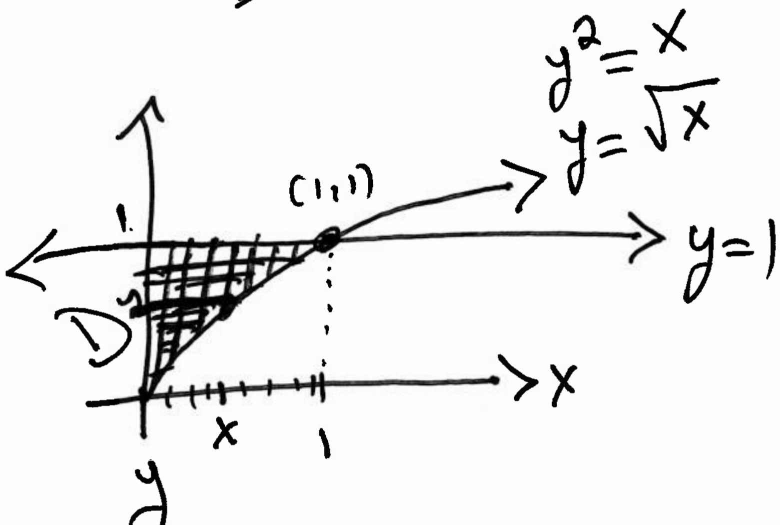
5.4 Changing the order of integration

① D region bounded by $y=\sqrt{x}$, $y=1$ and y-axis.
Compute $$\iint_D e^{y^3}\,dA$$



y-simple

$0 \le x \le 1$

$\sqrt{x} \le y \le 1$

x-simple

$0 \le y \le 1$

$0 \le x \le y^2$

y-simple integral

$$\iint_D e^{y^3}\,dA = \int_0^1 \int_{\sqrt{x}}^1 e^{y^3}\,dy\,dx$$ looks like a hard integral

x-simple integral

$$\iint_D e^{y^3}\,dA = \int_0^1 \int_0^{y^2} e^{y^3}\,dx\,dy = \int_0^1 x e^{y^3}\Big|_{x=0}^{x=y^2}\,dy$$

why can't we do

$$\int_{\sqrt{x}}^{1} \int_{0}^{1} e^{y^3}\,dx\,dy = \int_{\sqrt{x}}^{1} x e^{y^3} \Big|_{x=0}^{x=1} dy$$

$$= \int_{\sqrt{x}}^{1} e^{y^3}\,dy$$

WRONG because final answer would have a $\sqrt{x}$ in it

final answer ~~[illegible]~~ is a number

$$= \int_0^1 y^2 e^{y^3}\, dy \qquad u = y^3,\ du = 3y^2\, dy$$

$y=0,\ u=0,\quad y=1,\ u=1$

$$= \frac{1}{3}\int_0^1 e^u\, du = \frac{1}{3} e^u \Big|_0^1 = \boxed{\frac{1}{3}(e-1)}$$

② Change order of integration and compute

$$\int_{-1}^{1} \int_{|y|}^{1} (x+y)^2\, dx\, dy$$

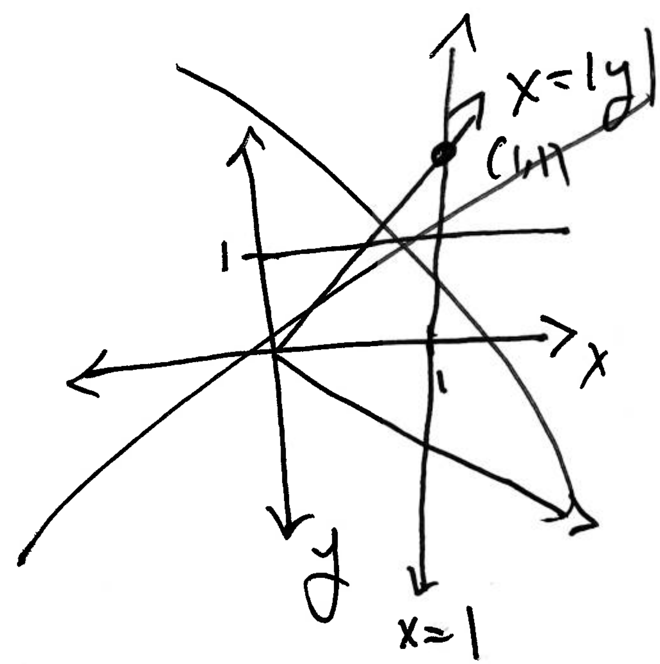
x-simple

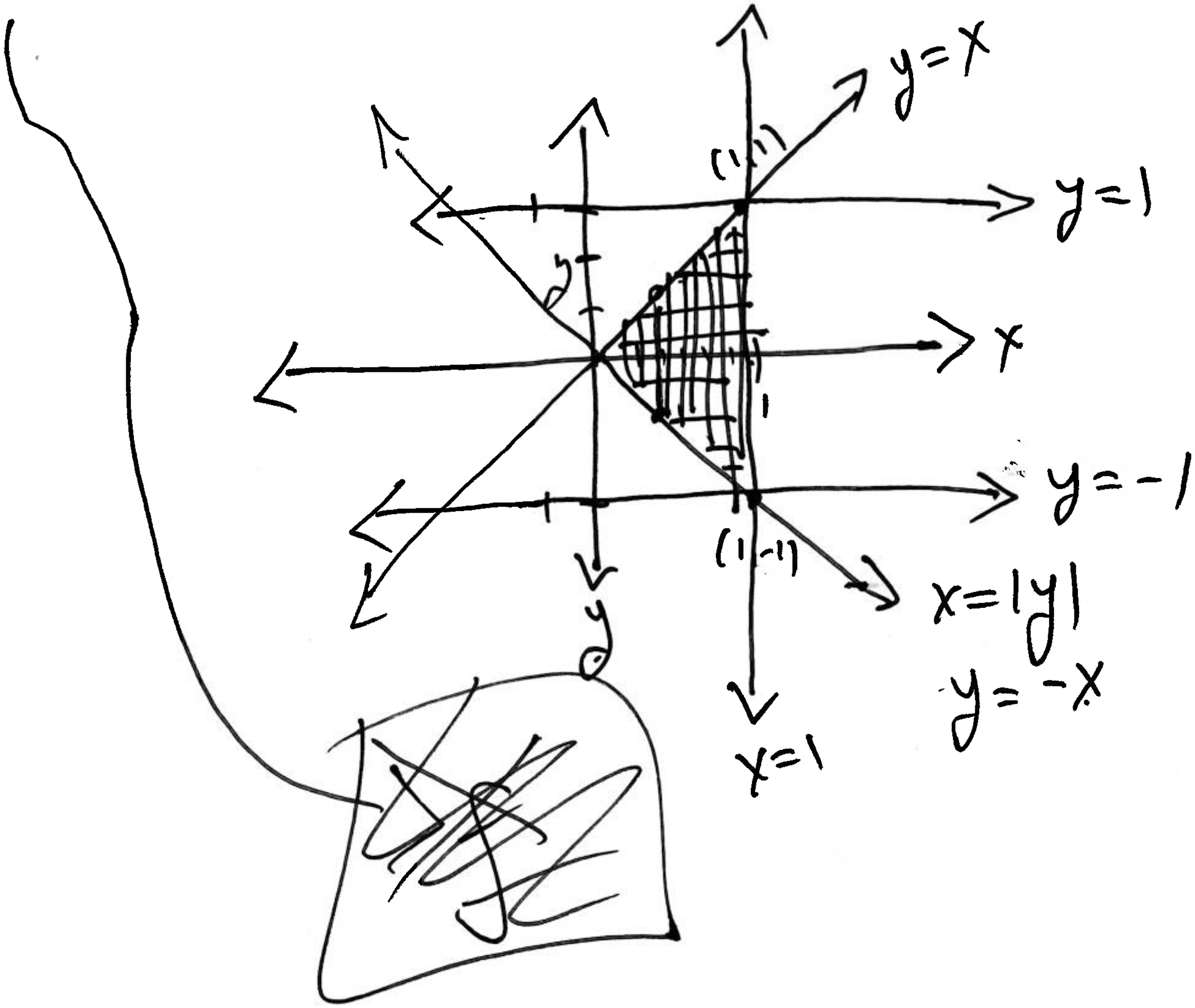
$-1 \le y \le 1$

$|y| \le x \le 1$

$y=-1,\ y=1$

$x=|y|,\ x=1$





y = simple

$0 \le x \le 1$

$-x \le y \le x$

$$\int_{-1}^{1} \int_{|y|}^{1} (x+y)^2 \, dx\, dy = \int_0^1 \int_{-x}^{x} (x+y)^2 \, dy\, dx$$

$$= \int_0^1 \left(\left. \frac{(x+y)^3}{3} \right|_{y=-x}^{y=x} \right) dx$$

$$= \int_0^1 \frac{(x+x)^3}{3} - \frac{(x-x)^3}{3}\, dx = \int_0^1 \frac{(2x)^3}{3}\, dx$$

$$= \int_0^1 \frac{8x^3}{3}\, dx = \frac{8}{3}\frac{x^4}{4}\Big|_0^1$$

$$= \frac{8}{3}\cdot\frac{1}{4} = \boxed{\frac{2}{3}}$$

③ Change order of integration and integrate

$$\int_0^1 \int_y^1 \sin(x^2)\, dx\, dy$$

<u>x-simple</u>

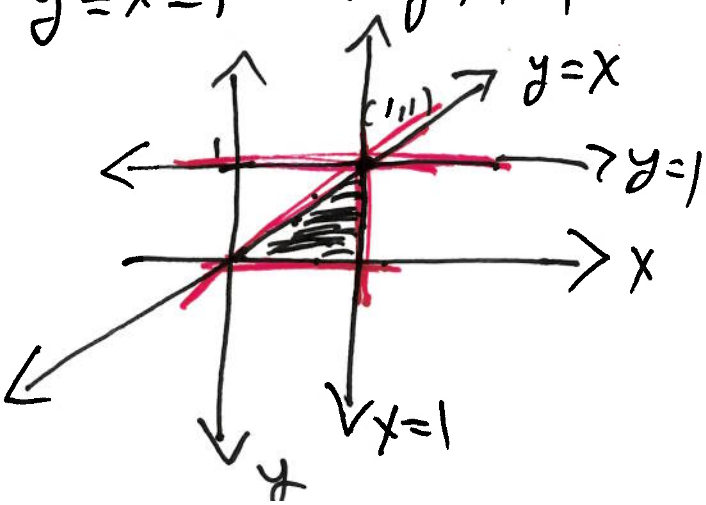
$0 \le y \le 1$ $y=0,\ y=1$

$y \le x \le 1$ $x=y,\ x=1$

<u>y-simple</u>

$0 \le x \le 1$

$0 \le y \le x$



$$\int_0^1 \int_0^x \sin(x^2)\, dy\, dx = \int_0^1 y\sin(x^2)\Big|_{y=0}^{y=x} dx$$

$$= \int_0^1 x\sin(x^2)\,dx \qquad \begin{aligned} u &= x^2 \\ du &= 2x\,dx \end{aligned} \qquad \begin{aligned} x=0,\ & u=0 \\ x=1,\ & u=1 \end{aligned}$$

$$= \frac{1}{2}\int_0^1 \sin(u)\,du = \frac{1}{2}\left(-\cos(u)\Big|_0^1\right)$$

$$= \frac{1}{2}\left(-\cos(1) + \cos(0)\right)$$

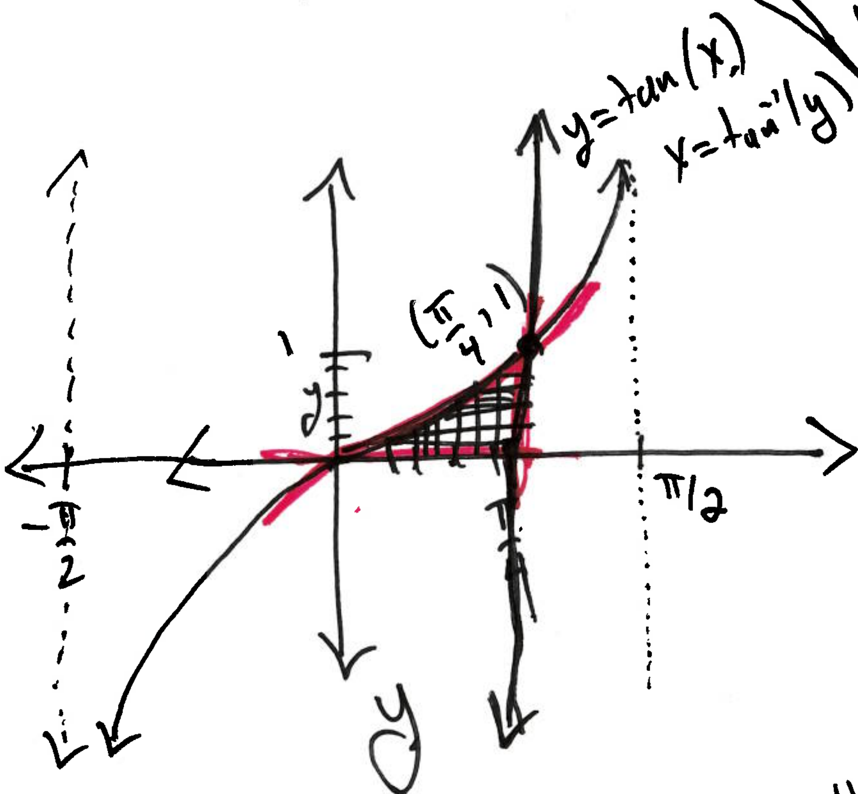
$$= \boxed{\frac{1-\cos(1)}{2}}$$

④ Let D be the region bounded by

$x = \tan^{-1}(y)$, $x = \pi/4$ and the x-axis

Calculate.

$$\iint_D \sec^5 x \, dA$$



y=tan(x)

X-simple

$0 \le y \le 1$

$\tan^{-1}(y) \le x \le \frac{\pi}{4}$

$$\int_0^1 \int_{\tan^{-1}(y)}^{\frac{\pi}{4}} \sec^5 x \, dx\,dy$$

y-simple

$0 \le x \le \frac{\pi}{4}$

$0 \le y \le \tan x$

Do this one

$$\int_0^{\frac{\pi}{4}} \int_0^{\tan x} \sec^5 x \, dy\,dx$$

$$\int_0^{\pi/4}\int_0^{\tan x} \sec^5 x \, dy\, dx = \int_0^{\pi/4} y \sec^5 x \Big|_{y=0}^{y=\tan x} dx$$

$$= \int_0^{\pi/4} \tan x \sec^5 x \, dx$$

$u = \sec x, \quad du = \sec x \tan x \, dx$

$x = 0, \quad u = \sec(0) = 1$

$x = \pi/4, \quad u = \sec(\pi/4) = \dfrac{1}{\sqrt{2}/2} = \dfrac{2}{\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}} = \sqrt{2}$

$$\int_1^{\sqrt{2}} u^4 \, du = \frac{u^5}{5}\Big|_1^{\sqrt{2}} = \frac{(\sqrt{2})^5}{5} - \frac{1}{5}$$

$$= \boxed{\frac{4\sqrt{2}-1}{5}}$$

5.5 Triple Integrals

$f(x,y,z)$ = real-valued function of 3-variables

$B = [a,b] \times [c,d] \times [p,q]$ rectangular parallelpiped

$a \le x \le b$

$c \le y \le d$

$p \le z \le q$

What is $\iiint_B f(x,y,z)\, dV$?

dz, dy, dx = ~~infinitesimal~~ length of an infinitesimal line segment

dA = area of infinitesimal rectangle

dV = volume of infinitesimal rectangular parallelpiped

$\iint_D f\, dA$ = volume under graph of f over D

$$\iiint_B f(x,y,z)\,dV = \text{4-dimension "area or volume" over } B \text{ under graph of } f \text{ in } \mathbb{R}^4$$

graph of $f(x,y,z)$ B in $\mathbb{R}^4$

$$\frac{1}{\text{volume}(B)} \iiint_B f(x,y,z)\,dV = \text{average value of } f \text{ on } B$$