Warm-Up Integral

Lecture 3

$$\int_1^3 \int_1^2 \frac{xy}{(x^2+y^2)^{3/2}}\,dx\,dy =$$

Inner integral: $\displaystyle \int \frac{xy}{(x^2+y^2)^{3/2}}\,dx$ $\qquad u = x^2+y^2$, $du = 2x\,dx$

$$\| $$

$$\frac{1}{2}\int \frac{y\,du}{u^{3/2}} = \frac{1}{2}\int y u^{-3/2}\,du = \frac{\frac{1}{2}y u^{-1/2}}{-1/2} = \cancel{y u^{-1/2}}$$

$$= \frac{-y}{u^{1/2}} = \frac{-y}{(x^2+y^2)^{1/2}}$$

$$\int_1^3 \left(\frac{-y}{(x^2+y^2)^{1/2}} \Bigg|_{x=1}^{x=2} \right) dy = \int_1^3 \frac{-y}{(4+y^2)^{1/2}} + \frac{y}{(1+y^2)^{1/2}}\,dy$$

To finish another u-sub with $u = 4+y^2$ and $u = 1+y^2$

Warning: Not every function $f(x,y)$ is integrable!

The order of integration ($dx\,dy$ or $dy\,dx$) can matter.

Fubini's Thm: If f is continuous, on $R = [a,b] \times [c,d]$, then

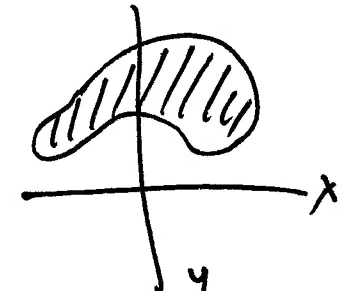
$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Section 5.2 #17 gives a function where $dx\,dy$ integration is different than $dy\,dx$.

5.3 Double Integrals over more general regions

Elementary Regions

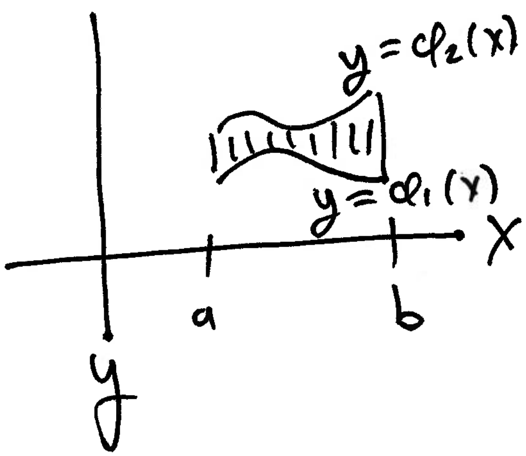
D = closed and bounded region in $\mathbb{R}^2$ (usually D is given by inequalities)



D is called <u>y-simple</u> if D can be expressed in the form

$$D: \begin{array}{l} a \le x \le b \\ \varphi_1(x) \le y \le \varphi_2(x) \end{array}$$

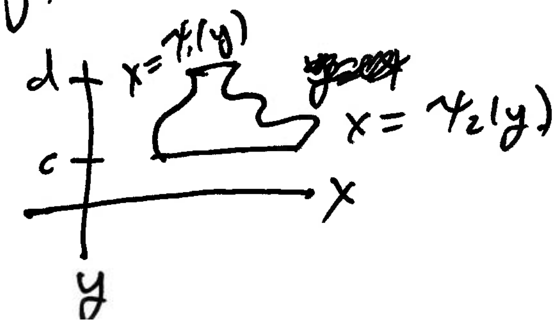
for some numbers a, b and functions $\varphi_1(x), \varphi_2(x)$



<u>x-simple</u>: D can be expressed in the form

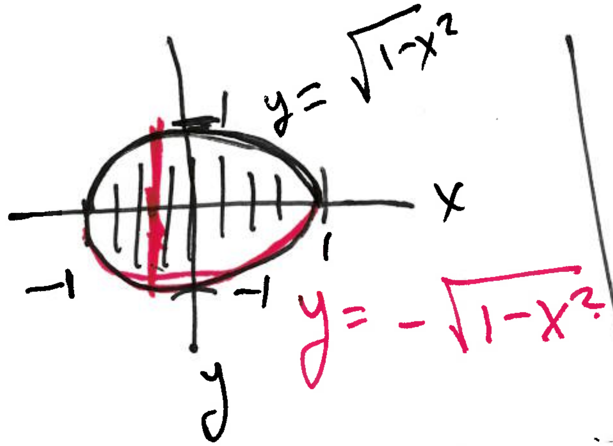
$$D: \begin{array}{l} c \le y \le d \\ \psi_1(y) \le x \le \psi_2(y) \end{array}$$

c, d numbers
ψ_1, ψ_2 functions



D is simple if D is x-simple and y-simple.

EX ① Express the unit disk, D: $x^2+y^2 \le 1$ as an x-simple and y-simple region.



y-simple

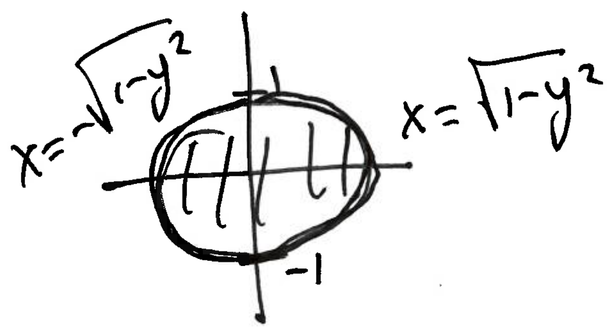
$-1 \le x \le 1$

$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$

x-simple

$-1 \le y \le 1$

$-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$



$x^2+y^2=1$

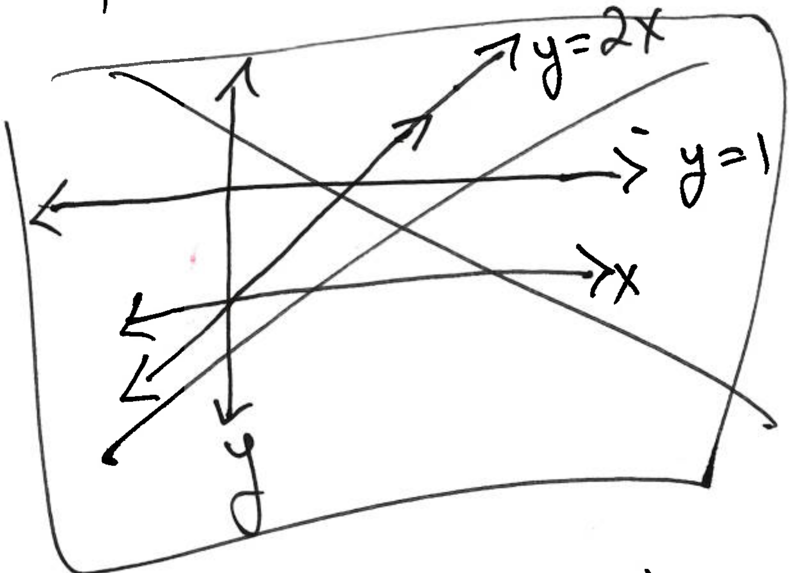
$x^2=1-y^2$

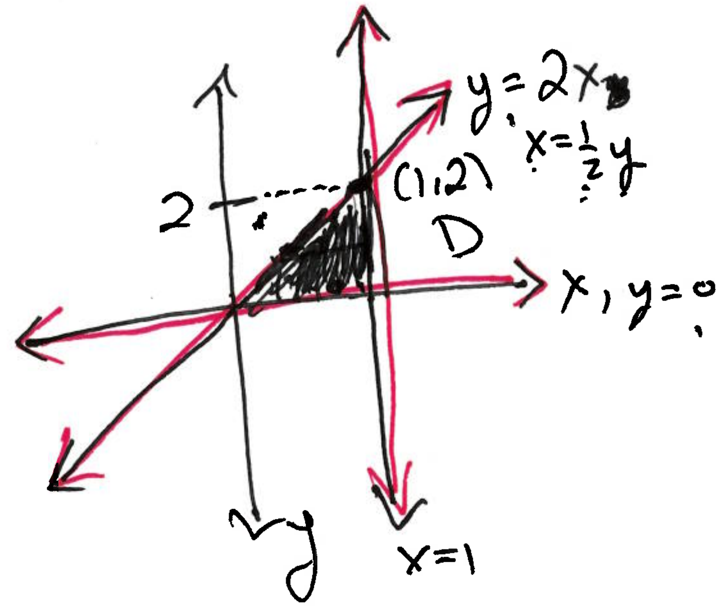
$x=\pm\sqrt{1-y^2}$

② Let D be the region bounded by $y=2x$, $y=0$ and $x=1$.

$x=\frac{1}{2}y$

Express D as x-simple and y-simple.





<u>y-simple</u>

$0 \le x \le 1$

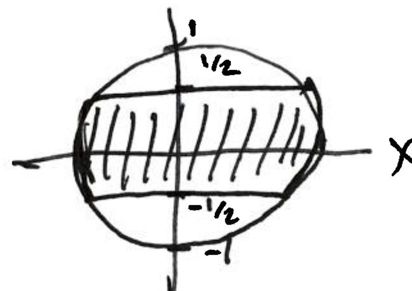
$0 \le y \le 2x$

<u>x-simple</u>

$0 \le y \le 2$

$\frac{1}{2}y \le x \le 1$

<u>Example of not simple region:</u>



<u>x-simple</u>

$-\frac{1}{2} \le y \le \frac{1}{2}$

$-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$

try y-simple: $-1 \le x \le 1$

two possibilities for y-boundaries

$\boxed{\text{Not } y\text{-simple}}$

Integrating over Elementary Regions

$f(x,y)$ - continuous real-valued function

D - elementary region

D y-simple

$a \le x \le b$

$\varphi_1(x) \le y \le \varphi_2(x)$

$$\iint_D f(x,y)\, dA = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\, dy\, dx$$

D x-simple

$c \le y \le d$

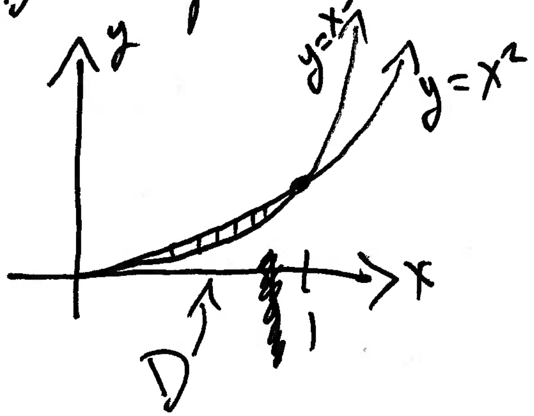
$\psi_1(y) \le x \le \psi_2(y)$

$$\iint_D f(x,y)\, dA = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\, dx\, dy$$

Examples

D = region bounded by $y=x^2$ and $y=x^3$. Calculate $\iint_D y\, dA$.

$(f(x,y) = y)$



y-simple

$0 \le x \le 1$

$x^3 \le y \le x^2$

$$\iint_D y\, dA = \int_0^1 \int_{x^3}^{x^2} y\, dy\, dx$$

$$\iint_D y\,dA = \int_0^1 \int_{x^3}^{x^2} y\,dy\,dx$$

$$= \int_0^1 \left(\frac{y^2}{2} \Big|_{y=x^3}^{y=x^2} \right) dx$$

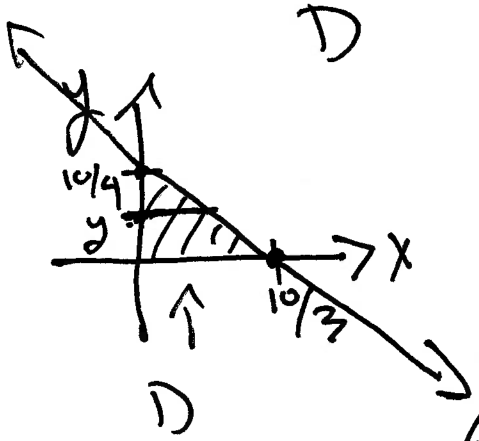
$$= \int_0^1 \frac{x^4}{2} - \frac{x^6}{2}\,dx$$

$$= \frac{x^5}{10} - \frac{x^7}{14} \Big|_0^1$$

$$= \boxed{\frac{1}{10} - \frac{1}{14}}$$

(2) D = region bounded by positive x-axis, positive y-axis and $3x+4y=10$. Compute

$$\iint_D x+y\,dA \qquad (f(x,y)=x+y)$$



$3x+4y=10$

Solve for x

$3x = 10-4y$

$x = \dfrac{10-4y}{3}$

<u>x-simple</u>

$0 \le y \le \frac{10}{4}$

$0 \le x \le \dfrac{10-4y}{3}$

$$\iint_D x+y\,dA = \int_0^{\frac{10}{4}} \int_0^{(10-4y)/3} x+y\,dx\,dy$$

$$\int_0^{10/4} \frac{x^2}{2} + xy \Big|_{x=0}^{x=\frac{10-4y}{3}} dy = \int_0^{10/4} \frac{1}{2}\left(\frac{10-4y}{3}\right)^2 + y\left(\frac{10-4y}{3}\right) dy$$