Gauss' Theorem (aka divergence theorem)

W - bounded solid in $\mathbb{R}^3$, ∂W - surface boundary of W with outward orientation

F - vector field in $\mathbb{R}^3$

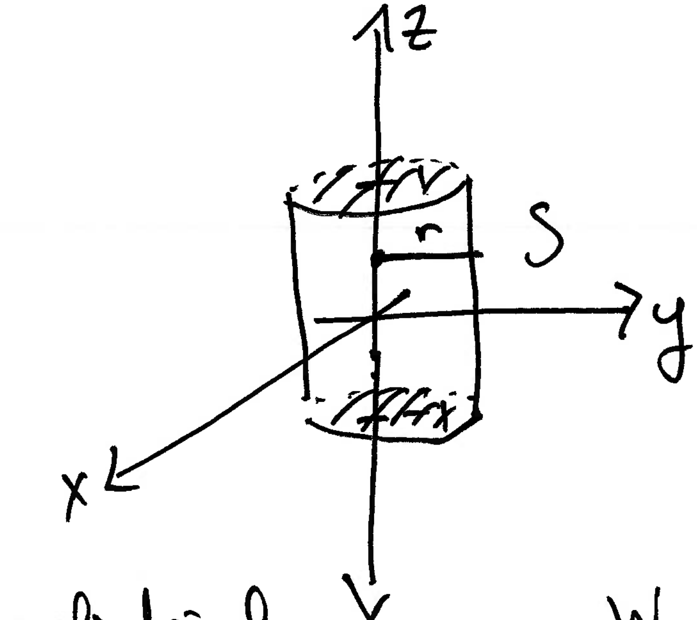
$$\iiint_W \operatorname{div} F \, dV = \iint_{\partial W} F \cdot d\vec{S}$$

$$\left(F = (P, Q, R), \quad \operatorname{div} F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right)$$

Examples

(1) Evaluate $\iint_S F \cdot d\vec{S}$ where S is the surface of the cylinder $x^2+y^2=1$, $-1 \le z \le 1$ with ~~*~~ the top and bottom of the cylinder, and

$$F = (xy^2, x^2y, y).$$



W = ~~inde~~ terior of cylinder

$\partial W = S$

Gauss Thm

$$\iint_S F\cdot d\vec{S} = \iiint_W \mathrm{div}\, F\, dV$$

cylindrical

$-1 \le z \le 1$

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

W as "elementary region"

$x^2+y^2 \le 1, \quad -1 \le z \le 1$

leave for now since may use cylindrical coord later

$$\mathrm{div}\, F = y^2 + x^2 + 0 = x^2+y^2$$

$$\iint_S F\cdot d\vec{S} = \iiint_W x^2+y^2\, dV \underset{\text{cylindrical}}{=} \int_0^{2\pi}\int_{-1}^{1}\int_0^{1} r^2\, r\,dr\,dz\,d\theta$$

$dV = r\,dr\,d\theta\,dz$

$x^2+y^2 = r^2$

$$= 2\pi \cdot 2 \cdot \frac{r^4}{4}\Big|_0^1$$

$$= 4\pi \cdot \frac{1}{4} = \boxed{\pi}$$

(2) Find the flux of the vector field $F = (x - y^2, y, x^3)$ out of the rectangular solid $[0,1] \times [1,2] \times [1,4]$.

$W = [0,1] \times [1,2] \times [1,4]$

(means $0 \le x \le 1$, $1 \le y \le 2$, $1 \le z \le 4$.)

∂W = surface of W (aka boundary of W) (with outward orientation).

want $\iint_{\partial W} F \cdot d\vec{S}$

Gauss Thm

$$\iint_{\partial W} F \cdot d\vec{S} = \iiint_W \operatorname{div} F \, dV$$

$\operatorname{div} F = 1 + 1 + 0 = 2$, $\quad F = (x - y^2, y, x^3)$

$$\iiint_W \operatorname{div} F \, dV = \int_0^1 \int_1^2 \int_1^4 2 \, dz \, dy \, dx = 2 \cdot 3 \cdot 1 \cdot 1$$

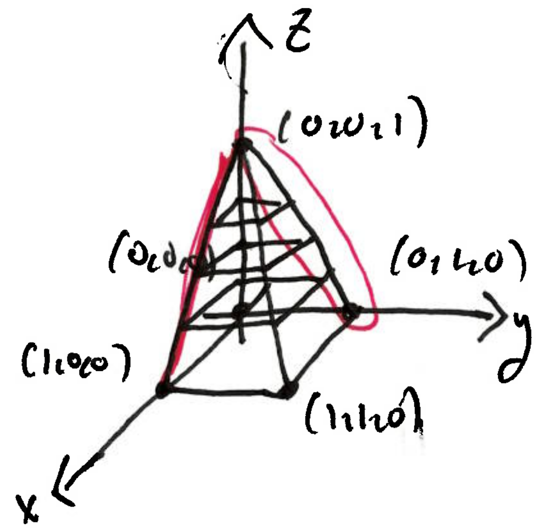
$$= 2 \text{ volume}(W) = 2 \cdot 1 \cdot 1 \cdot 3 = \boxed{6}$$

(3) $W =$ solid pyramid with top vertex $(0,0,1)$, and base vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(1,1,0)$.

$S = \partial W$ boundary of W with outward orientation

$F = (x^2 y, 3y^2 z, 9z^2 x)$

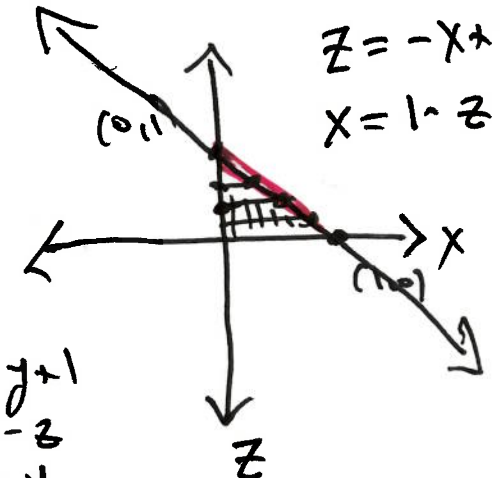
Calculate $\iint_S F \cdot d\vec{S}$.



Gauss Thm $\iint_S F \cdot d\vec{S} = \iiint_W \operatorname{div} F \, dV$

$$\operatorname{div} F = \frac{\partial(x^2y)}{\partial x} + \frac{\partial(3y^2z)}{\partial y} + \frac{\partial(9z^2x)}{\partial z}$$

$$= 2xy + 6yz + 18xz$$

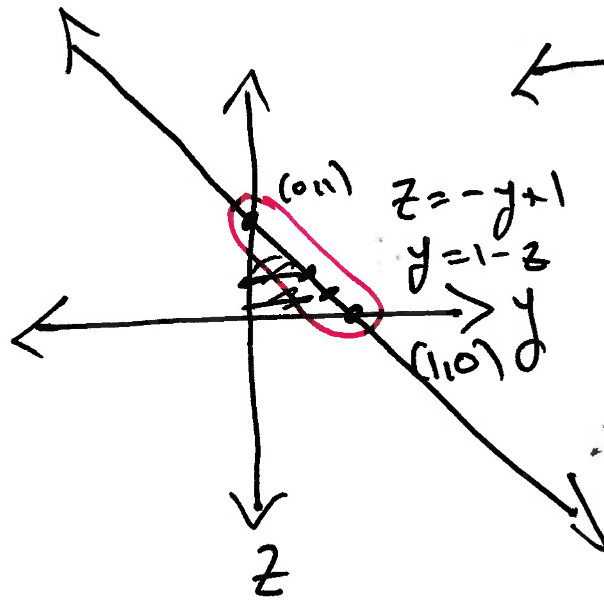


W as elementary region

$0 \le z \le 1$

$0 \le x \le 1-z$

$0 \le y \le 1-z$



$$\iiint_W \operatorname{div} F \, dV =$$

$$= \int_0^1 \int_0^{1-z} \int_0^{1-z} 2xy + 6yz + 18xz \; dy\, dx\, dz$$

$$= \int_0^1 \int_0^{1-z} xy^2 + 3y^2z + 18xyz \Big|_{y=0}^{y=1-z} \, dx\, dz$$

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$$= \int_0^1 \int_0^{1-z} x(1-z)^2 + 3(1-z)^2 z + 18x(1-z)z \, dx\, dz$$

$$= \int_0^1 \frac{x^2(1-z)^2}{2} + 3x(1-z)^2 z + 9x^2(1-z)z \Big|_{x=0}^{x=1-z} dz$$

$$= \int_0^1 \frac{(1-z)^4}{2} + 3(1-z)^3 z + 9(1-z)^3 z \, dz$$

$$= \int_0^1 \frac{(1-z)^4}{2} + 12(1-z)^3 z \, dz$$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

$$= \int_0^1 \frac{(1-z)^4}{2} + 12(1-3z+3z^2-z^3)z \, dz$$

$$= \int_0^1 \frac{(1-z)^4}{2} + 12z - 36z^2 + 36z^3 - 12z^4 \, dz$$

$$= \left. \frac{-(1-z)^5}{10} + 6z^2 - 12z^3 + 9z^4 - \frac{12z^5}{5} \right|_0^1$$

$$= 0 + 6 - 12 + 9 - \frac{12}{5} - \left(-\frac{1}{10} + 0\right)$$

$$= 3 - \frac{24}{10} + \frac{1}{10} = \frac{30}{10} - \frac{23}{10} = \boxed{\frac{7}{10}}$$

Conservative Vector Fields

FTOC from single variable calculus

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

~~" Integration and deriv~~

" integrating and taking the derivative are inverse operations "

what's the antiderivative of $\cos(x)$

$$\int_a^b \cos(x)\,dx \overset{?}{=} \sin(x)\Big|_a^b$$

$\int_a^b f(x)\,dx =$ what is the function $F(x)$ such that $F'(x) = f(x)$?

Multivariable FTOC

$$\int_{c(t)} \nabla f \cdot d\vec{s} = f(c(b)) - f(c(a))$$

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA = \int_{\partial D} P\,dx + Q\,dy \qquad \text{(Green)}$$

$$\iint_{S^+} \operatorname{curl} F \cdot d\vec{S} = \int_{\partial S^+} F \cdot d\vec{s} \qquad \text{(Stokes)}$$

$$\iiint_W \operatorname{div} F \, dV = \iint_{\partial W} F \cdot d\vec{S} \qquad \text{(Gauss)}$$

Given $\int_{c(t)} F \cdot d\vec{s}$ is $F = \nabla f$?

If the answer is yes, ~~the~~ and we can find f, then the integral is easier with f.

Stokes $\iint_{S^+} \text{curl}\, F \cdot d\vec{S} = \int_{\partial S^+} F \cdot d\vec{s}$

Given $\iint_{S^+} G \cdot d\vec{S}$, is $G = \text{curl}\, F$? If yes then

$\iint_{S^+} G \cdot d\vec{S} = \int_{\partial S^+} F \cdot d\vec{s}$ and $\int_{\partial S^+} F \cdot d\vec{s}$ may be easier.

Gauss $\iiint_W \text{div}\, F \, dV = \iint_{\partial W} F \cdot d\vec{S}$.

Given $\iiint_W f\,dV$, is $f = \operatorname{div} F$? If yes, then

$\iiint_W f\,dV = \iint_{\partial W} F \cdot d\vec{S}$, and $\iint_{\partial W} F \cdot d\vec{S}$ may

be easier.