4.4 Curl

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$ del or nabla differential operator

$F = (P, Q, R)$ vector field in $\mathbb{R}^3$

$$\text{curl}(F) = \nabla \times F = \det\begin{pmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{pmatrix}$$

$$\text{curl}(F) = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

If $F(x,y) = (P(x,y), Q(x,y))$ is a vector field in $\mathbb{R}^2$, extend F to a vector field in $\mathbb{R}^3$:

$$F(x,y,z) = (P(x,y), Q(x,y), 0)$$

Then $\frac{\partial P}{\partial z} = 0$, $\frac{\partial Q}{\partial z} = 0$, $R = 0$, so

$$\operatorname{curl}(F) = \left(0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$\operatorname{curl}(F)\cdot k = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$ called <u>scalar curl</u> of F

<u>Greens Thm</u>

$$\iint_D \operatorname{curl}(F)\cdot k \, dA = \int_{\partial D^+} F\cdot d\vec{s}$$

<u>Examples of Curl</u>

$$F(x,y,z) = (xy, -\sin z, 1)$$

$$\operatorname{curl}(F) = \det\begin{pmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & -\sin z & 1 \end{pmatrix} = \left(0 - -\cos z, -(0-0), 0 - x\right)$$

$$\operatorname{curl}(F) = (\cos z, 0, -x)$$

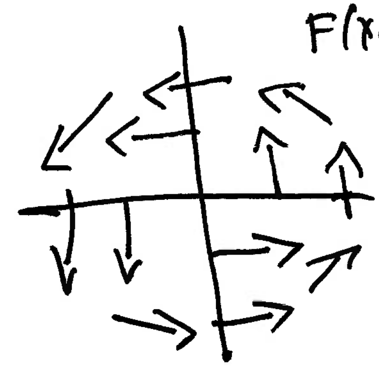
Geometric Interpretation of Curl: F = vector field

curl F = vector field which points in direction of the axis of which F is rotating around at a point (twice the angular velocity of a rigid body that rotates as F does)

Examples

(1) $F(x,y,z) = (-y, x, 0)$

$F(x,y) = (-y,x)$



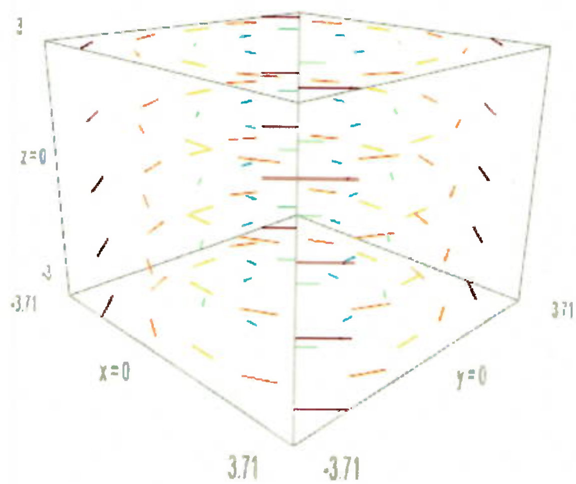
$$\operatorname{curl}(F) = \det \begin{pmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{pmatrix}$$

$$= (0, -0+0, 1--1) = (0,0,2)$$

2019-11-20-124305.sagews

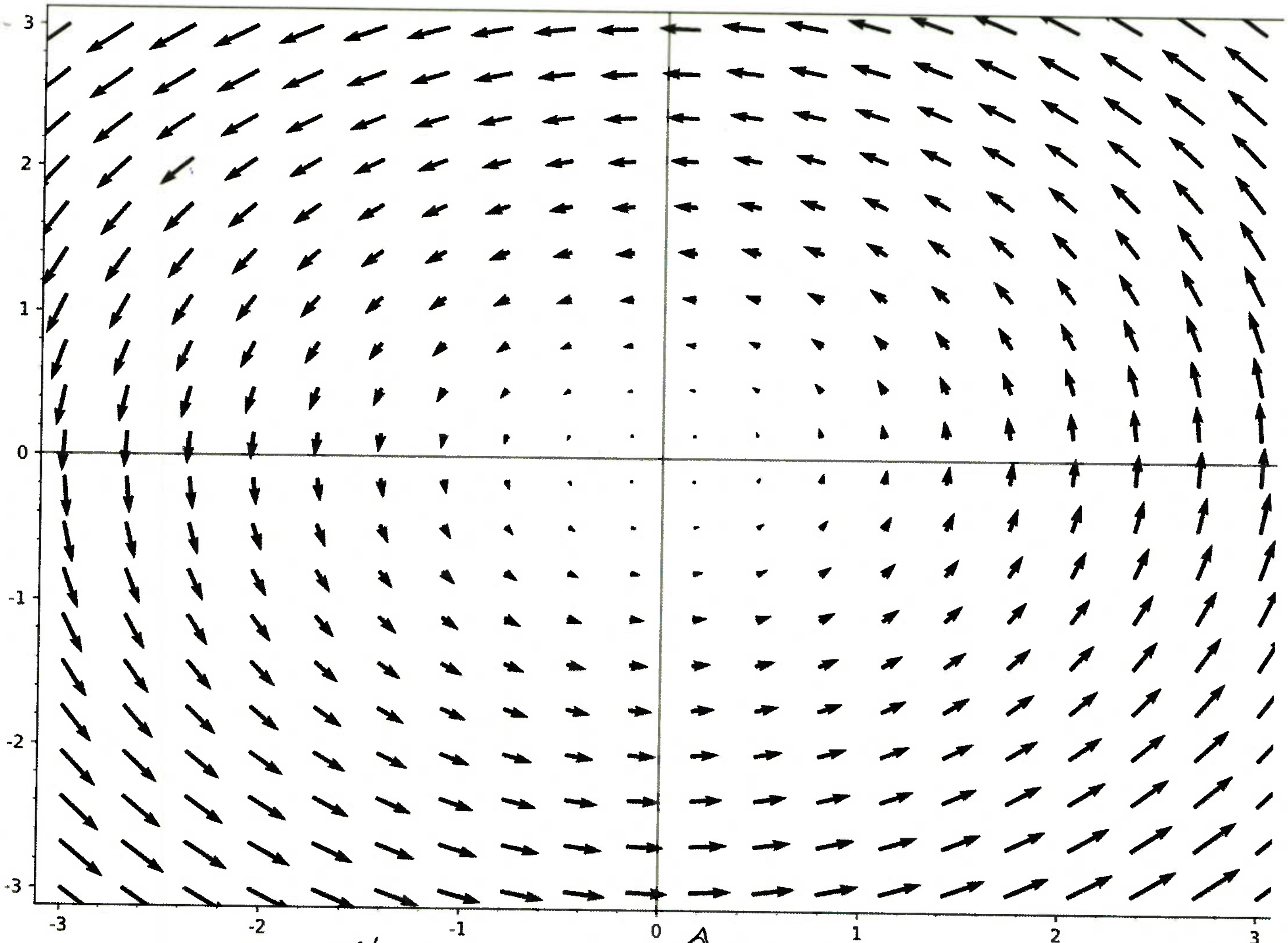
Author	Joe Ferrara
Date	2019-11-20T20:46:40
Project	f5945acc-5481-42f4-b70f-cdfaa43d6ef5
Location	2019-11-20-124305.sagews
Original file	2019-11-20-124305.sagews

```
x,y,z = var('x y z')
plot_vector_field3d((-y, x, 0), (x,-3,3), (y,-3,3), (z, -3, 3), figsize = 10)
```

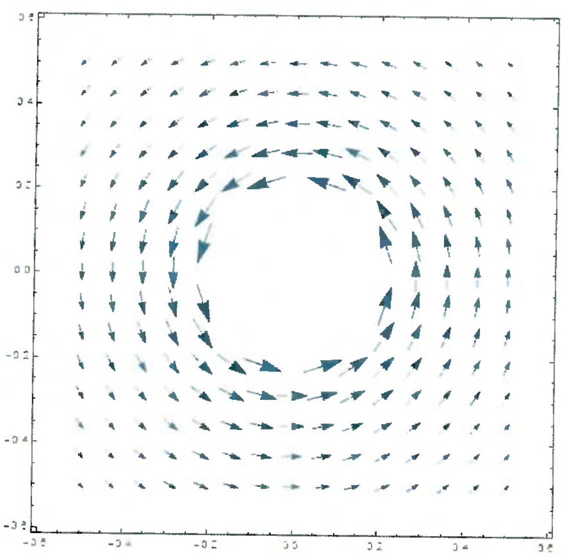


$F(x,y,z) = (-y, x, 0)$

```
x,y = var('x y')
plot_vector_field((-y, x), (x,-3,3), (y,-3,3), figsize = 10)
```



$F(x,y) = (-y, x)$



$F(x,y) = \left(\dfrac{y}{x^2+y^2}, \dfrac{-x}{x^2+y^2} \right)$

~~doesn~~ doesn't model rotation

(2) $F(x,y,z) = (1,2,3)$

$\text{curl}(F) = (0,0,0)$

(3) $F(x,y,z) = (x,y,z)$

$$\text{curl}(F) = \det\begin{pmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{pmatrix}$$

$$= (0-0, -(0-0), 0-0)$$

$$= (0,0,0)$$

(4) $F(x,y,z) = \left(\frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}, 0 \right)$

/5

$$\operatorname{curl}(F) = \det\begin{pmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{y}{x^2+y^2} & \frac{-x}{x^2+y^2} & 0 \end{pmatrix}.$$

$$= \left(0-0, -(0-0), \frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right).$$

$$\frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right) = \frac{(x^2+y^2)\cdot(-1) - (-x)\cdot(2x)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \frac{(x^2+y^2)\cdot(1) - y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\operatorname{curl}(F) = (0,0,0)$$

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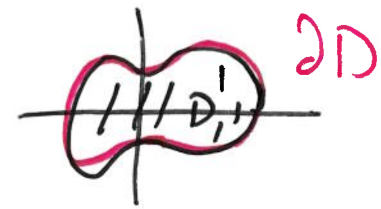
8.2 Stokes Theorem

S = surface in $\mathbb{R}^3$

∂S = boundary of S

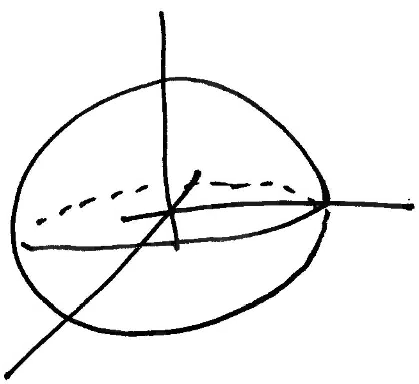
(curve that bounds S)

Green



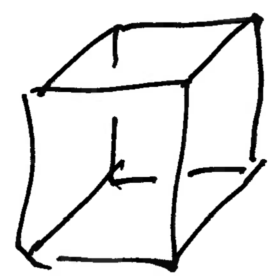
Examples

① S = sphere $x^2+y^2+z^2=1$



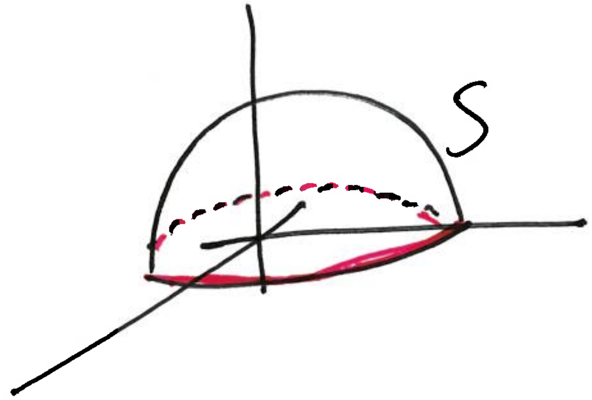
∂S = nothing, sphere has no boundary

② S = box = $[0,1]\times[0,1]\times[0,1]$ (surface of)



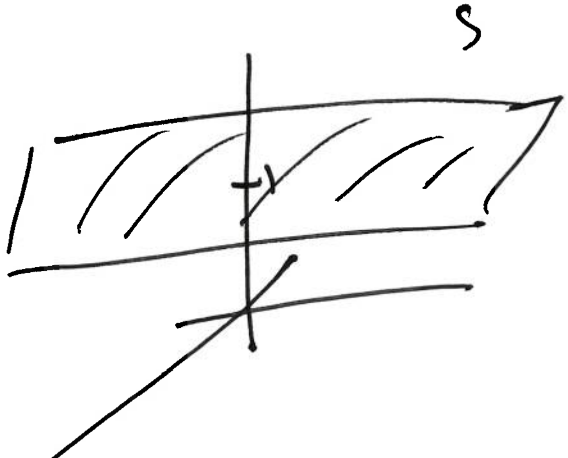
∂S = nothing
no boundary

(3) $S =$ hemisphere $x^2+y^2+z^2=1,\ z \geq 0$



$\partial S =$ circle $x^2+y^2=1,\ z=0$

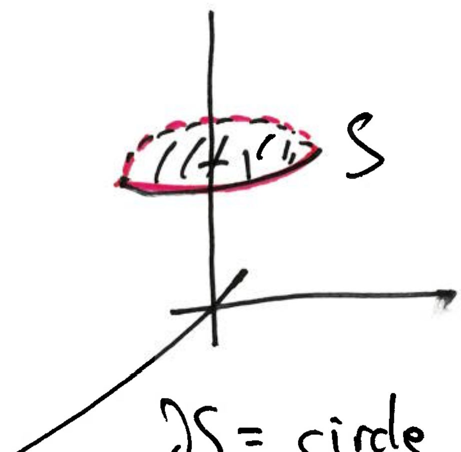
(4) $S =$ plane $z=1$



$\partial S =$ nothing

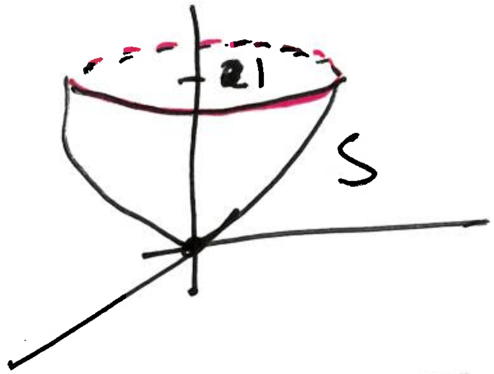
no boundary because goes forever in every direction

(5) S: $z=1,\ x^2+y^2 \leq 1$



$\partial S =$ circle $x^2+y^2=1$
$z=1$

⑥ $S = \text{graph of } z = x^2+y^2,\ x^2+y^2 \le 1$



$\partial S: x^2+y^2=1,\ z=1$

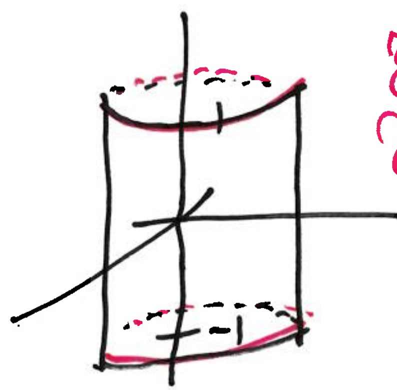
⑦ $S = \text{cone } z = \sqrt{x^2+y^2},\ x^2+y^2 \le 1$

$\partial S: x^2+y^2=1,\ z=1$

⑧ $S = \text{cylinder}$

$x^2+y^2=1$

$-1 \le z \le 1$



$\partial S: x^2+y^2=1,\ z=1$

and

$x^2+y^2=-1,\ z=-1$

Stokes Theorem Preliminary Version

S = surface, ∂S = boundary of S, F = vector field in $\mathbb{R}^3$

$$\iint_S \operatorname{curl}(F)\cdot d\vec{S} = \int_{\partial S} F\cdot d\vec{s}$$

Question: What's missing / wrong? What's the issue?

Answer: Both integrals are oriented integrals. (The orientations of S and ∂S matter.) We must give S and ∂S compatible orientations in order for equality to be true.

Compatible Orientations

S^+ = oriented surface, ∂S = boundary of S

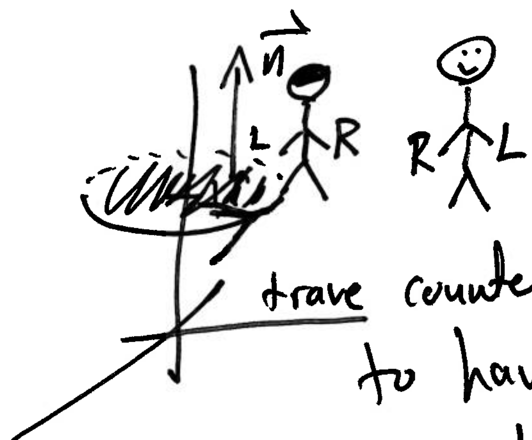
Compatible orientation of ∂S: Give ∂S the orientation that if you are walking on ∂S with your head pointing in same direction of normal vector to S^+, then S is on your left.

$\partial S^+ = \partial S$ with compatible orientation.

Example

S^+: $z=1$, $x^2+y^2 \leq 1$

$\vec{n} = (0,0,1)$



trave counter clockwise to have surface on left

S^+ $\quad z = 1 \quad x^2 + y^2 \le 1$

$\vec{n} = (0, 0, -1)$