Exam

- To prepare: Review homework
  - Do old exam (no solutions)
  - Examples from class and in book
  - Problems from book

- Tuesday in here from 3:00 - 5:59 pm

- 15 problems for 104 total points

- Many problems are shorter which is why there are more problems.

- About 40% is from after midterm 2

- Cumulative in general.

- Bring photo ID

- 8.5 x 11 inch front and back page of hand-written notes

- No calculators, phones, or electronic aids

- Office hours on Monday 11:30 am - 1:30 pm AP&M 6321
8.3 Conservative Vector Fields

Question: Given \( F \) in \( \mathbb{R}^3 \) a vector field, does there exist a vector field \( G \) such that \( F = \text{curl}(G) \)?

Fact: If \( F = \text{curl}(G) \), then \( \text{div}(F) = 0 \).

Proof: \( G = (P, Q, R) \). Then

\[
F = \text{curl}(G) = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)
\]

\[
\text{div}(F) = \frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial z} + \frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 R}{\partial y \partial x} + \frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial y} = 0.
\]

Contrapositive: If \( \text{div}(F) \neq 0 \) then \( F \neq \text{curl}(G) \) for any \( G \).

Theorem: Let \( F \) be a vector field in \( \mathbb{R}^3 \) that is defined everywhere, and such that \( \text{div}(F) = 0 \). Then there exists a \( G \) such that \( F = \text{curl}(G) \).
Example: (really non-example)

\[
F = \left( \frac{-x}{(x^2+y^2+z^2)^{3/2}}, \frac{-y}{(x^2+y^2+z^2)^{3/2}}, \frac{-z}{(x^2+y^2+z^2)^{3/2}} \right)
\]

\[F\] is not defined at \((0,0,0)\)

\[
\text{div}(F) = \cdots \text{a lot of algebra} \cdots = 0
\]

We'll show that there does not exist a \(G\) such that \(F = \text{curl}(G)\).

Suppose \(F = \text{curl}(G)\) for some \(G\).

\(S^+ = \text{sphere } x^2+y^2+z^2 = 1 \) with outward orientation.

Calculate \(\iint_S F \cdot d\mathbf{S}\) \text{ two ways.}

1. \(S^+ = S^+_1 \cup S^+_2\) where \(S^+_1 = \text{upper hemisphere}\)
   \(S^+_2 = \text{lower hemisphere}\)
\[ \iint_{S^+} F \cdot d\vec{S} = \iint_{S_1^+} F \cdot d\vec{S} + \iint_{S_2^+} F \cdot d\vec{S} \]

\[ = \int_{S_1^+} G \cdot d\vec{S} + \int_{S_2^+} G \cdot d\vec{S} \]

\[ F = \text{curl}(G), \text{ so } \iint_{S^+} F = \iint_{S_1} \text{curl} G = \int_{S_1^+} G \cdot d\vec{S} + \int_{S_2^+} G \cdot d\vec{S} \]

\[ C = \text{equator} \text{ so } C = \partial S_1 = \partial S_2 \]

\[ C^+ = \text{travel counterclockwise} \]

\[ = \int_{S_1^+} G \cdot d\vec{S} + \int_{S_2^+} G \cdot d\vec{S} = \int_{C^+} G \cdot d\vec{S} - \int_{C^-} G \cdot d\vec{S} = 0 \]

We've shown that \[ \iint_{S^+} F \cdot d\vec{S} = 0. \]
(2) Calculate \( \iint_{S^+} F \cdot dS \) directly.

\( \vec{e}(\theta, \phi) = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi) \) \quad D: \quad 0 \leq \phi \leq \pi \quad 0 \leq \theta \leq \pi \\

\( T_\theta \times T_\phi = (-\sin^2 \phi \cos \theta, -\sin^2 \phi \sin \theta, -\sin \phi \cos \phi) \)

\[ F(x, y, z) = \left( \frac{-x}{(x^2+y^2+z^2)^{3/2}} \right) - \frac{y}{(x^2+y^2+z^2)^{3/2}} - \frac{z}{(x^2+y^2+z^2)^{3/2}} \]

\[ F(\vec{e}(\theta, \phi)) = (\sin \phi \cos \phi, \sin \phi \sin \theta, -\cos \phi) \]

\[ F(\vec{e}(\theta, \phi)) \cdot T_\theta \times T_\phi = \sin^2 \phi \cos^2 \theta + \sin^2 \phi \sin^2 \theta + \sin \phi \cos^2 \phi \]

\[ = \sin^2 \phi (\cos^2 \theta + \sin^2 \theta) + \sin \phi \cos^2 \phi \]

\[ = \sin \phi (\sin^2 \phi + \cos^2 \phi) \]

\[ = \sin \phi \]
\[ \int_{S^+} F \cdot d\vec{S} = \pm \int_0^{2\pi} \int_0^{\pi} \sin \phi \, d\phi \, d\theta \]

\[ = \pm 2\pi (\cos \phi) \bigg|_0^{\pi} \]

\[ = \pm 2\pi (1 - 1) \]

\[ = \pm 4\pi \]

This shows that \( F \) has inward pointing normal so \( S^+ \) orientation reversing. Then

\[ \int_{S^+} F \cdot d\vec{S} = -4\pi \]

We showed that \( \int_{S^+} F \cdot d\vec{S} = 0 \) and \( \int_{S^+} F \cdot d\vec{S} = 4\pi - 4\pi \)

which is a contradiction, so \( F \neq \text{curl}(G) \).
8.5 Differential Forms (Not on Final)

\[ n \geq 1, \quad k \leq n \]

\[ M \subseteq \mathbb{R}^n \]

\[ \text{\(n\)-dimensional space} \]

\[ \text{k-dimensional geometric object} \]

\( \text{(k-dimensional manifold)} \)

\[ n=1 \]

\[ \mathbb{R} = \text{intervals } [a,b] \]

\[ \text{points a} \]

\[ 0 \text{-dimensional} \]

\[ n=2 \]

\[ \mathbb{R}^2 \]

\[ \text{regions D} \]

\[ 2\text{-dim.} \]

\[ \text{curves } C \]

\[ 1\text{-dim.} \]
$n = 3$

$\mathbb{R}^3$

Points $(x_0, y_0, z_0)$
0-dim.

Solid sphere $x^2 + y^2 + z^2 \leq 1$
W is 3-dim.

Sphere $x^2 + y^2 + z^2 = 1$
$S$ is 2-dim.
$M \in \mathbb{R}^n$, k-dim geometric object $\subset \mathbb{R}^n$.

Define integration on $M$. Integrate things called k-forms $\omega$.

Define universal differential operator $d$ such that if $\omega$ is a k-form then $d\omega$ is a $k+1$ form.
$\mathbb{R}^3$

0-forms are scalar valued functions \( f(x,y,z) \)

\[ \int_{(x,y,z)} f = f(x,y,z) \]

1-form is a vector field \( F(x,y,z) \) and

\[ d(f) = \nabla f \]

\[ \int_{C} F \cdot ds = \text{line integrals} \]

2-form is also a vector field \( F(x,y,z) \) and

\[ d(\text{1-form } F) = \text{curl}(F) \]

\[ \iint_{S} F \cdot d\vec{S} = \text{surface integral} \]

3-forms is also a scalar valued function \( f(x,y,z) \) and

\[ d(\text{2-form } F) = \text{div}(F), \quad \iiint_{W} f \, dV = \text{triple integral} \]
Universal Stokes Theorem

\[ \int_M d\omega = \int_{\partial M} \omega \]

\( \omega = k\)-form

\( M = (k+1)\)-dim. manifold\)