{1}{2}\int_{C_1^+} x\,dy - y\,dx + \frac{1}{2}\int_{C_2^+} x\,dy - y\,dx$$

$$\frac{1}{2}\int_{C_1^+} x\,dy - y\,dx = ?$$

$$\| $$

$$-\frac{1}{2}\int_{C_1^-} x\,dy - y\,dx$$

$$\|$$

Parametrization of C_1^-

$c(t) = \left(a(t-\sin t),\ a(1-\cos t)\right),\ 0 \le t \le 2\pi$

(given in problem)

$c'(t) = \left(a(1-\cos t),\ a\sin t\right)$

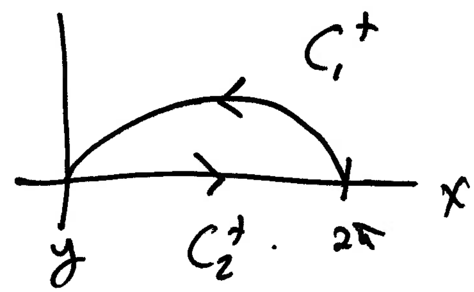
$$-\frac{1}{2}\int_0^{2\pi} a(t-\sin t)\cdot a\sin t - a(1-\cos t)\,a(1-\cos t)\,dt$$

$$\|$$

$$-\frac{1}{2}\int_0^{2\pi} a^2 t\sin t - a^2\sin^2 t - a^2(1-2\cos t+\cos^2 t)\,dt$$

$$-\frac{1}{2}\int_0^{2\pi} \underbrace{a^2 t\sin t}_{\text{IBP}} - 2a^2 + 2a^2\cos t \, dt = \text{something}$$

$$\frac{1}{2}\int_{C_2^+} x\,dy - y\,dx = ?$$



parametrize C_2^+: $c(t) = (t,0)$, $0 \le t \le 2\pi$

$c'(t) = (1,0)$

$$\frac{1}{2}\int_0^{2\pi} t\cdot 0 - 0\cdot 1 \, dt = 0$$

4.4 Curl

$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

Curl of a vector field: $F(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$

The curl of F is the vector field

$$\operatorname{curl} F = \nabla \times F = \det\begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{pmatrix}$$

$$\operatorname{curl} F = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$