Stokes Thm

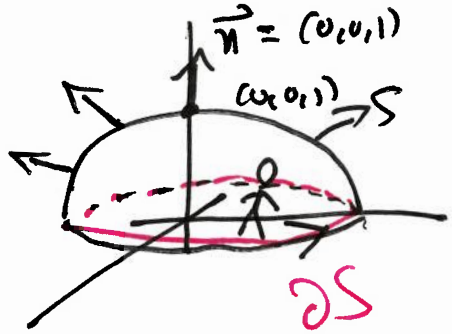
S^+ = oriented surface
∂S^+ = boundary of S with compatible orientation
F = vector field in $\mathbb{R}^3$

$$\iint_{S^+} \operatorname{curl}(F) \cdot d\vec{S} = \int_{\partial S^+} F \cdot d\vec{s}$$

Example

S = hemisphere $x^2+y^2+z^2=1$, $z>0$ with normal $(0,0,1)$ at $(0,0,1)$ ("outward" pointing normal)

$F = (y, -x, e^{xz})$. Calculate $\displaystyle\iint_{S^+} \nabla \times F \cdot d\vec{S}$.



Stokes

$$\iint_{S^+} \nabla \times F \cdot d\vec{S} = \int_{\partial S^+} F \cdot d\vec{s}$$

$\partial S^+ \quad x^2+y^2=1, \; z=0$ orientation counterclockwise

$c(t) = (\cos t, \sin t, 0), \quad 0 \le t \le 2\pi$ parametrizes ∂S^+

$$\int_{\partial S^+} F \cdot d\vec{s} = \int_0^{2\pi} F(c(t)) \cdot c'(t)\, dt, \quad F = (y, -x, e^{xz})$$

$$= \int_0^{2\pi} F(\cos t, \sin t, 0) \cdot (-\sin t, \cos t, 0)\, dt$$

$$= \int_0^{2\pi} (\sin t, -\cos t, e^{\cos t \cdot 0}) \cdot (-\sin t, \cos t, 0)\, dt$$

$$= \int_0^{2\pi} -\sin^2 t - \cos^2 t \, dt$$

$$= \int_0^{2\pi} -1 \, dt = \boxed{-2\pi}$$

Intuition/Meaning for Stokes Thm

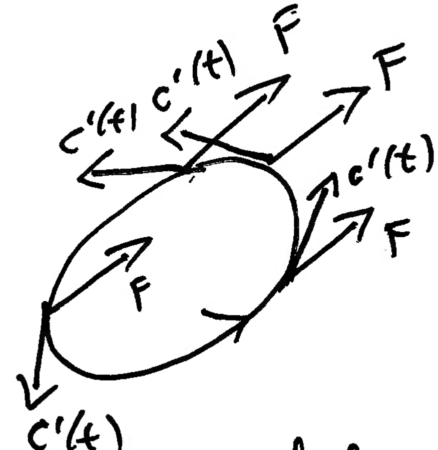
$$\int_{\partial S^+} F\cdot d\vec{s} = \int_a^b F(c(t))\cdot c'(t)\,dt =$$ measures the tangential component of F along ∂S^+

$$\iint_{S^+} \mathrm{curl}(F)\cdot d\vec{S} = \iint_D \mathrm{curl}(F)(\Phi(u,v))\cdot T_u\times T_v\,du\,dv =$$ measures the normal component of F over S^+

Simple Examples

No rotation so

$$\iint_{S^+} \mathrm{curl}(F)\cdot d\vec{S} = 0$$

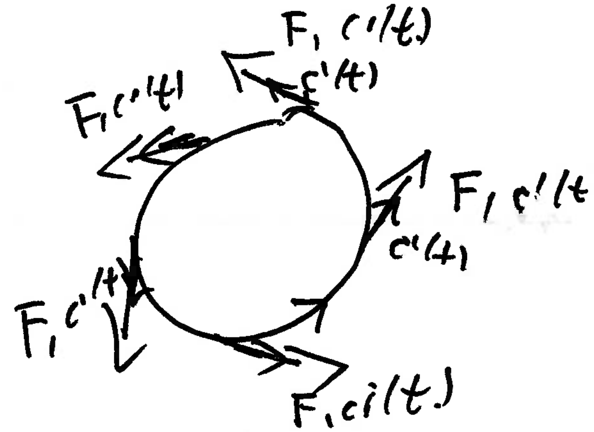


All tangential components of F cancel out so

$$\int_{\partial S^+} F\cdot d\vec{s} = 0$$

F has counterclockwise rotation

so $\displaystyle\iint_{S^+} \text{curl}(F)\cdot d\vec{S} > 0$

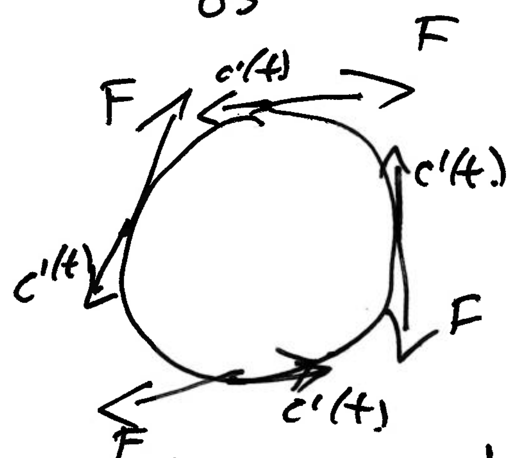


F points in same direction as tangent vectors

$\displaystyle\int_{\partial S^+} F\cdot d\vec{s} > 0$

F has clockwise rotation

so $\displaystyle\iint_{S^+} \text{curl}(F)\cdot d\vec{S} < 0$

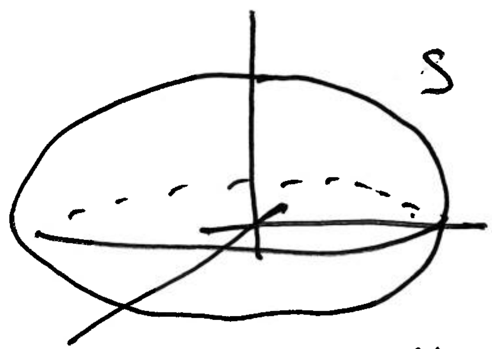


F points in opposite direction of tangent vectors, so

$\displaystyle\int_{\partial S^+} F\cdot d\vec{s} < 0$

Example

Find $\iint_S \nabla \times F \cdot d\vec{S}$ where S is the ellipsoid $x^2 + y^2 + 2z^2 = 10$ and F is the vector field $F = (\sin(xy), e^x, -yz)$. Give S normal that points away from origin.



$\partial S =$ nothing

no boundary of S

Stokes Thm

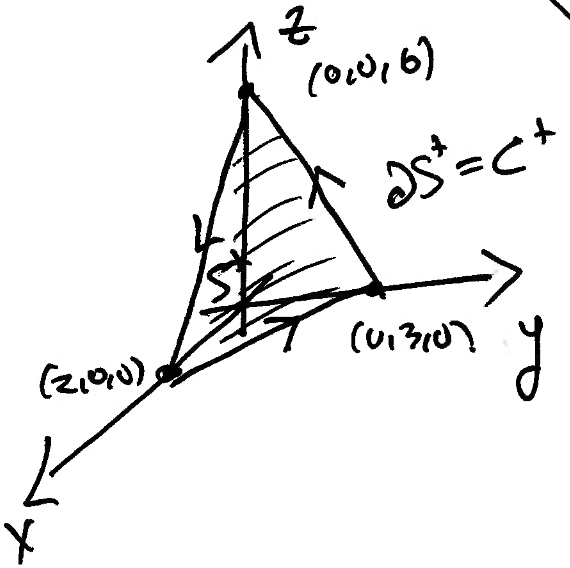
$$\iint_{S^+} \nabla \times F \cdot d\vec{S} = \int_{\partial S^+} F \cdot d\vec{s} = 0$$

because no boundary

Example

Evaluate $\displaystyle\int_C (x+y)\,dx + (2x-z)\,dy + (y+z)\,dz$

where C is the perimeter of triangle connecting $(2,0,0)$, $(0,3,0)$ and $(0,0,6)$ in that order.



normal of S is pointing away from origin

$F = (x+y,\ 2x-z,\ y+z)$

Stokes

$$\int_{C^+} F\cdot d\vec{s} = \iint_{S^+} \mathrm{curl}(F)\cdot d\vec{S}$$

Let's calculate $\displaystyle\iint_{S^+} \mathrm{curl}(F)\cdot d\vec{S}$

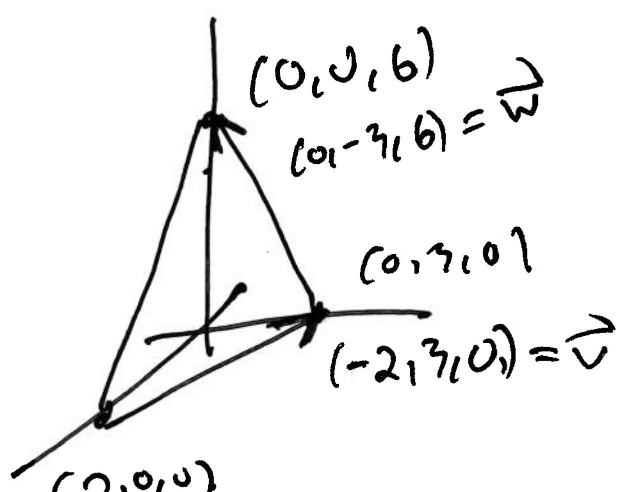
$$\operatorname{curl}(F) = \det\begin{pmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y & 2x-z & y+z \end{pmatrix}$$

$$= (1 - -1, -(0-0), 2-1) = (2,0,1)$$

Parametrize S

S is inside plane containing $(0,0,6)$, $(2,0,0)$, and $(0,3,0)$

normal to plane is



$$\vec{v} \times \vec{w} = \det\begin{pmatrix} i & j & k \\ -2 & 3 & 0 \\ 0 & -3 & 6 \end{pmatrix}$$

$$= (18, -(-12), 6)$$

$$= (18, 12, 6)$$

$$= 6 \cdot (3,2,1)$$

Equation of plane containing (x_0, y_0, z_0) with normal vector (a,b,c) is $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

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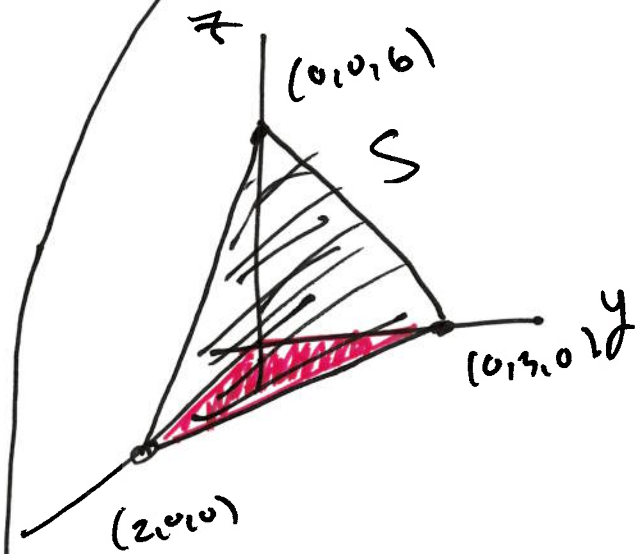
S is in the plane

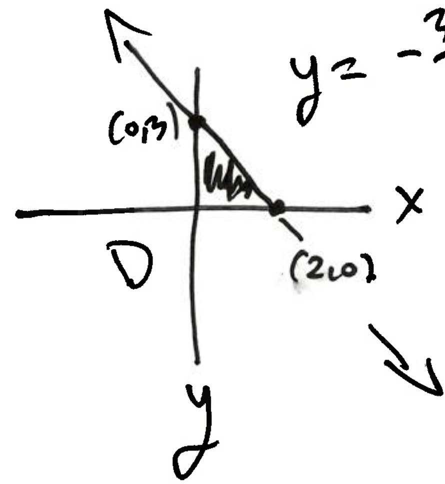
$$3(x-2) + 2y + z = 0$$

Parametrize plane $z = -3(x-2) - 2y$

$$\Phi(u,v) = (u, v, -3(u-2) - 2v)$$



$D: \; 0 \le u \le 2$
$\quad\;\; 0 \le v \le -\frac{3}{2}u + 3$



$0 \le x \le 2$
$0 \le y \le -\frac{3}{2}x + 3$

remember $\operatorname{curl}(F) = (2, 0, 1)$ integrating $\operatorname{curl}(F)$ over S with $\Phi(u,v)$

$T_u = (1, 0, -3)$

$T_v = (0, 1, -2)$

$T_u \times T_v = (3, 2, 1)$ - pointing away from origin so Φ gives correct orientation of S

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$$\int_{C^+} F \cdot d\vec{s} \overset{\text{Stokes}}{=} \iint_{S^+} \text{curl}(F) \cdot d\vec{S}$$

$$= \iint_D \text{curl}(F)(\Phi(u,v)) \cdot T_u \times T_v \, du\, dv$$

$$= \int_0^2 \int_0^{-\frac{3}{2}u+3} (2,0,1) \cdot (3,2,1) \, dv\, du$$

$$= \int_0^2 \int_0^{-\frac{3}{2}u+3} 7 \, dv\, du$$

$$= 7 \int_0^2 -\frac{3}{2}u + 3 \, du = 7\left(-\frac{3u^2}{4} + 3u \Big|_0^2 \right)$$

$$= 7(-3+6)$$

$$= \boxed{21}$$

/9.

13 and 20 from 8.2

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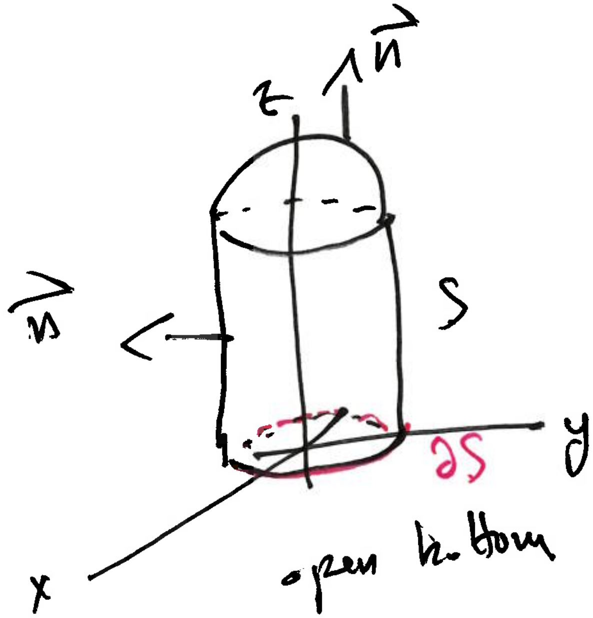

want to integrate

$$\iint_{S^+} \nabla \times F \cdot d\vec{S}$$

Use Stokes

$$\int_{\partial S^+} F \cdot d\vec{s}$$

<u>OR</u>

Change S to another surface with same boundary.

new S: $x^2+y^2 \le 1$, $z=0$

$\Phi(u,v) = (u,v,0)$, D: $u^2+v^2 \le 1$

Then surface integral is easier.