1.4 Cylindrical and Spherical Coordinates

Cylindrical Coordinates

Every point \((x, y, z)\) in \(\mathbb{R}^3\) can be determined by an 
\((\rho, \theta, \phi)\) cylindrical coordinate

cylinder \(y\) radius \(\rho\) every point on cylinder can be determined by an angle \(\theta\) from 
\(x\)-axis and a \(z\)-value which is the height

cylindrical coordinates are polar coordinates for \(xy\), then add \(z\).
Relations between cylindrical and rectangular coordinates

\[ x = r \cos \theta \]
\[ y = r \sin \theta \]
\[ z = z \]

\[ x^2 + y^2 = r^2 \]

to solve for \( \theta \), draw triangle and think

\textbf{Note}: By convention take \( r \geq 0 \).

\textbf{Examples}

1. Find cylindrical coordinates of \((-6, 6, 8)\)

\[ (-6)^2 + 6^2 = r^2 \]
\[ 36 + 36 = r^2 \]
\[ 72 = r^2 \]
\[ \sqrt{72} = r \]

\( \alpha = \frac{\pi}{4} \)
\( \theta = \frac{3\pi}{4} \)
\( \alpha = \sin^{-1} \left( \frac{6}{\sqrt{72}} \right) \)
\( \theta = \pi - \sin^{-1} \left( \frac{6}{\sqrt{72}} \right) \)

\[ x = \sin \alpha \]
\[ \sin \alpha = \frac{6}{\sqrt{72}} \]
\[ x = \frac{6}{\sqrt{72}} \]

\textbf{Answer:} \( \left( \sqrt{72}, \pi - \sin^{-1} \left( \frac{6}{\sqrt{72}} \right), 8 \right) \)
2) Find rectangular coordinates of $(8, \frac{\pi}{3}, -1)$

\[
\begin{align*}
x &= r \cos \theta = 8 \cos \left( \frac{\pi}{3} \right) = 8 \cdot \frac{1}{2} = 4 \\
y &= r \sin \theta = 8 \sin \left( \frac{\pi}{3} \right) = 8 \cdot \frac{\sqrt{3}}{2} = 4\sqrt{3} \\
z &= -1
\end{align*}
\]

\text{Answer: } (4, 4\sqrt{3}, -1)
Spherical Coordinates \((\rho, \theta, \phi)\)

- \(\rho\) = radius
- \(\theta\) = latitude = same as \(\theta\) from polar/cylindrical coordinates
- \(\phi\) = longitude = angle from \(z\)-axis

Positive \(\phi\)

\[
\sin \left( \frac{\pi}{2} - \phi \right) = \frac{z}{\rho} = \rho \sin (\frac{\pi}{2} - \phi) = \rho \cos (\phi)
\]

\[
\cos \left( \frac{\pi}{2} - \phi \right) = \frac{r}{\rho} = \rho \cos (\frac{\pi}{2} - \phi) = r
\]

\[
\rho \cos (\frac{\pi}{2} - \phi) = r
\]

Remember:

\[
\sin \left( \frac{\pi}{2} - \phi \right) = \cos (\phi)
\]

\[
\cos \left( \frac{\pi}{2} - \phi \right) = \sin (\phi)
\]

\[
x = r \cos \theta = \rho \cos \left( \frac{\pi}{2} - \phi \right) \cos \theta = \rho \sin (\phi) \cos \theta
\]

\[
y = r \sin \theta = \rho \sin (\phi) \sin \theta
\]
Relations

\[ x = \rho \sin(\phi) \cos(\theta) \]
\[ y = \rho \sin(\phi) \sin(\theta) \]
\[ z = \rho \cos(\phi) \]

\[ x^2 + y^2 + z^2 = \rho^2 \]
\[ \phi = \cos^{-1} \left( \frac{z}{\rho} \right) \]

For \( \theta \) draw picture as for polar/cylindrical coord.

**Note:** \( \rho \geq 0 \)

**Examples**

1. Find spherical coordinates \( \rho(1, -2, 2) \)

\[ \rho^2 = 1^2 + (-2)^2 + (2)^2 = 1 + 4 + 4 = 9 \Rightarrow \rho = 3 \]

\[ \phi = \cos^{-1} \left( \frac{2}{3} \right) \]

**Answer:** \( (3, 2\pi - \tan^{-1}(2), \cos^{-1}(\frac{2}{3})) \)

\[ \alpha + \theta = 2\pi \]
\[ \theta = 2\pi - \alpha \]
\[ \phi = \tan^{-1}(\frac{2}{1}) \]
\[ \theta = 2\pi - \tan^{-1}(2) \]
\[ \phi = \phi = \phi \]

(2) Find rectangular coordinates \( r \left( \frac{\pi}{3}, \frac{\pi}{6} \right) \)

\[
X = \rho \sin \phi \cos \theta = 1 \cdot \sin \left( \frac{\pi}{6} \right) \cos \left( \frac{\pi}{3} \right) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}
\]

\[
y = \rho \sin \phi \sin \theta = 1 \cdot \sin \left( \frac{\pi}{6} \right) \sin \left( \frac{\pi}{3} \right) = 1 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}
\]

\[
z = \rho \cos \phi = 1 \cdot \cos \left( \frac{\pi}{6} \right) = \frac{\sqrt{3}}{2}
\]

Answer: \( \left( \frac{1}{4}, \frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{2} \right) \)
Think of cylindrical and spherical coordinates as transformations or mappings from $\mathbb{R}^3$ to $\mathbb{R}^3$.

**Cylindrical**

$$T(r, \theta, z) = (r \cos \theta, r \sin \theta, z)$$

$$x = r \cos \theta, y = r \sin \theta, z = z$$

**Spherical**

$$T(\rho, \theta, \phi) = (\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi)$$

Both maps are onto: given any $(x, y, z)$ there is a cylindrical or spherical coordinate.
How can we restrict the domains to make the maps one-to-one?

For cylindrical domain would be:

\[ r > 0 \]
\[ 0 \leq \theta \leq 2\pi \]
\[ -\infty < z < \infty \]
\[ (0, 0, 0) \]

For spherical think about it.

Given \((x, y, z)\), what's the best choice for \((\rho, \theta, \phi)\)?