Announcements

- Final Tuesday 3:00-5:59pm in this room
- Extra office hours

  Fri 10:30am - 12:30pm

  Mon. 11:30am - 1:30pm AP&M 6321
- Posted old final exam on Canvas
  - Typo in problem #4, should say

    $$f(x,y,z) = \frac{1}{\left(1+(x^2+y^2+z^2)^{3/2}\right)^{3/2}}$$
  - Will not post solutions
- Everyone gets 100% on homework 8

8.3 Conservative Vector Fields

Question 1: Given a vector field F in $\mathbb{R}^2$ or $\mathbb{R}^3$, does there exist a scalar valued function f such that
$F = \nabla f$?

Question 2: Given a vector field F in $\mathbb{R}^3$, does there exist a vector field G such that $F = \text{curl}(G)$?

Single Variable calculus: Given $f(x)$, define

$$g(x) = \int_0^x f(t)\,dt$$

Then $g'(x) = f(x)$.

Question 1: Answer is <u>not</u> always yes.

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

$$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

Example $F(x,y) = (y, -x)$

Suppose $F = \nabla f$,

then $\frac{\partial f}{\partial x} = y$ and $\frac{\partial f}{\partial y} = -x$

$$\frac{\partial^2 f}{\partial y \partial x} = 1 \qquad \frac{\partial^2 f}{\partial x \partial y} = -1$$

Since $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$ we got a contradiction,

so f cannot exist.

Fact 1: Let F be a vector field in $\mathbb{R}^3$ such that $F = \nabla f$. Then $\operatorname{curl}(F) = (0,0,0)$.

proof: $F = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$$\operatorname{curl}(F) = \det \begin{pmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \end{pmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, -\frac{\partial^2 f}{\partial x \partial z} + \frac{\partial^2 f}{\partial z \partial x}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right)$$

$$= (0, 0, 0). \quad \square$$

<u>Fact 2</u>: Let F be a vector field in $\mathbb{R}^2$ such that $F = \nabla f$. Then the scalar curl of F is 0.

<u>proof</u>: $F = (P, Q)$ then the scalar curl of F is $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$. Let $F = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$. Then the scalar curl of F is $\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} = 0$. $\square$

Theorem: Let F be a vector field in $\mathbb{R}^3$ defined everywhere except a finite number of points. If $\operatorname{curl}(F) = (0,0,0)$, then $F = \nabla f$ for some scalar function $f(x,y,z)$

proof: It's complicated ... see book

Idea (same as in single variable calculus)

Define $f(x,y,z)$ as follows: Given (x,y,z), let $c(t)$ be any curve from $(0,0,0)$ to (x,y,z) and define

$$f(x,y,z) = \int_{c(t)} F \cdot d\vec{s} \qquad \left(\underset{\text{single var.}}{\underbrace{\text{an}}}\ g(x) = \int_0^x f(t)\,dt \right)$$

Show ① $f(x,y,z)$ does not depend on $c(t)$.

② $\nabla f = F$ $\square$

Example

$$F(x,y,z) = -\frac{(x,y,z)}{\|(x,y,z)\|^3} = \left(\frac{-x}{(x^2+y^2+z^2)^{3/2}}, \frac{-y}{(x^2+y^2+z^2)^{3/2}}, \frac{-z}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$\left(F(\vec{v}) = -\frac{\vec{v}}{\|\vec{v}\|^3} \right), \quad \|F(x,y,z)\| = \frac{(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^{3/2}} = \frac{1}{(x^2+y^2+z^2)}$$

$$= \frac{1}{\|(x,y,z)\|^2}$$

$F(x,y,z)$ is not defined $(0,0,0)$.

check with a lot of algebra $\text{curl}(F) = (0,0,0)$

$$f(x,y,z) = \frac{1}{\|(x,y,z)\|} = \frac{1}{(x^2+y^2+z^2)^{1/2}} = (x^2+y^2+z^2)^{-1/2}$$

$$\nabla f = \left(-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2x,\ -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2y,\ -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2z\right)$$

$$= \left(\frac{-x}{(x^2+y^2+z^2)^{3/2}}, \frac{-y}{(x^2+y^2+z^2)^{3/2}}, \frac{-z}{(x^2+y^2+z^2)^{3/2}}\right) = F$$

Corollary: Let F be a vector field in $\mathbb{R}^2$ defined everywhere, and such that the scalar curl of F is 0. Then $F = \nabla f$ for some scalar function $f(x,y)$.

proof: $F = (P, Q)$, define $\tilde{F} = (P, Q, 0)$ and apply theorem to $\tilde{F}$. ∎

Examples

(1) $F(x,y) = \left(\dfrac{-x}{(x^2+y^2)^{3/2}}, \dfrac{-y}{(x^2+y^2)^{3/2}} \right)$ not defined at $(0,0)$ but still gradient vector field

$f(x,y) = \dfrac{1}{(x^2+y^2)^{1/2}}$ then $\nabla f = F$.

~~also~~ Can check that the scalar curl of F is 0.

(2) $F(x,y) = \left(\dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right)$

Remember
$F(x,y) = (-y, x)$
scalar curl $1 - -1 = 2$

$$\text{scalar curl} = \frac{\partial\left(\frac{x}{x^2+y^2}\right)}{\partial x} - \frac{\partial\left(\frac{-y}{x^2+y^2}\right)}{\partial y}$$

$$= \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2} - \frac{(x^2+y^2)\cdot(-1) + y\cdot 2y}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2x^2+x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$= \frac{0}{(x^2+y^2)^2} = 0$$

$$F(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

scalar curl of F is 0

F is not defined at $(0,0)$

Suppose $F = \nabla f$.

Let $\mathcal{C}^+$ be the unit circle $x^2+y^2=1$ traveled counterclockwise.

Let's calculate $\displaystyle\int_{\mathcal{C}^+} F\cdot d\vec{s}$ two different ways.

$$\boxed{\int_{c(t)} \nabla f \cdot d\vec{s} = f(c(b)) - f(c(a))}$$

(1) Use that $F = \nabla f$. Then

$$\int_{\mathcal{C}^+} F\cdot d\vec{s} = \int_{\mathcal{C}^+} \nabla f\cdot d\vec{s} = f(1,0) - f(1,0) = \boxed{0}$$

(2) Calculate directly. Let $c(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$

$$\int_{\mathcal{C}^+} F\cdot d\vec{s} = \int_0^{2\pi} F(c(t))\cdot c'(t)\,dt$$

$$= \int_0^{2\pi} \left(\frac{-\sin t}{1}, \frac{\cos t}{1}\right)\cdot(-\sin t, \cos t)\,dt$$

$$= \int_0^{2\pi} \sin^2 t + \cos^2 t\,dt = \int_0^{2\pi} dt = \boxed{2\pi}$$

We've obtained a contradiction so $F \neq \nabla f$.

We've shown that $F(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$ is

<u>not</u> a gradient vector field even though it has scalar curl 0.

<u>Fact 3</u>: If $F = \text{curl}(G)$, then $\text{div}(F) = 0$.

<u>Theorem</u>: Let F be a vector field in $\mathbb{R}^3$ defined everywhere, and such that $\text{div}(F) = 0$. Then there exists a vector field G such that $F = \text{curl}(G)$.