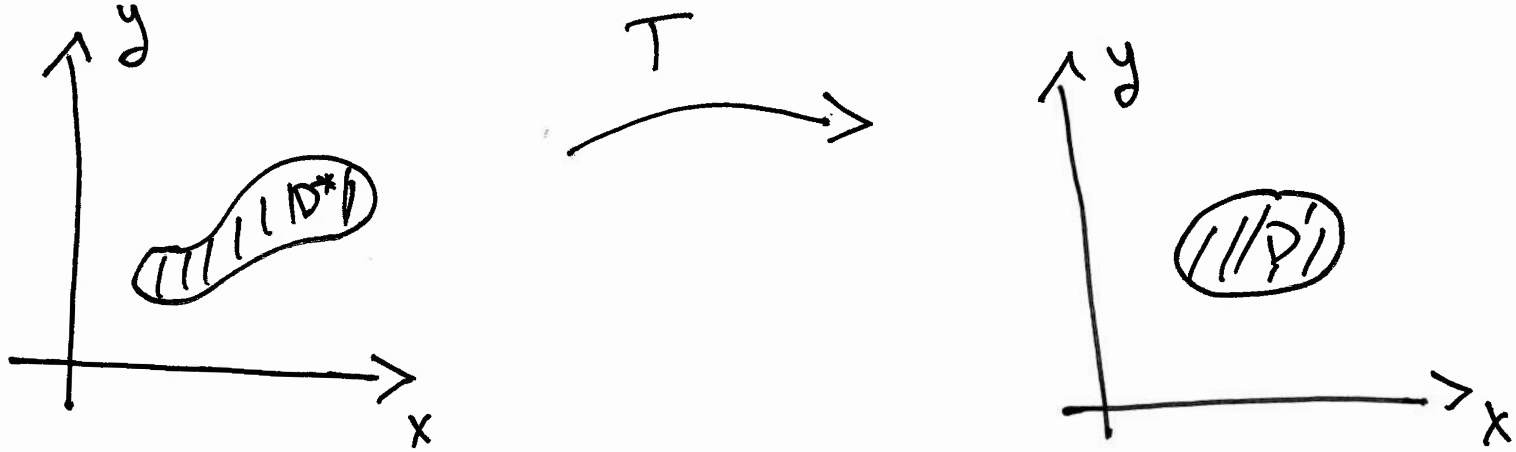
6.1 The Geometry of Maps from $\mathbb{R}^2$ to $\mathbb{R}^2$

$D^* \subseteq \mathbb{R}^2$ region in $\mathbb{R}^2$

$T : D^* \longrightarrow \mathbb{R}^2$ function

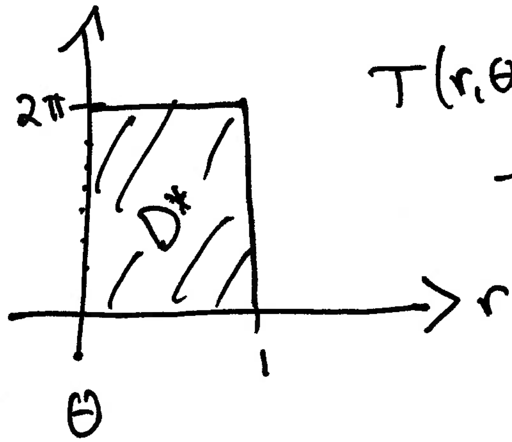
$D = T(D^*) =$ all points (x,y) such that $(x,y) = T(x^*, y^*)$ for some (x^*, y^*) in D^*



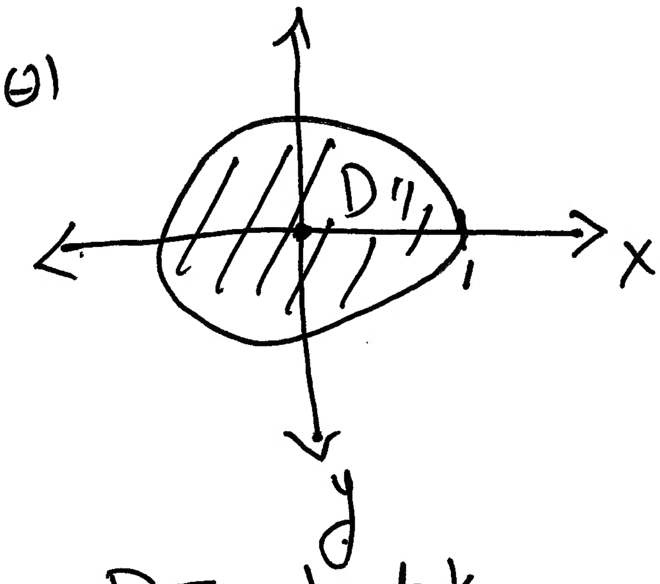
Examples

① $T(r, \theta) = (r\cos\theta, r\sin\theta)$ polar coordinate change of variables

$D^* = [0,1] \times [0, 2\pi]$. Determine $T(D^*) = D$.



$T(r,\theta) = (r\cos\theta, r\sin\theta)$



$x = r\cos\theta$

$y = r\sin\theta$

D^*: $0 \leq r \leq 1$

$0 \leq \theta \leq 2\pi$

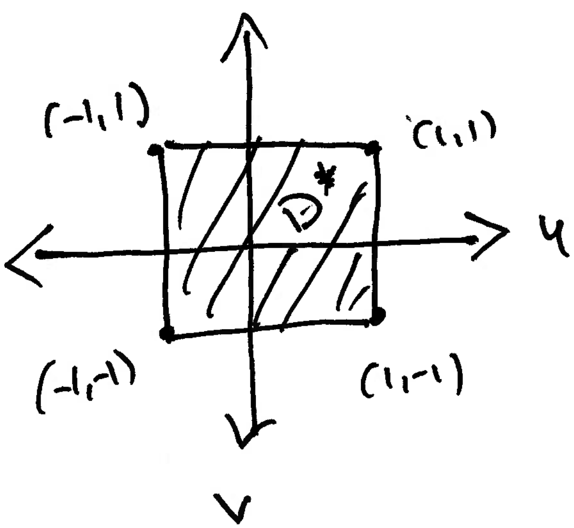
D = unit disk
(points inside and including unit circle)

$D: \; x^2 + y^2 \leq 1$

② $T(u,v) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \left(\frac{1}{2}u + \frac{1}{2}v, \; \frac{1}{2}u - \frac{1}{2}v\right)$

② $T(u,v) = \left(\frac{1}{2}u + \frac{1}{2}v,\ \frac{1}{2}u - \frac{1}{2}v\right)$

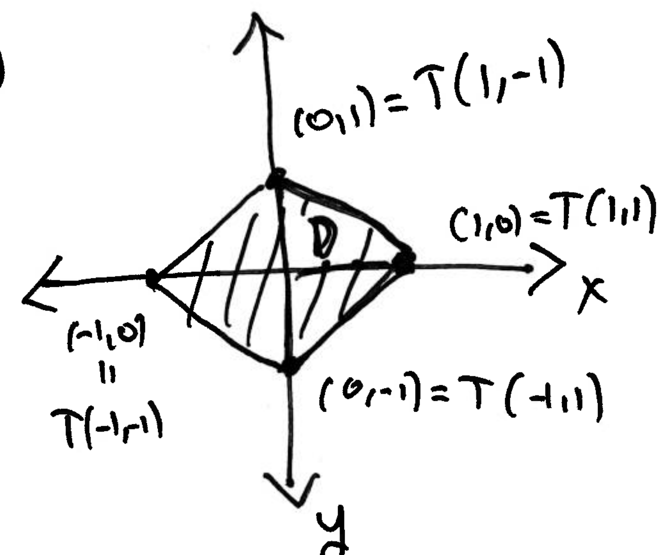
$D^* = [-1,1] \times [-1,1]$ Determine $D = T(D^*)$



$T(u,v) = \left(\frac{1}{2}u + \frac{1}{2}v, \frac{1}{2}u - \frac{1}{2}v\right) = (x,y)$ $\longrightarrow$

$x = \frac{1}{2}u + \frac{1}{2}v$

$y = \frac{1}{2}u - \frac{1}{2}v$



$D^*:\ -1 \le u \le 1$
$\quad\ -1 \le v \le 1$

$D =$ square with vertices $(0,1), (1,0), (-1,0), (0,-1)$

(*) T is a linear map so T sends parallelograms to parallelograms, so we just need to determine where vertices go. (*)

$T(1,1) = \left(\frac{1}{2} + \frac{1}{2}, \frac{1}{2} - \frac{1}{2}\right) = (1,0)$, $T(-1,1) = \left(-\frac{1}{2} + \frac{1}{2}, -\frac{1}{2} - \frac{1}{2}\right) = (0,-1)$

$T(-1,-1) = \left(-\frac{1}{2} - \frac{1}{2}, -\frac{1}{2} + \frac{1}{2}\right) = (-1,0)$, $T(1,-1) = \left(\frac{1}{2} - \frac{1}{2}, \frac{1}{2} + \frac{1}{2}\right) = (0,1)$

Linear Maps: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ 2x2 matrix defines a linear map

$$T(u,v) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = (au+bv, cu+dv)$$

Linear maps send parallelograms to parallelograms and vertices to vertices.

Aside: $T(u,v) = \left(u^2+uv+e^{v+u}, \cos(u)\cos(v)e^{\cos(u+v)} \right)$

This is a random $T(u,v)$

We usually consider linear $T(u,v)$'s or polar coord.
On homework you'll see other examples.

<u>One-to-one</u> maps: A mapping $T: D^* \to \mathbb{R}^2$ is one-to-one if for (u,v) and (u^*,v^*) in D^* if $T(u,v) = T(u^*,v^*)$ then $u = u^*$ and $v = v^*$.

<u>Examples</u>

(1) $T(r,\theta) = (r\cos\theta, r\sin\theta)$, $D^* = [0,1] \times [0,2\pi]$

Is T one-to-one? No because

$T(0,\theta) = (0,0)$ no matter what θ is.

(for instance $T(0,0) = (0,0) = T(0,2\pi)$
but $(0,0) \neq (0,2\pi)$)

$x = r\cos\theta$
$y = r\sin\theta$

can't solve for r, θ in terms of x, y uniquely

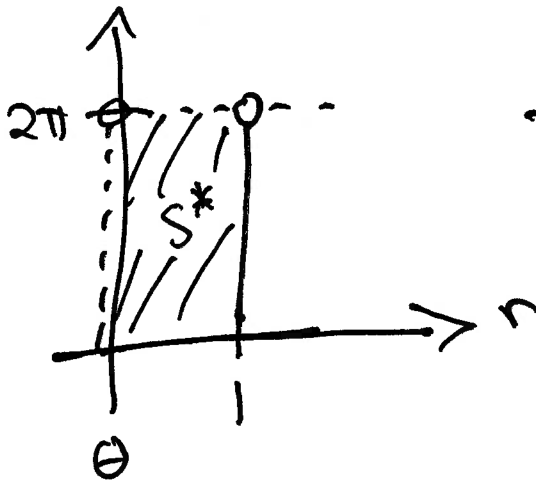
$x^2 + y^2 = r^2$ and $\theta = \cos^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right)$ ⇐ doesn't make sense if $x=0$ and $y=0$

$\sqrt{x^2+y^2} = r$

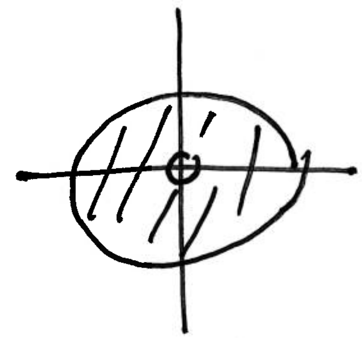
T is one-to-one on $S^* = (0,1] \times [0,2\pi)$

$0 < r \leq 1$

$0 \leq \theta < 2\pi$



$T(S^*) = S$



$S =$ unit disk without $(0,0)$

$S: 0 < x^2+y^2 \leq 1$

(2) Linear map given by $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$(T(u,v) = (au+bv, cu+dv))$

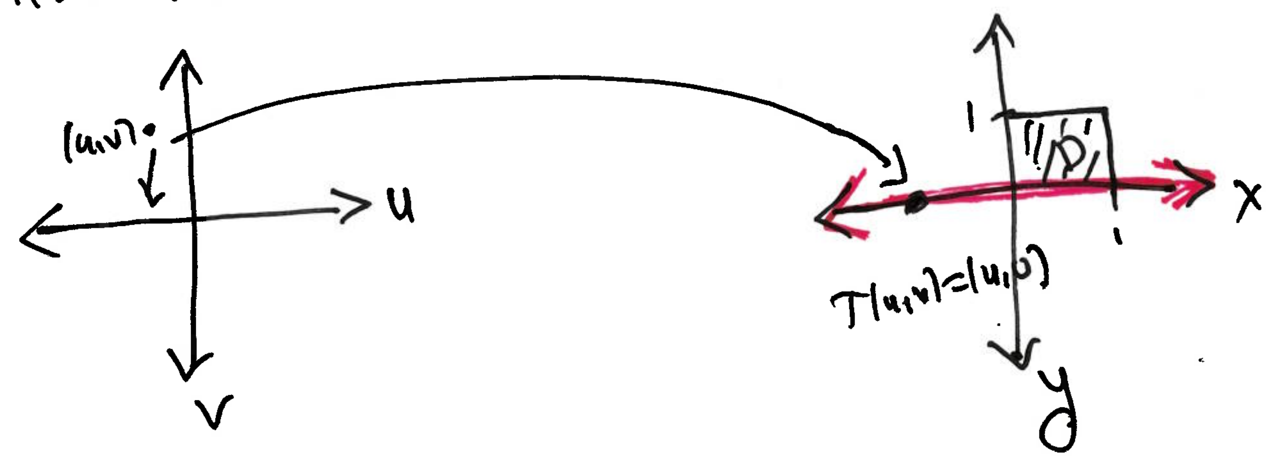
one-to-one if and only if $\det A = ad-bc \neq 0$

(3) $T(u,v) = (\frac{1}{2}u + \frac{1}{2}v, \frac{1}{2}u - \frac{1}{2}v)$ $\det = \frac{1}{2} \cdot -\frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$

<u>Onto maps</u>: The mapping T is <u>onto</u> D if for every (x,y) in D, there exists at least one point in the domain (u,v) such that $T(u,v) = (x,y)$.

<u>Examples</u>

(1) $T(u,v) = (u,0)$. Is T onto $D = [0,1] \times [0,1]$?

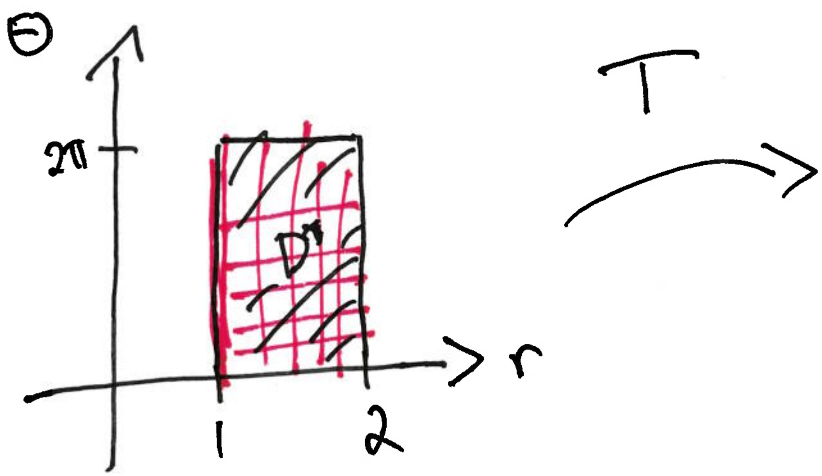


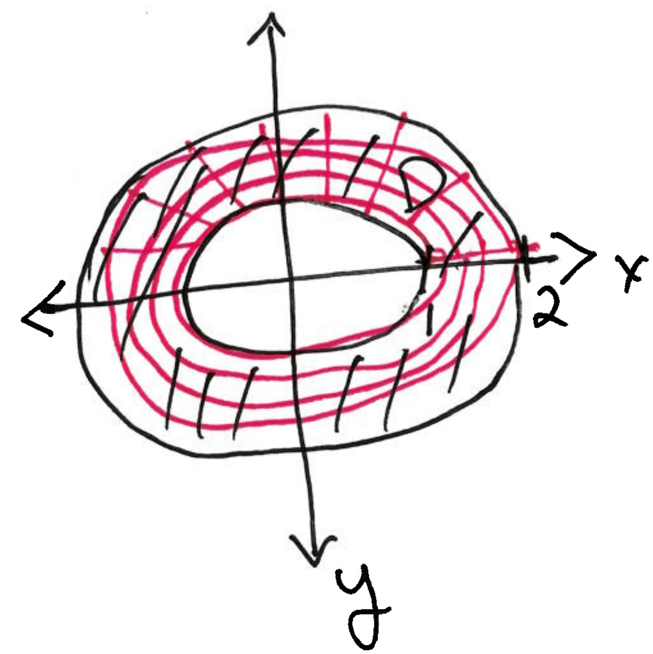
T is not onto D because the image of T is the x-axis. For instance $(\frac{1}{2}, \frac{1}{2})$ is in D but there does not exist (u,v) such that $T(u,v) = (\frac{1}{2}, \frac{1}{2})$ because $T(u,v) = (u, 0)$

(2) $T(r, \theta) = (r\cos\theta,\ r\sin\theta)$

D = region between $x^2+y^2=1$ and $x^2+y^2=4$.

Find D^* such that T maps D^* onto D.





D^*: $1 \le r \le 2$
$0 \le \theta \le 2\pi$

$x = r\cos\theta$
$y = r\sin\theta$