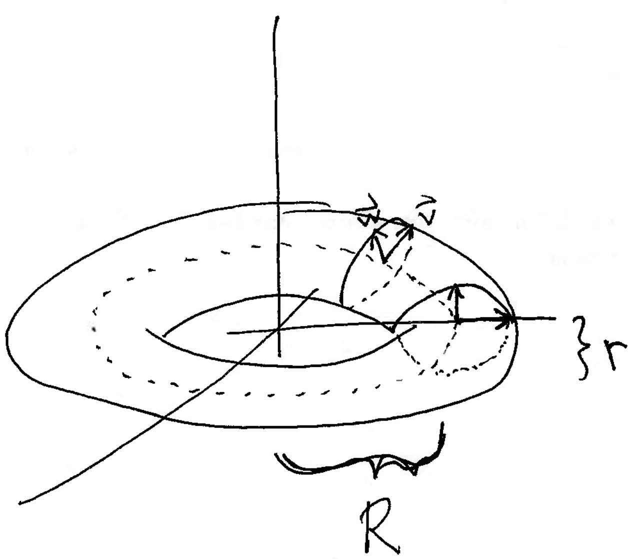
Torus



Parametrizing torus

circle of radius R

$$c(u) = (R\cos u, R\sin u, 0), \qquad 0 \le u \le 2\pi$$

circle of radius r around $c(u)$

$$c(u) + r\cos(v)\vec{v} + r\sin(v)\vec{w}, \qquad 0 \le v \le 2\pi$$

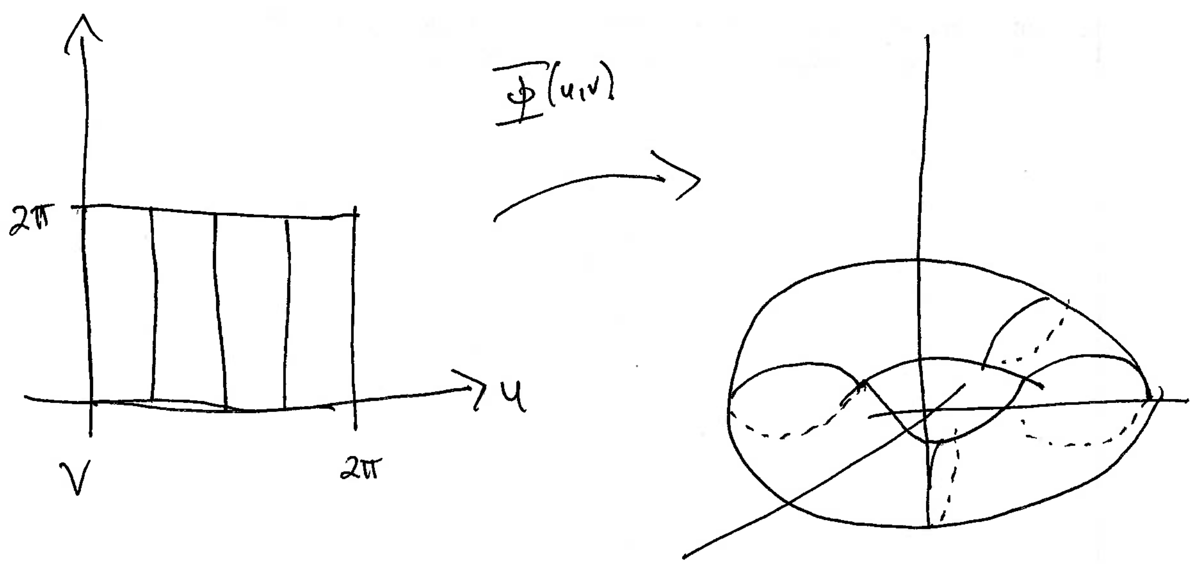
$$\vec{w} = (0,0,1), \quad \vec{v} = \frac{c(u)}{\|c(u)\|} = (\cos u, \sin u, 0)$$

$$\Phi(u,v) = (R\cos u, R\sin u, 0) + r\cos(v)(\cos u, \sin u, 0) + r\sin(v)(0,0,1)$$

$$= (R\cos u, R\sin u, 0) + (r\cos^2 u, r\cos u \sin u, 0) + (0,0,r\sin(v))$$

$$= (R\cos u + r\cos v\cos u, R\sin u + r\cos v\sin u, r\sin(v))$$

$$\Phi(u,v) = (\cos u(R + r\cos v), \sin u(R + r\cos v), r\sin v)$$



Surface area of Torus?

$$T_u = (-\sin u(R + r\cos v), \cos u(R + r\cos v), 0)$$

$$T_v = (\cos u(-r\sin v), \sin u(-r\sin v), r\cos v)$$

$$T_u \times T_v = (r\cos v\cos u(R + r\cos v), + r\cos v\sin u(R + r\cos v),$$

$$+ \sin^2 u(R + r\cos v) r\sin v + \cos^2 u(R + r\cos v) r\sin v)$$

$$= (r\cos v\cos u(R + r\cos v), r\cos v\sin u(R + r\cos v), (R + r\cos v) r\sin v)$$

$$\|T_u \times T_v\| = \sqrt{r^2\cos^2 v\cos^2 u\,(R+r\cos v)^2 + r^2\cos^2 v\sin^2 u\,(R+r\cos v)^2 + (R+r\cos v)^2 r^2\sin^2 v}$$

$$= (R+r\cos v)\sqrt{r^2\cos^2 v\,(\cos^2 u+\sin^2 u) + r^2\sin^2 v}$$

$$= (R+r\cos v)\sqrt{r^2\cos^2 v + r^2\sin^2 v}$$

$$= (R+r\cos v)\,r$$

Surface Area

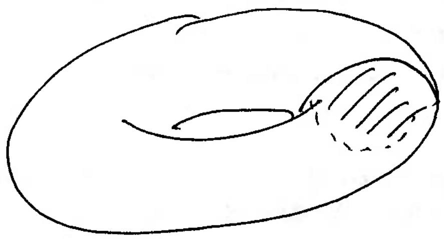
$$\int_0^{2\pi}\int_0^{2\pi}(R+r\cos v)\,r\,du\,dv = 2\pi r\left(Rv + r\sin v\,\Big|_0^{2\pi}\right)$$

$$= 2\pi r\,R\,2\pi$$

$$= Rr(2\pi)^2$$

/3

Volume of Torus

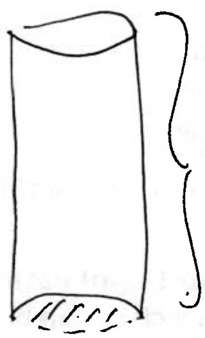


slice has area

πr^2

circumference of big circle of radius R is

$2\pi R$



$2\pi R$

πr^2

$$\text{Volume} = 2\pi R \cdot \pi r^2$$

Fact: Torus is solutions to equation

$$\left(R - \sqrt{x^2+y^2}\right)^2 + z^2 = r^2$$

Example

Integrate $F(x,y,z) = \left(\frac{1}{3}x + yz,\ \frac{1}{3}y + \sin(xz),\ \frac{1}{3}z + e^{xy}\right)$

over torus T with outward orientation.

Gauss. Theorem

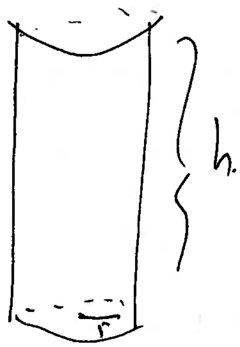
$$\iint_T F \cdot d\vec{S} = \iiint_W \operatorname{div} F \, dV$$

where W is solid that T bounds.

$$\operatorname{div} F = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$$

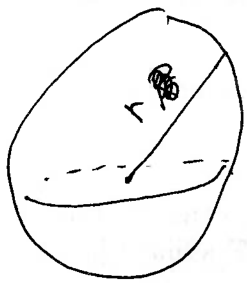
so

$$\iint_T F \cdot d\vec{S} = \iiint_W 1 \cdot dV = \text{volume}(W) = 2\pi R \cdot \pi r^2$$



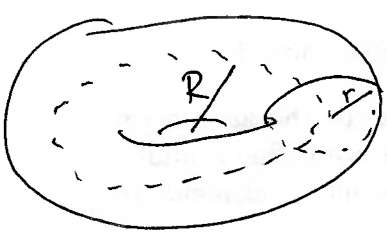
volume $= \pi r^2 h$ = ~~area~~ volume of W

surface area $= 2\pi r h$ = area of ∂W
(without top and bottom)



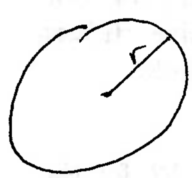
volume $= \frac{4}{3}\pi \cancel{R^3} r^3$ = volume of W

surface area $= 4\pi \cancel{R} r^2$ = area of ∂W



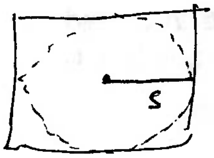
volume $= 2\pi^2 R r^2$ = volume of W

surface area $= 4\pi^2 R r$ = area of ∂W



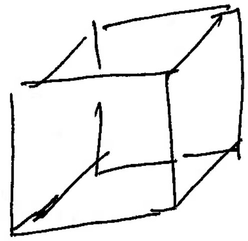
area $\cancel{\varnothing}$ $= \pi r^2$ = area of D

perimeter $= 2\pi r$ = length of ∂D



area $= (2s)^2 = 4s^2$ = area of D

perimeter $= 4 \cdot 2s = 8s$ = length of ∂D



2s

volume $= (2s)^3 = 8s^3$ = volume of W

surface area $= 6\cdot(2s)^2 = 6\cdot 4\cdot s^2 = 24s^2$ = area of ∂W

Relationship does not hold for all shapes.

Motivation for notation ∂ for boundary.