Exam

To prepare: Review homework
Do ~~pra~~ old exam (no solutions)
Examples from class and in book
Problems from book

- Tuesday in here from 3:00 - 5:59pm
- 15 problems for 104 total points
  graded out of 100 points
- Many problems are shorter, which is why
  there are more problems.
- About 40% is from after midterm 2
- Cumulative in general.
- Bring photo ID
- 8.5 x 11 inch front and back page of hand-written notes
- No calculators, phones, or electronic aids
- office hours on Monday 11:30am - 1:30pm AP&M 6321

8.3 Conservative Vector Fields

Question: Given F in $\mathbb{R}^3$ a vector field, does there exist a vector field G such that $F = \text{curl}(G)$?

Fact: If $F = \text{curl}(G)$, then $\text{div}(F) = 0$.

proof: $G = (P, Q, R)$. Then

$$F = \text{curl}(G) = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

$$\text{div}(F) = \frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial z} + \frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 R}{\partial y \partial x} + \frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial y} = 0. \square$$

cancel cancel cancel

Contrapositive: If $\text{div}(F) \neq 0$ then $F \neq \text{curl}(G)$ for any G.

Theorem: Let F be a vector field in $\mathbb{R}^3$ that is defined everywhere, and such that $\text{div}(F) = 0$. Then there exists a G such that $F = \text{curl}(G)$.

<u>Example</u> (really non-example)

$$F = \left(\frac{-x}{(x^2+y^2+z^2)^{3/2}}, \frac{-y}{(x^2+y^2+z^2)^{3/2}}, \frac{-z}{(x^2+y^2+z^2)^{3/2}} \right)$$

F is not defined at $(0,0,0)$

div$(F) = \dots$ a lot of algebra $\dots = 0$

We'll show that there does not exist a G such that $F = \text{curl}(G)$.

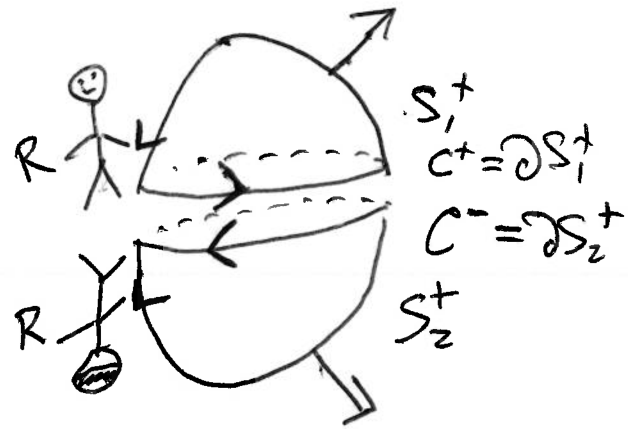
Suppose $F = \text{curl}(G)$ for some G.

$S^+ =$ sphere $x^2+y^2+z^2=1$ with outward orientation

Calculate $\iint_{S^+} F \cdot d\vec{S}$ is two ways.

(1) $S^+ = S_1^+ \cup S_2^+$ where $S_1^+ =$ upper hemisphere
$S_2^+ =$ lower hemisphere

$$\iint_{S^+} F\cdot d\vec{S} = \iint_{S_1^+} F\cdot d\vec{S} + \iint_{S_2^+} F\cdot d\vec{S}$$



$$= \int_{\partial S_1^+} G\cdot d\vec{s} + \int_{\partial S_2^+} G\cdot d\vec{s}$$

$F = \text{curl}(G)$, so $\displaystyle\iint_{S_i^+} F = \iint_{S_i^+} \text{curl}\, G = \int_{\partial S_i^+} G$

C = equator so $C = \partial S_1 = \partial S_2$

C^+ = travel counterclockwise

$$= \int_{C^+} G\cdot d\vec{s} + \int_{C^-} G\cdot d\vec{s} = \int_{C^+} G\cdot d\vec{s} - \int_{C^+} G\cdot d\vec{s} = 0$$

We've shown that $\displaystyle\iint_{S^+} F\cdot d\vec{S} = 0.$

(2) Calculate $\iint_{S^+} F \cdot d\vec{S} = \pm \iint_D F(\Phi(\theta,\varphi)) \cdot T_\theta \times T_\varphi \, d\theta \, d\varphi$ directly.

$$\Phi(\theta, \varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi) \quad D: \begin{matrix} 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi \end{matrix}$$

$$T_\theta \times T_\varphi = (-\sin^2\varphi\cos\theta, -\sin^2\varphi\sin\theta, -\sin\varphi\cos\varphi)$$

$$F(x, y, z) = \left(\frac{-x}{(x^2+y^2+z^2)^{3/2}}, \frac{-y}{(x^2+y^2+z^2)^{3/2}}, \frac{-z}{(x^2+y^2+z^2)^{3/2}} \right)$$

$$F(\Phi(\theta, \varphi)) = (-\sin\varphi\cos\theta, -\sin\varphi\sin\theta, -\cos\varphi)$$

$$F(\Phi(\theta,\varphi)) \cdot T_\theta \times T_\varphi = \sin^3\varphi\cos^2\theta + \sin^3\varphi\sin^2\theta + \sin\varphi\cos^2\varphi$$

$$= \sin^3\varphi + \sin\varphi\cos^2\varphi$$

$$= \sin\varphi(\sin^2\varphi + \cos^2\varphi)$$

$$= \sin\varphi$$

$$\iint_{S^+} F \cdot d\vec{S} = \pm \int_0^{2\pi} \int_0^{\pi} \sin\varphi \, d\varphi \, d\theta$$

$$= \pm 2\pi \left(-\cos\varphi \Big|_0^{\pi} \right)$$

$$= \pm 2\pi \left(- -1 - -1 \right)$$

$$= \pm 4\pi$$

$\Phi(\theta, \varphi)$ has inward pointing normal so is orientation reversing.

Then $\iint_{S^+} F \cdot d\vec{S} = -4\pi$

We showed that $\iint_{S^+} F \cdot d\vec{S} = 0$ and $\iint_{S^+} F \cdot d\vec{S} = \cancel{4\pi}\ -4\pi$ which is a contradiction, so $F \neq \operatorname{curl}(G)$.

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8.5 Differential Forms (Not on Final)

$n \geq 1, \quad k \leq n$

$$M \subseteq \mathbb{R}^n$$

n-dimensional space

k-dimensional geometric object
(k-dimensional manifold)

$n = 1$

$\mathbb{R} \supseteq$ intervals $[a,b]$ 1-dimensional

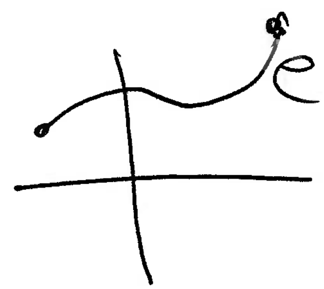
points a 0-dimensional

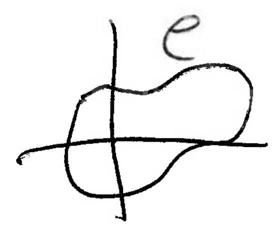
$n = 2$

$\mathbb{R}^2$

regions D

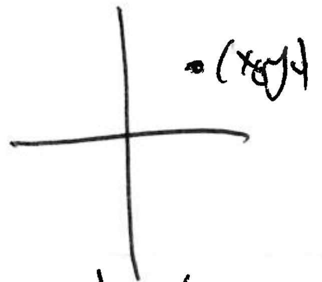
2-dim.





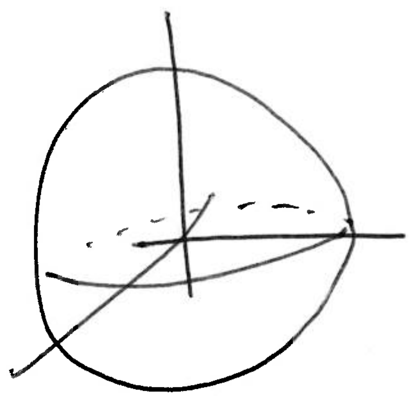
curves e

1-dim.

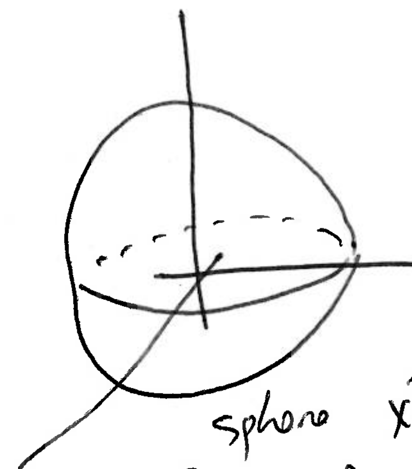


points (x_0,y_0)
0-dim.

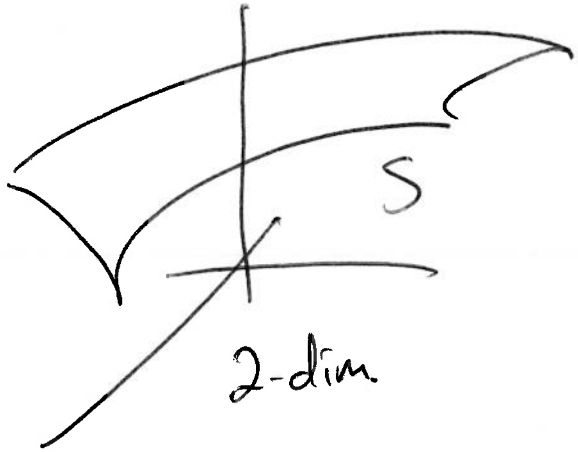
$n=3$
$\mathbb{R}^3$



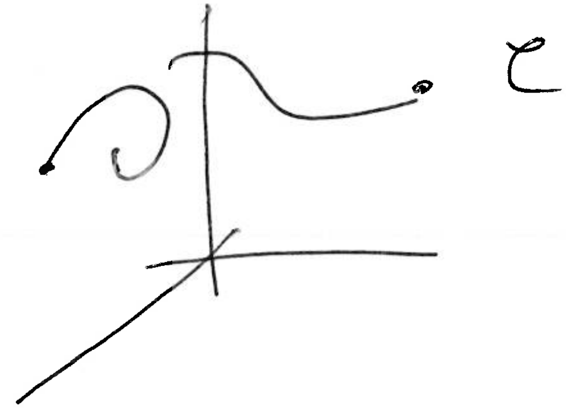
solid sphere $x^2+y^2+z^2\le 1$
W is 3-dim.

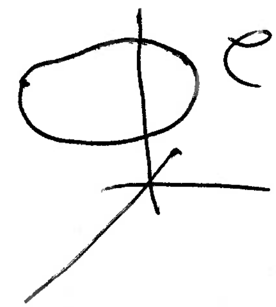


sphere $x^2+y^2+z^2=1$
S is 2-dim.



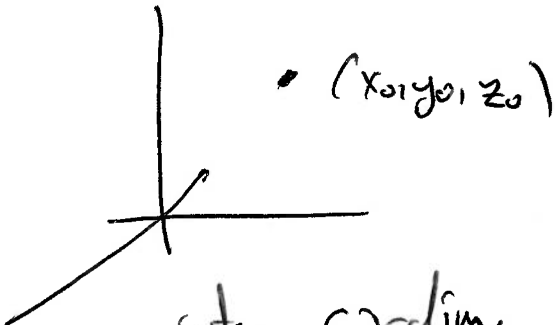
2-dim.





curves C

1-dim.



points 0-dim.

$M \subseteq \mathbb{R}^n$ k-dim geometric object $\swarrow$ $\int_M \omega$

Define integration on M. Integrate things called k-forms ω.

Define universal differential operator d such that if ω is a k-form then $d\omega$ is a k+1 form.

$\mathbb{R}^3$

0-forms are scalar valued functions $f(x,y,z)$

$$\int_{\{(x,y,z)\}} \omega = \int_{\{(x,y,z)\}} f = f(x,y,z)$$

1-form is a vector field $F(x,y,z)$ and

$$d(f) = \nabla f$$

$$\int_{\mathcal{C}} \omega = \int_{\mathcal{C}} F \cdot d\vec{s} = \text{line integrals}$$

2-form is also a vector field $F(x,y,z)$ and

$$d(\text{1-form } F) = \text{curl}(F)$$

$$\int_S \omega = \iint_S F \cdot d\vec{S} = \text{surface integral of vector field}$$

3-forms is also a scalar valued function $f(x,y,z)$ and

$$d(\text{2-form } F) = \text{div}(F), \quad \int_W \omega = \iiint_W f\, dV = \text{triple integral}$$

Universal Stokes Thm

$\omega =$ k-form

$M = (k+1)$-dim. manifold

$$\int_M d\omega = \int_{\partial M} \omega$$