Midterm 2 scores on Gradescope
Solutions on Canvas

Gauss Thm Preview (aka Divergence Thm)

$$\iiint_W \operatorname{div} F \, dA = \iint_{\partial W} F \cdot d\vec{S}$$

W ← solid region in $\mathbb{R}^3$

∂W ← boundary / bounding surface of W

4.4 Divergence

$F = (P, Q, R)$ vector field in $\mathbb{R}^3$

The **divergence** of F is defined to be the scalar function

$$\operatorname{div} F = \nabla \cdot F = \frac{\partial (P)}{\partial x} + \frac{\partial (Q)}{\partial y} + \frac{\partial (R)}{\partial z}$$

Example

$$F = (x^2y,\ z,\ xyz)$$

$$\operatorname{div} F = \frac{\partial(x^2y)}{\partial x} + \frac{\partial(z)}{\partial y} + \frac{\partial(xyz)}{\partial z} = 2xy + 0 + xy = 3xy$$

Meaning of Divergence: F = velocity field of a gas or fluid

$\operatorname{div} F$ = the rate of expansion per unit volume under the flow of the gas or fluid

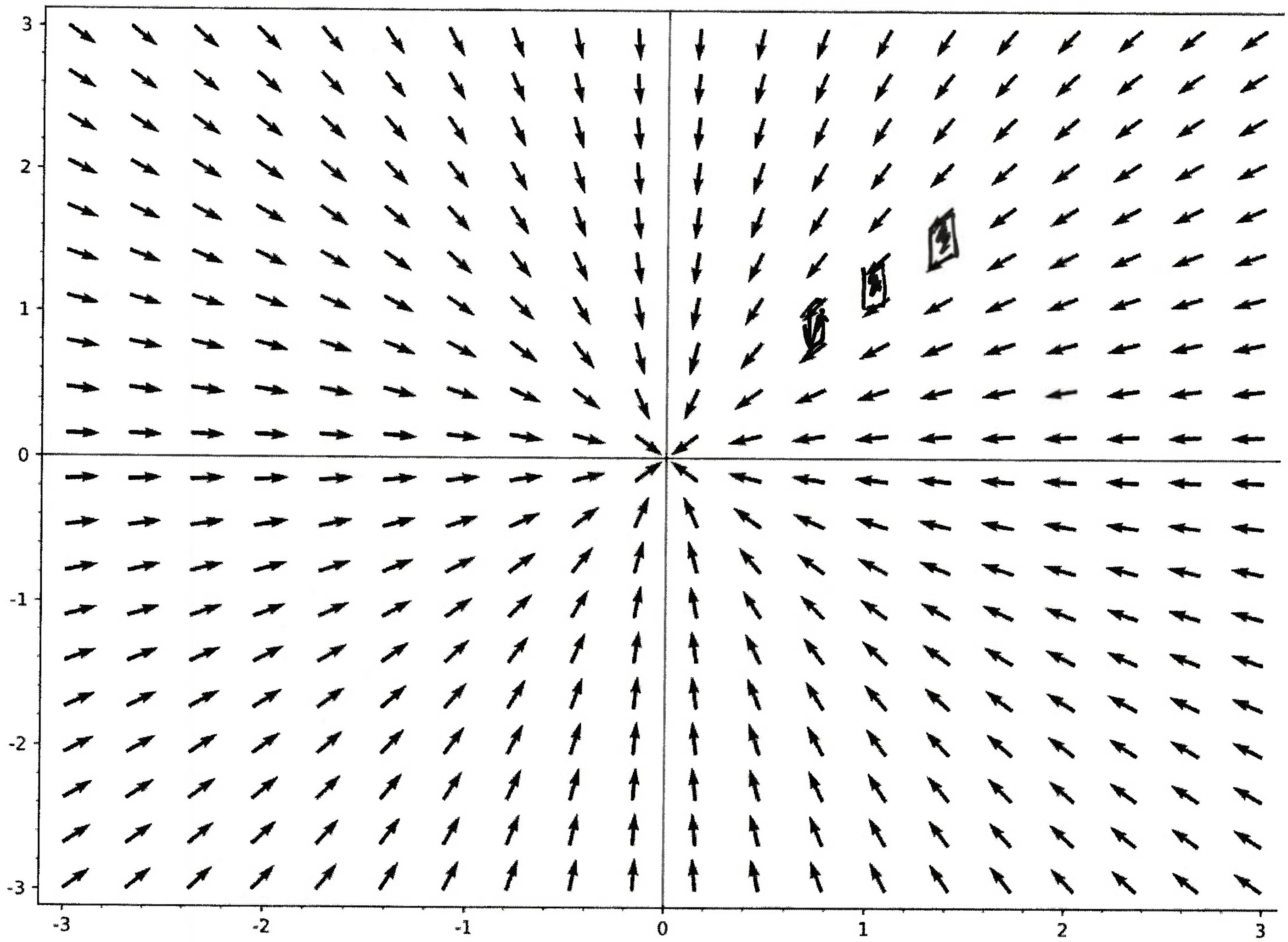
$\operatorname{div} F > 0$ expanding

$\operatorname{div} F < 0$ compressing

$F = (P, Q)$ vector field in $\mathbb{R}^2$, $\quad \operatorname{div} F = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$

in meaning, replace volume with area

```
x,y = var('x y')
plot_vector_field((-x/sqrt(x^2 + y^2), -y/sqrt(x^2 + y^2)), (x,-3,3), (y,-3,3), figsize = 10)
```



$$F(x,y) = \left(\frac{-x}{(x^2+y^2)^{1/2}}, \frac{-y}{(x^2+y^2)^{1/2}} \right)$$

$$\mathrm{div}(F) = \frac{(x^2+y^2)^{1/2}\cdot(-1) - (-x)\cdot 2x \cdot (x^2+y^2)^{-1/2}}{(x^2+y^2)^{1}} +$$

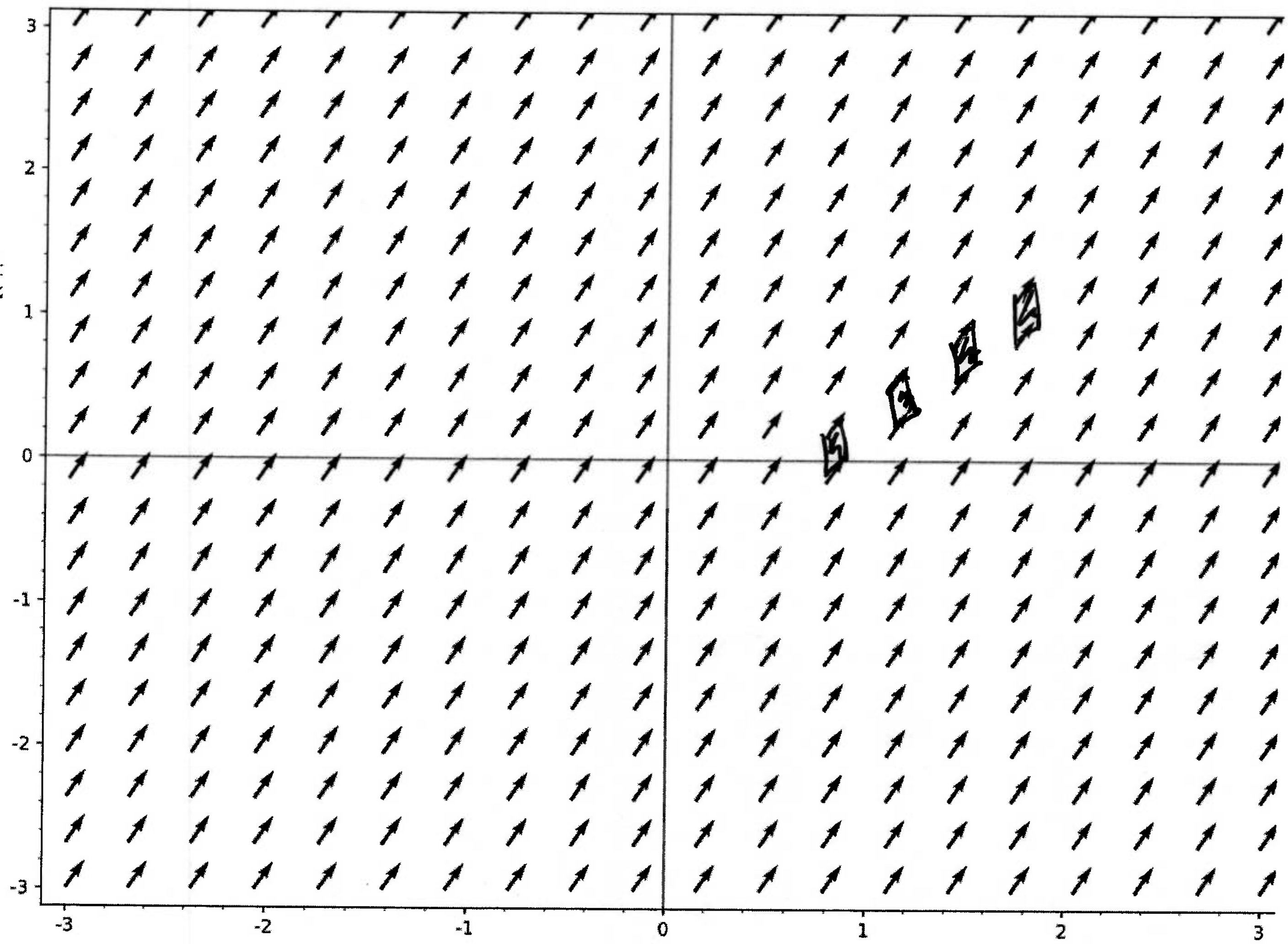
$$+ \frac{(x^2+y^2)^{1/2}\cdot(-1) - (-y)\cdot 2y \cdot (x^2+y^2)^{-1/2}}{(x^2+y^2)^{1}}$$

$$=$$

$$\frac{\dfrac{-(x^2+y^2)+2x^2}{(x^2+y^2)^{1/2}}}{(x^2+y^2)}+\frac{\dfrac{-(x^2+y^2)+2y^2}{(x^2+y^2)^{1/2}}}{(x^2+y^2)}$$

$$=\frac{-x^2-y^2+2x^2-x^2-y^2+2y^2}{(x^2+y^2)^{3/2}}=0$$

```
x,y = var('x y')
plot_vector_field((1,2), (x,-3,3), (y,-3,3), figsize = 10)
```



$$F(x,y) = (1,2)$$

$$\operatorname{div} F = \frac{\partial(1)}{\partial x} + \frac{\partial(2)}{\partial y} = 0$$

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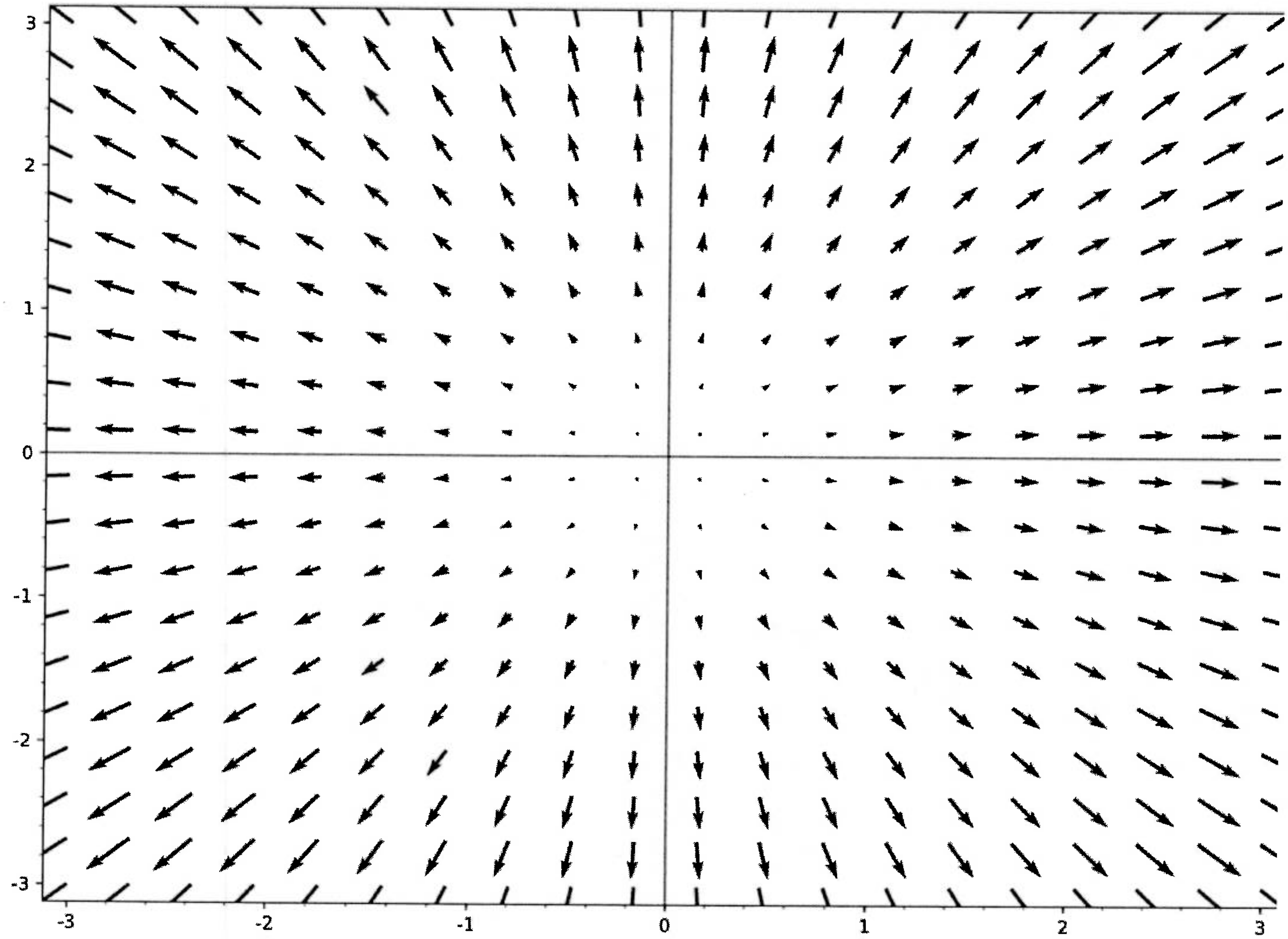
```
x,y = var('x y')
plot_vector_field((x,y), (x,-3,3), (y,-3,3), figsize = 10)
```

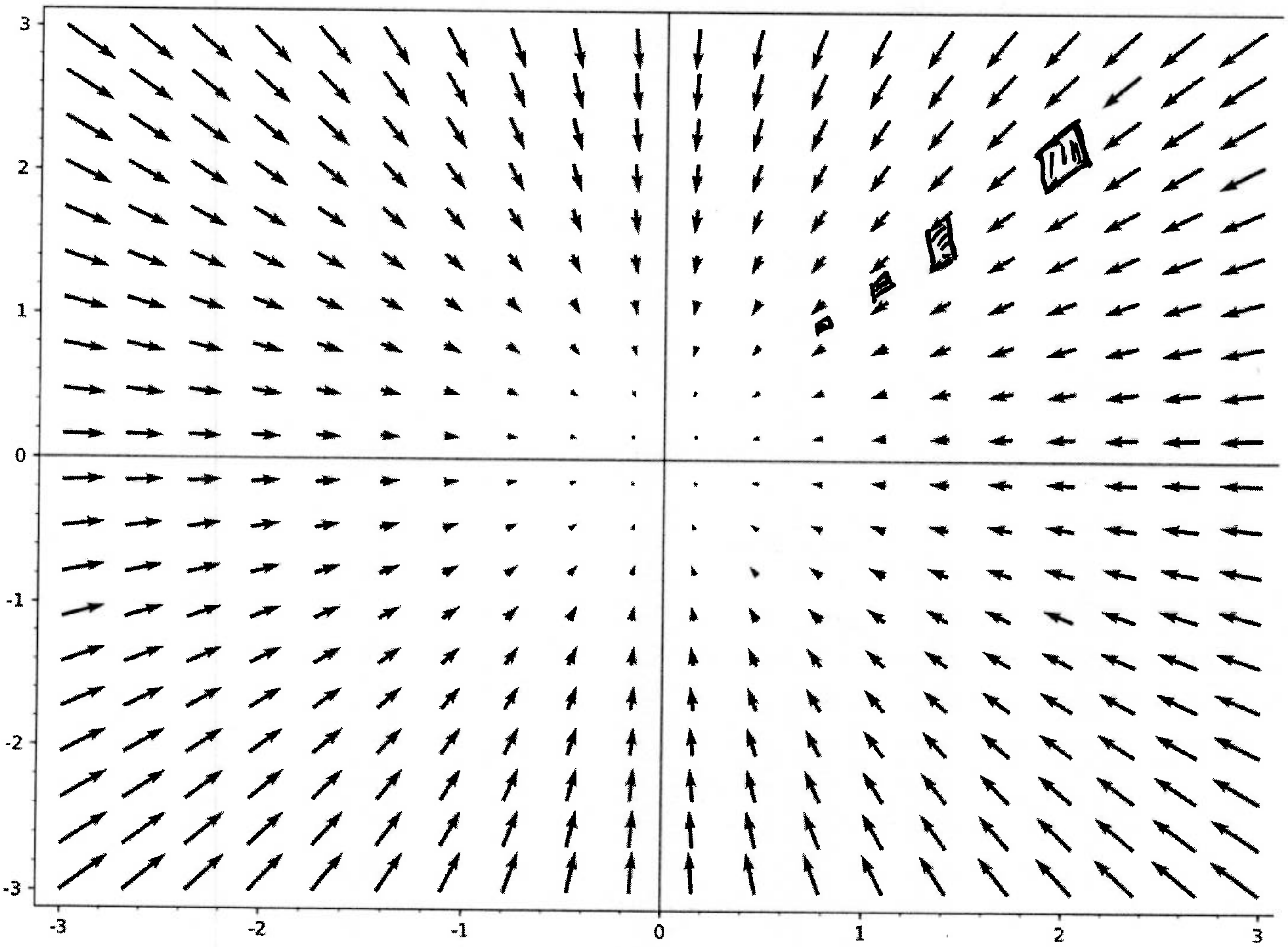


$F(x,y) = (x,y)$

$\text{div}(F) = \frac{\partial(x)}{\partial x} + \frac{\partial(y)}{\partial y} = 2$

/4

```
x,y = var('x y')
plot_vector_field((-x, -y), (x,-3,3), (y,-3,3), figsize = 10)
```



$$F(x,y) = (-x,-y)$$

$$\operatorname{div}(F) = \frac{\partial(-x)}{\partial x} + \frac{\partial(-y)}{\partial y} = -2$$

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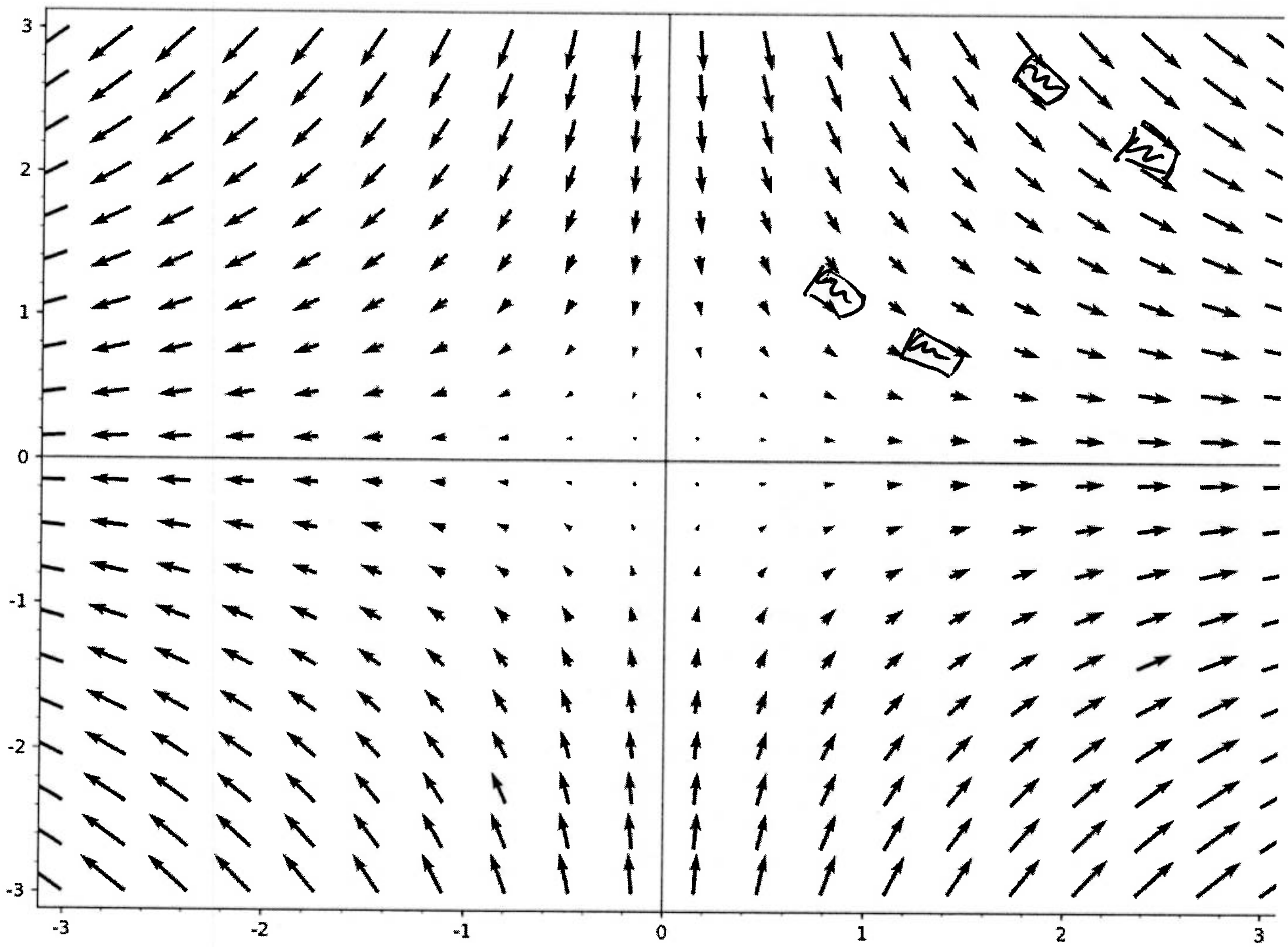
```
x,y = var('x y')
plot_vector_field((x, -y), (x,-3,3), (y,-3,3), figsize=10)
```



$F(x,y) = (x, -y)$

$\operatorname{div}(F) = 1 + -1 = 0$

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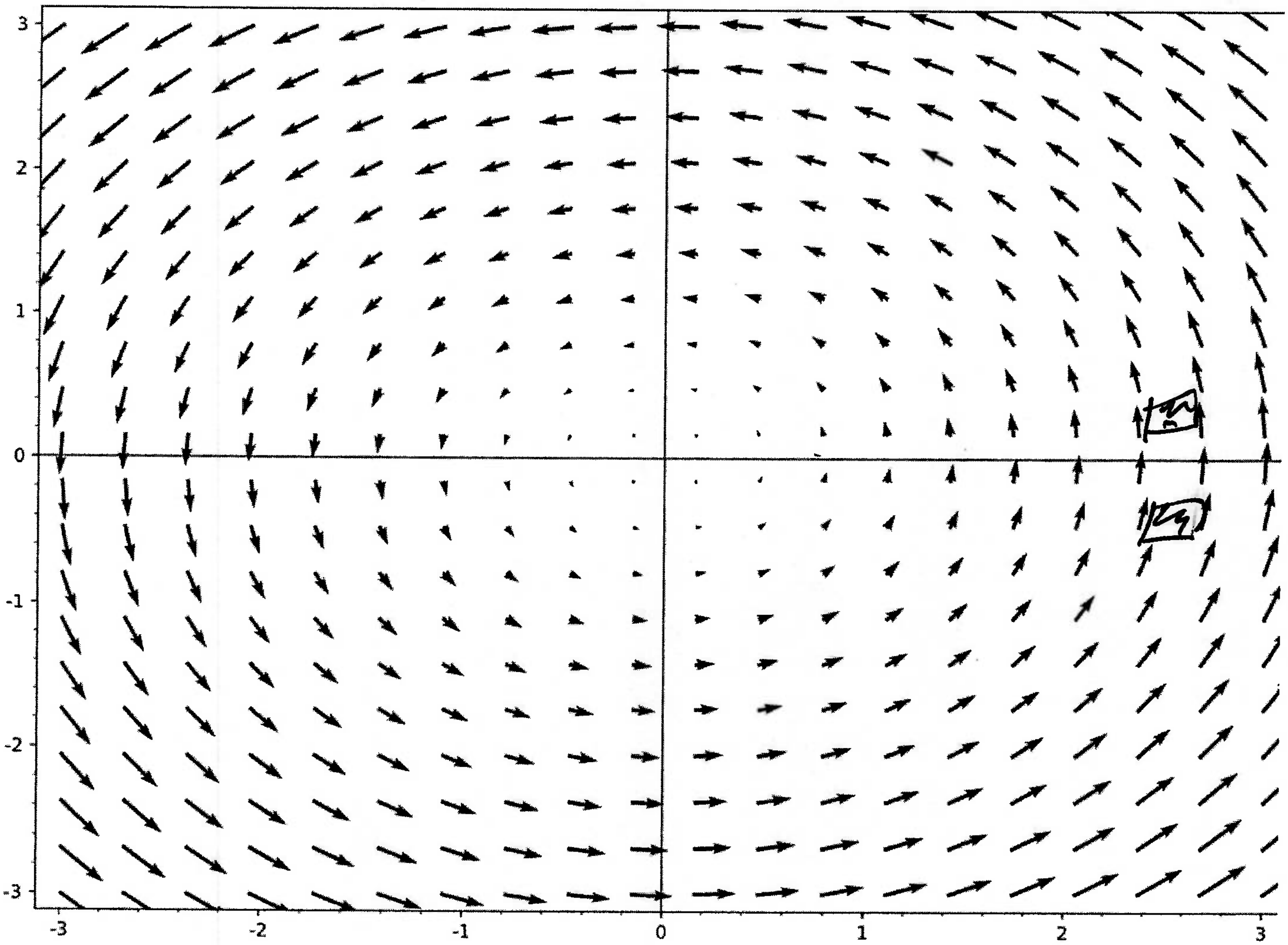
```
x,y = var('x y')
plot_vector_field((-y, x), (x,-3,3), (y,-3,3), figsize = 10)
```



$$F(x,y) = (-y, x)$$

$$\mathrm{div}(F) = 0 + 0 = 0$$

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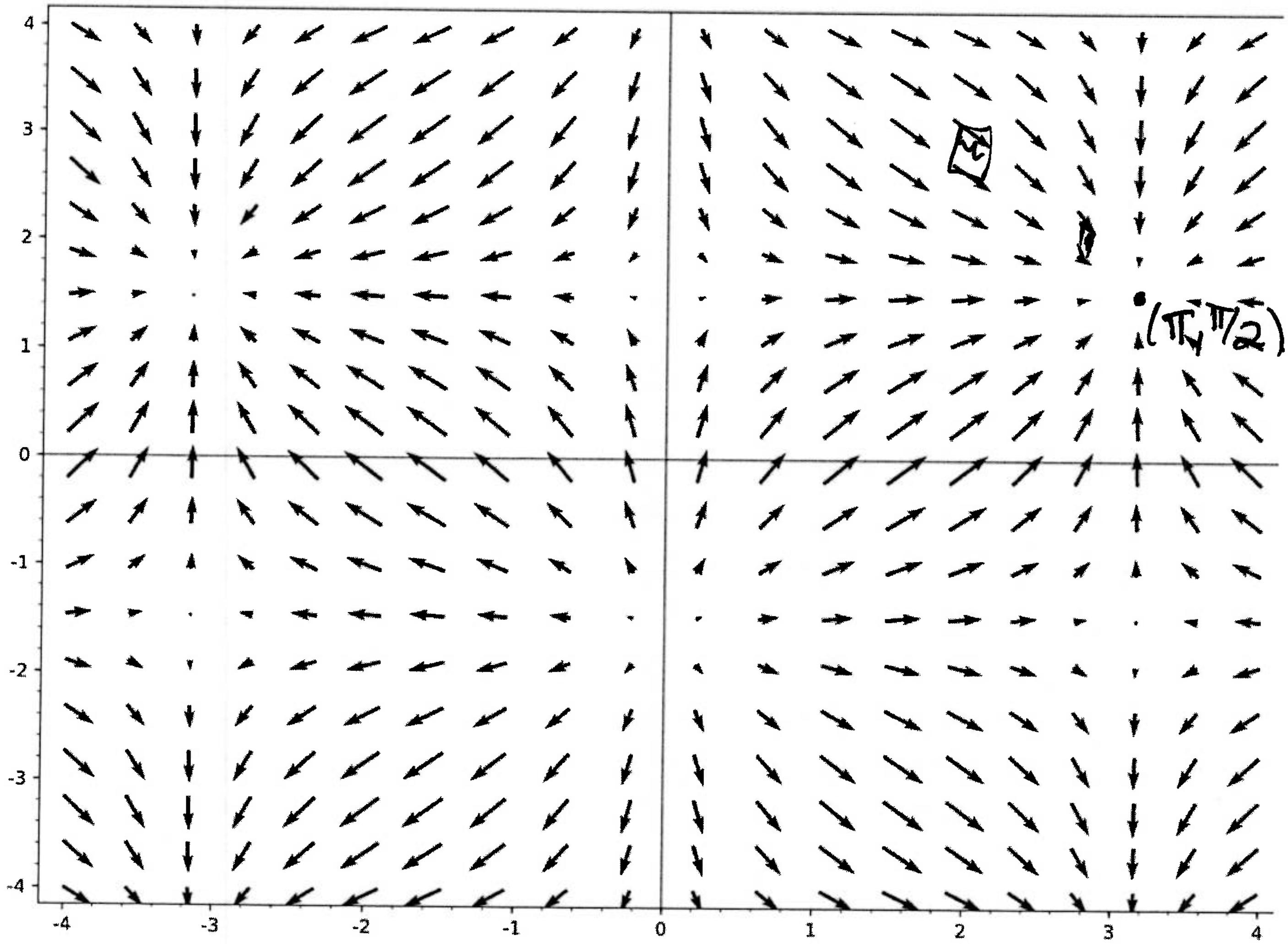
```
x,y = var('x y')
plot_vector_field((sin(x),cos(y)), (x,-4,4), (y,-4,4), figsize = 10)
```



$F(x,y) = (\sin(x), \cos(y))$

$\text{div}(F) = \cos(x) - \sin(y)$

$\cos(\pi) - \sin(\pi/2) = -1 - 1 = 2$

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8.4 Gauss' Theorem

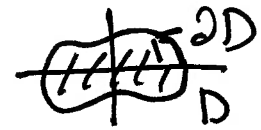
$W =$ closed and bounded elementary region in $\mathbb{R}^3$
(solid in $\mathbb{R}^3$ so have volume)

for instance

$$W: \quad a \le x \le b$$
$$\gamma_1(x) \le y \le \gamma_2(x)$$
$$\eta_1(x,y) \le z \le \eta_2(x,y)$$

Green $F = (P, Q)$

$$\iint_D \operatorname{curl} F \cdot k \, dA = \int_{\partial D} F \cdot d\vec{s}$$



$\partial W =$ boundary of W

∂W is the surface that enclosed W

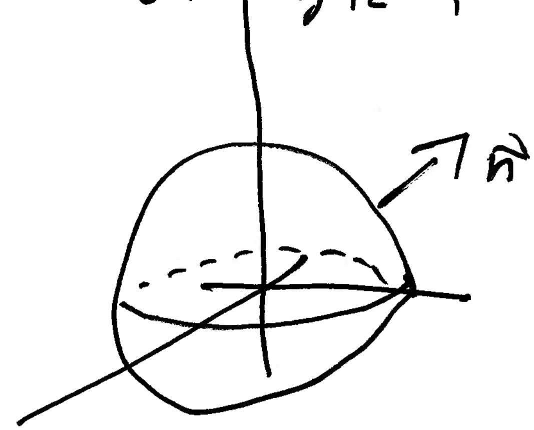
∂W is called a closed surface.

Give ∂W the orientation with outward pointing normal (makes sense since ∂W enclosed W, so W is the inside).

Example (1)

$W: x^2 + y^2 + z^2 \le 1$

$\partial W: x^2 + y^2 + z^2 = 1$



② $W = [0,1] \times [0,1] \times [0,1]$ inside and surface of box

$\partial W =$ boundary of W surface of box

six faces

$z = 1, \quad 0 \le x, y \le 1$

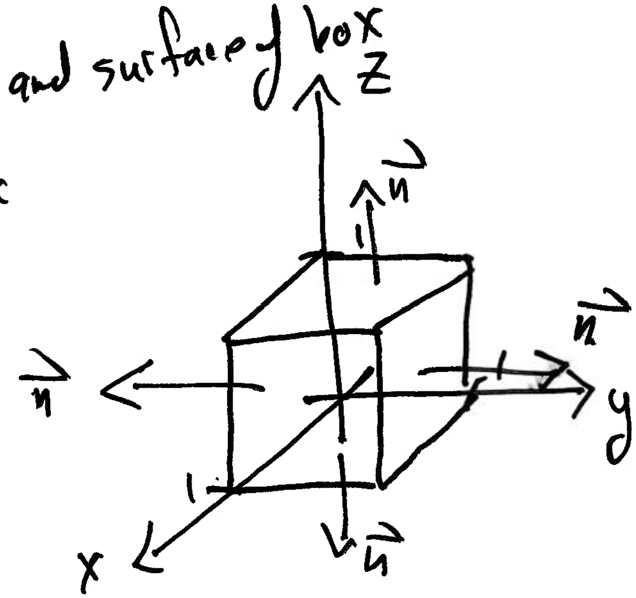
$z = 0, \quad$ " "

$x = 1, \quad 0 \le y, z \le 1$

$x = 0, \quad$ " "

$y = 1, \quad 0 \le x, z \le 1$

$y = 0, \quad 0 \le x, z \le 1$



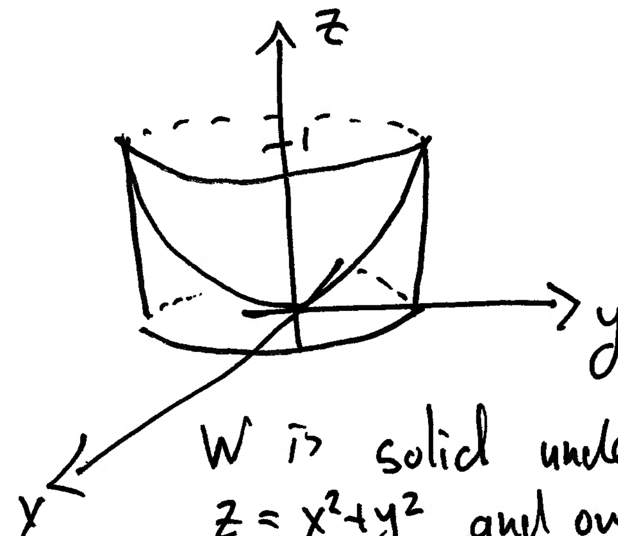
③ $W: \quad 0 \le z \le x^2 + y^2$

$x^2 + y^2 \le 1$

$\partial W: \quad z = x^2 + y^2, \quad x^2 + y^2 \le 1$

$z = 0, \quad x^2 + y^2 \le 1$

$x^2 + y^2 = 1, \quad 0 \le z \le 1$



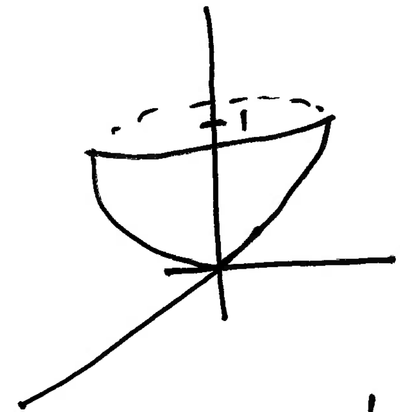
W is solid under $z = x^2 + y^2$ and over $x^2 + y^2 \le 1$, $z = 0$

④ W: • $x^2+y^2 \le z \le 1$

$x^2+y^2 \le 1$

∂W: $z = x^2+y^2$, $x^2+y^2 \le 1$

$z=1$, $x^2+y^2 \le 1$



W is solid above $z = x^2+y^2$ and below $z=1$

Gauss' Theorem (aka Divergence Theorem)

W = elementary in $\mathbb{R}^3$, ∂W = boundary of W with outward normal orientation,
F = vector field in $\mathbb{R}^3$. Then

$$\iiint_W \operatorname{div} F \, dV = \iint_{\partial W} F \cdot d\vec{S}$$

In words: Flux of a vector field out of a closed surface equals the integral of the divergence of that vector field over the volume enclosed by the surface.

Example

Evaluate $\iint_S F \cdot d\vec{S}$ where S is

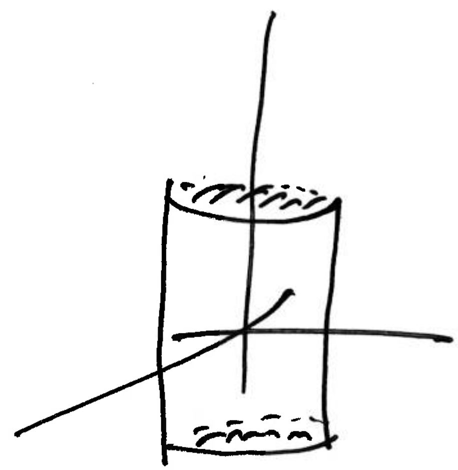
$x^2+y^2=1$, $-1 \le z \le 1$

$x^2+y^2 \le 1$, $z=-1$

$x^2+y^2 \le 1$, $z=1$

and $F = (xy^2, x^2y, y)$

Gauss



$$\iint_S F \cdot d\vec{S} = \iiint_W \operatorname{div} F \, dV$$

$$\operatorname{div} F = y^2 + x^2 + 0$$

W: $x^2+y^2 \le 1$, $-1 \le z \le 1$

$\iiint_W x^2+y^2 \, dV \leftarrow$ easy integral with cylindrical coord.