5.5 Triple Integrals

Calculate triple integrals just like double integrals

$B = [a,b] \times [c,d] \times [p,q]$

$a \leq x \leq b$
$c \leq y \leq d$
$p \leq z \leq q$

$$\iiint_B f(x,y,z)\,dV = \int_a^b \int_c^d \int_p^q f(x,y,z)\,dz\,dy\,dx$$

6 different orders of integration ($dx\,dy\,dz$, $dy\,dx\,dz$, etc.)

Example $B = [0,1] \times [0,2] \times [-1,0]$, $f(x,y,z) = xyz$

$$\iiint_B f(x,y,z)\,dV = \int_0^1 \int_0^2 \int_{-1}^0 xyz\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^2 \left.\frac{xyz^2}{2}\right|_{z=-1}^{z=0}$$

$$= \int_0^1 \int_0^2 -\frac{xy}{2}\,dy\,dx = \int_0^1 \left.-\frac{xy^2}{4}\right|_{y=0}^{y=2} dx$$

$$= \int_0^1 -x\,dx = -\frac{x^2}{2}\Big|_0^1 = \boxed{-\frac{1}{2}}$$

Elementary Regions: Change either

① top and bottom of B $\qquad \gamma_1(x,y) \le z \le \gamma_2(x,y)$

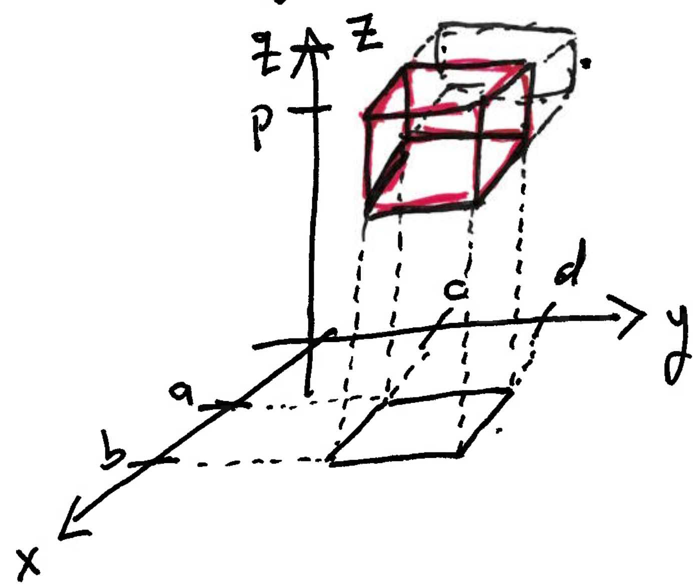
OR

② left and right of B $\qquad \delta_1(x,z) \le y \le \delta_2(x,y)$

OR

③ front and back of B $\qquad \rho_1(y,z) \le x \le \rho_2(y,z)$

to surfaces instead of just being flat



$B = [a,b] \times [c,d] \times [p,q]$

$a \le x \le b$

$c \le y \le d$

$p \le z \le q$

Case ① , say region W is given by inequalities

$$a \leq x \leq b$$

$$\varphi_1(x) \leq y \leq \varphi_2(x)$$

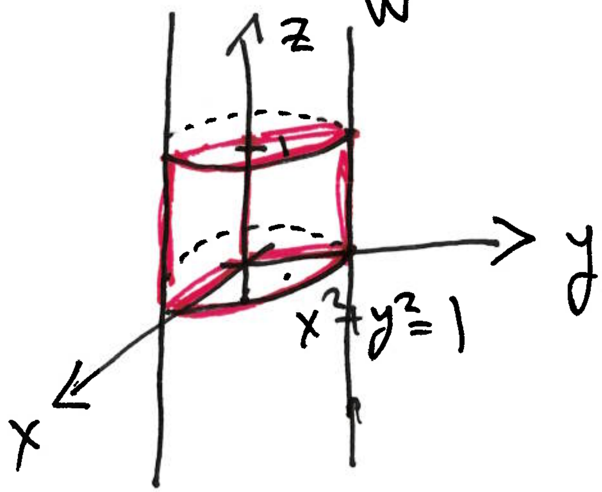
$$\gamma_1(x,y) \leq z \leq \gamma_2(x,y)$$

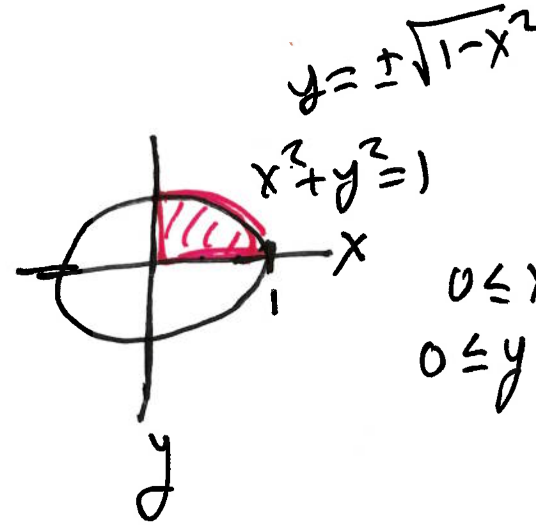
Then $$\iiint_W f(x,y,z)\, dV = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} \int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x,y,z)\, dz\, dy\, dx$$

Examples

(1) W = region bounded by planes $x=0$, $y=0$, $z=0$, $z=1$, the cylinder $x^2+y^2=1$, and in the quadrant $x\geq 0$, $y \geq 0$

Compute $\iiint_W z\,dV$ (given that $f(x,y,z)=z$)





$0 \leq x \leq 1$

$0 \leq y \leq \sqrt{1-x^2}$

W as elementary region is given by

$$\left.\begin{aligned} 0 &\leq x \leq 1 \\ 0 &\leq y \leq \sqrt{1-x^2} \\ 0 &\leq z \leq 1 \end{aligned}\right) \text{ same}$$

$$\iiint_W z\,dV = \int_0^1 \int_0^1 \int_0^{\sqrt{1-x^2}} z \, dy\, dx\, dz$$

$$= \int_0^1 \int_0^1 yz \Big|_{y=0}^{y=\sqrt{1-x^2}} \, dx\, dz$$

$$= \int_0^1 \int_0^1 z(1-x^2)^{1/2} \, dx\, dz$$

$$= \int_0^1 \int_0^1 z(1-x^2)^{1/2} \, dz\, dx$$

$$= \int_0^1 \frac{z^2}{2}(1-x^2)^{1/2} \Big|_{z=0}^{z=1} \, dx$$

$$= \frac{1}{2}\int_0^1 (1-x^2)^{1/2} \, dx \qquad \begin{array}{l} x = \sin\theta, \\ dx = \cos\theta \, d\theta \end{array} \qquad \begin{array}{l} x=0,\ \theta=0 \\ x=1,\ \theta=\pi/2 \end{array}$$

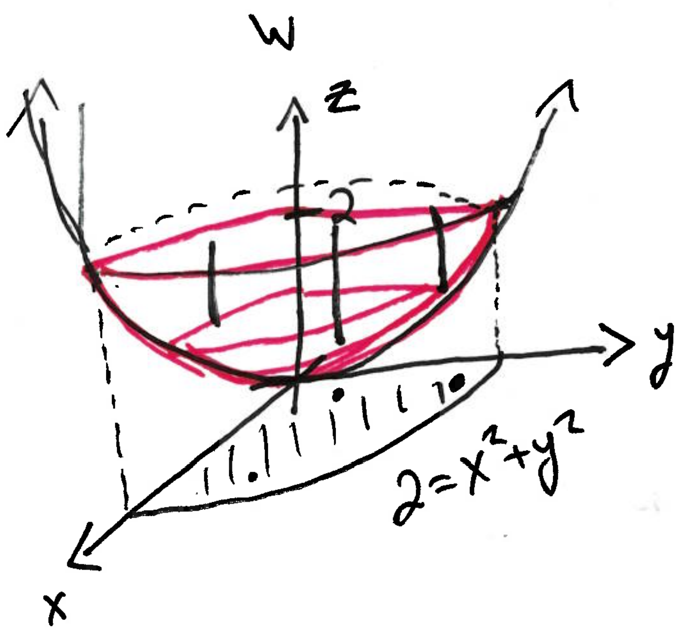
$$= \frac{1}{2}\int_0^{\pi/2} (1-\sin^2\theta)^{1/2} \cos\theta \, d\theta \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

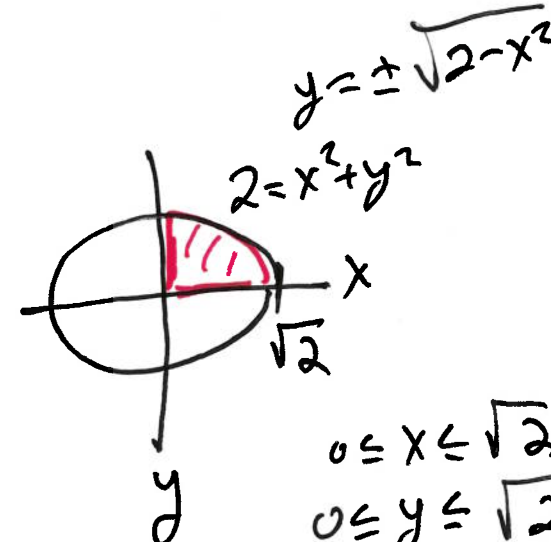
$$= \frac{1}{2}\int_0^{\pi/2} \cos^3\theta \, d\theta = \frac{1}{2}\int_0^{\pi/2} \frac{1+\cos 2\theta}{2} \, d\theta = \cdots$$

final answer $\frac{\pi}{2}$

② W = region bounded by planes $x=0$, $y=0$, $z=2$, the surface $z=x^2+y^2$, and lying in quadrant $x \geq 0$, $y \geq 0$.

Compute $\iiint x\,dV$ (given $f(x,y,z) = x$)





$0 \leq x \leq \sqrt{2}$

$0 \leq y \leq \sqrt{2-x^2}$

W as elementary region

$$0 \leq x \leq \sqrt{2}$$

$$0 \leq y \leq \sqrt{2-x^2}$$

$$x^2+y^2 \leq z \leq 2$$

$$\iiint_W x\,dV = \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} \int_{x^2+y^2}^{2} x \, dz\,dy\,dx$$

$$= \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} xz \Big|_{z=x^2+y^2}^{z=2} dy\,dx$$

$$= \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} 2x - x(x^2+y^2) \, dy\,dx$$

$$= \int_0^{\sqrt{2}} 2xy - yx^3 - \frac{xy^3}{3} \Big|_{y=0}^{y=(2-x^2)^{1/2}} dx$$

$$= \int_0^{\sqrt{2}} 2x(2-x^2)^{1/2} - (2-x^2)^{1/2}x^3 - \frac{x(2-x^2)^{3/2}}{3} \, dx$$

$$= \int_0^{\sqrt{2}} x(2-x^2)^{1/2}(2-x^2)^{2/2} - \frac{x(2-x^2)^{3/2}}{3} \, dx$$

$$= \int_0^{\sqrt{2}} \frac{3x(2-x^2)^{3/2}}{3} - \frac{x(2-x^2)^{3/2}}{3}\, dx$$

$$= \int_0^{\sqrt{2}} \frac{2x(2-x^2)^{3/2}}{3}\, dx$$

$u = 2 - x^2$ $\quad x=0,\ u=2$

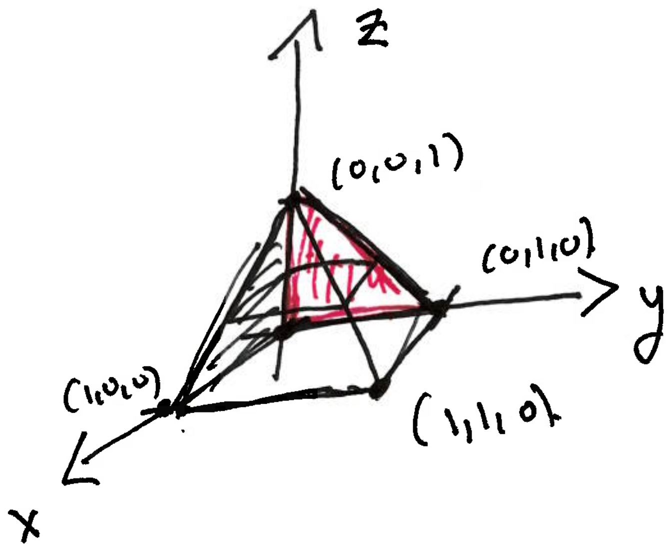
$du = -2x\,dx$ $\quad x=\sqrt{2},\ u=0$

$$= -\int_2^0 \frac{u^{3/2}}{3}\, du = -\left.\frac{u^{5/2}}{\frac{5}{2}\cdot 3}\right|_2^0 =$$

$$= \frac{2\cdot 2^{5/2}}{15} = \frac{2\cdot 2^2\cdot 2^{1/2}}{15} = \boxed{\frac{8\sqrt{2}}{15}}$$

(3) W = pyramid with top vertex at $(0,0,1)$ and base vertices at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(1,1,0)$.

Compute $\displaystyle\iiint_W (1-z^2)\,dV$

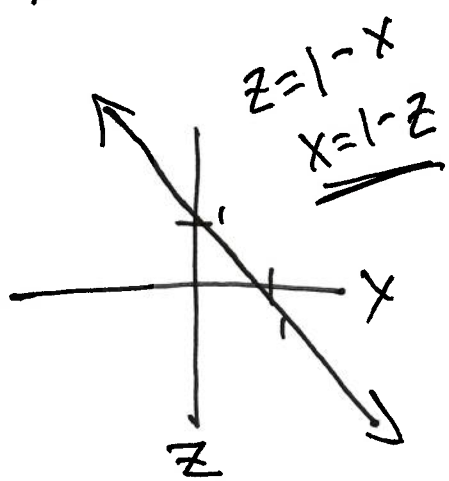


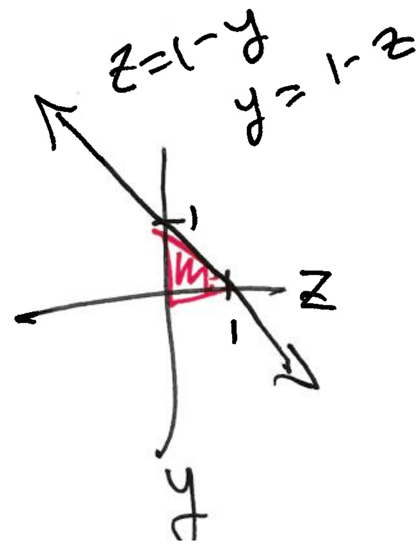
<u>W elementary region</u>

$0 \le z \le 1$

$0 \le y \le 1-z$

$0 \le x \le 1-z$





$0 \le z \le 1$

$0 \le y \le 1-z$