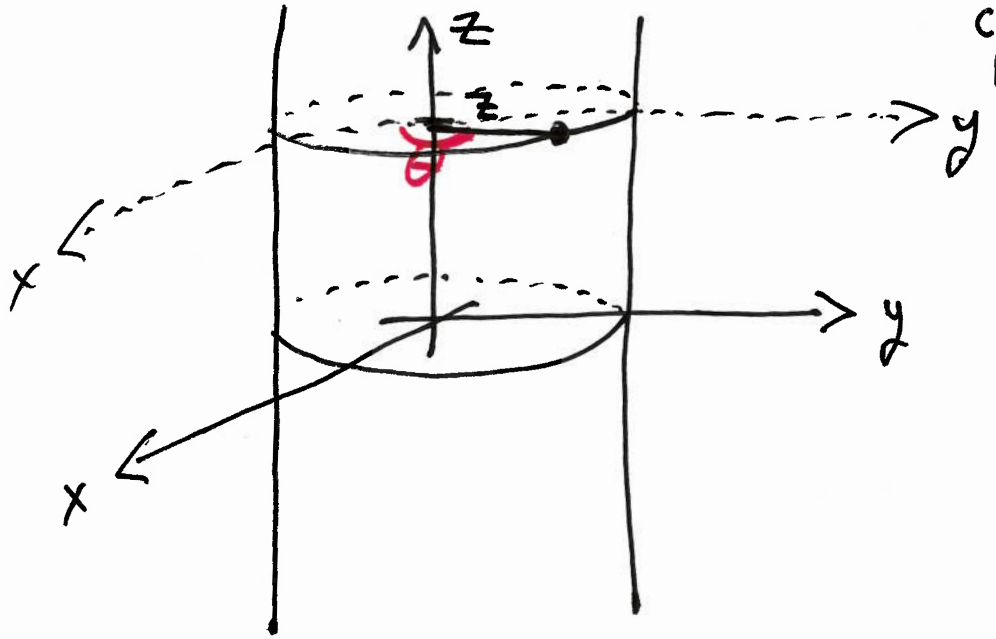
1.4 Cylindrical and Spherical Coordinates

Cylindrical Coordinates

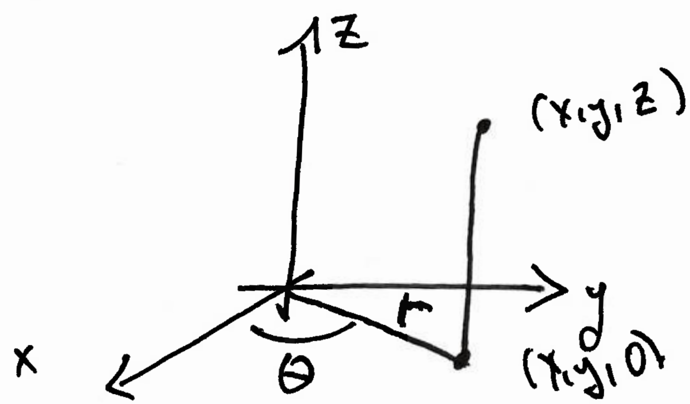
$(x^2+y^2=r^2)$

cylinder of radius r



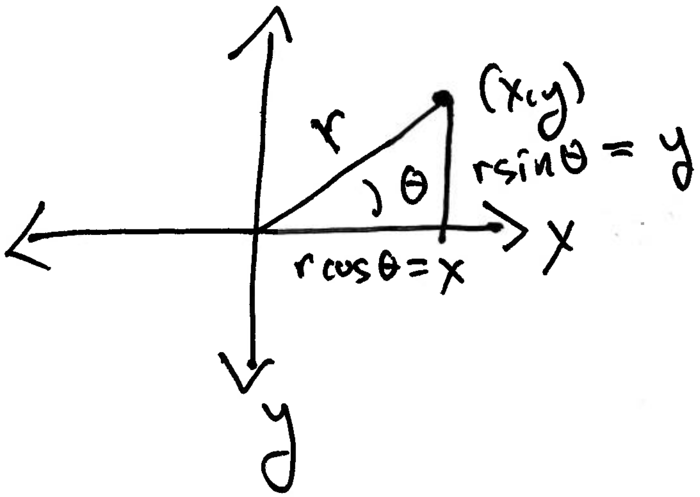
every point on cylinder can be determined by an angle θ from x-axis and a z-value which is the height

Every point (x,y,z) in $\mathbb{R}^3$ can be determined by an $(r, \theta, z) \leftarrow$ cylindrical coordinate



cylindrical coordinates are polar coordinates for x,y, then add z.

Relations between cylindrical and rectangular coordinates



$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$x^2 + y^2 = r^2$

to solve for θ, draw triangle and think

Note: By convention take $r \geq 0$.

Examples

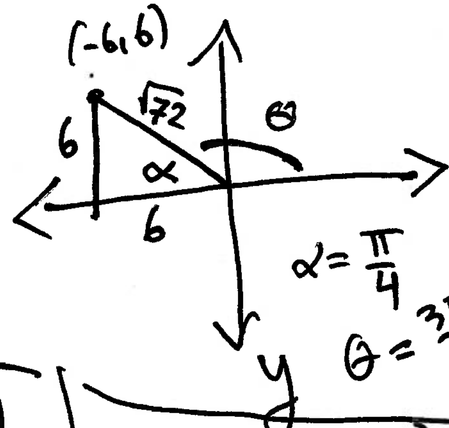
① Find cylindrical coordinates of $(-6, 6, 8)$

$(-6)^2 + 6^2 = r^2$

$36 + 36 = r^2$

$72 = r^2$

$\sqrt{72} = r$



$\alpha + \theta = \pi$

$\theta = \pi - \alpha$

$\sin\alpha = \frac{6}{\sqrt{72}}$

$\alpha = \frac{\pi}{4}$

$\theta = \frac{3\pi}{4}$

$\alpha = \sin^{-1}\left(\frac{6}{\sqrt{72}}\right)$

Answer: $\left(\sqrt{72}, \pi - \sin^{-1}\left(\frac{6}{\sqrt{72}}\right), 8\right) = \left(\sqrt{72}, \frac{3\pi}{4}, 8\right)$

$\theta = \pi - \sin^{-1}\left(\frac{6}{\sqrt{72}}\right)$

② Find rectangular coordinates of $(8, \frac{\pi}{3}, -1)$

$x = r\cos\theta = 8\cos\left(\frac{\pi}{3}\right) = 8\,\frac{1}{2} = 4$

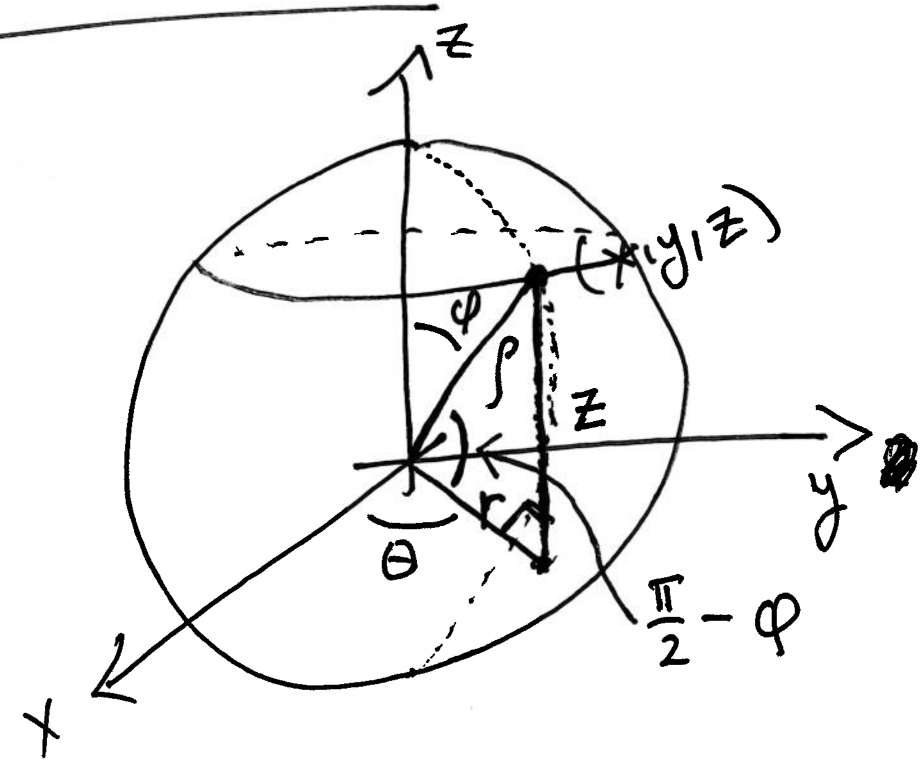
$y = r\sin\theta = 8\sin\left(\frac{\pi}{3}\right) = 8\cdot\frac{\sqrt{3}}{2} = 4\sqrt{3}$

$z = -1$

Answer: $(4, 4\sqrt{3}, -1)$

/3

Spherical Coordinates (ρ, θ, φ)



sphere of radius ρ

θ = lattitude = same θ from polar/cylindrical coordinates

φ = longitude = angle from positive z-axis

Remember

$\sin(\frac{\pi}{2} - \varphi) = \cos(\varphi)$

$\cos(\frac{\pi}{2} - \varphi) = \sin(\varphi)$

How to relate (ρ, θ, φ) to (x, y, z)?

$$\sin\left(\frac{\pi}{2} - \varphi\right) = \frac{z}{\rho}$$

$$z = \rho \sin\left(\frac{\pi}{2} - \varphi\right) = \rho\cos(\varphi)$$

$$\cos\left(\frac{\pi}{2} - \varphi\right) = \frac{r}{\rho}$$

$$\rho\cos\left(\frac{\pi}{2} - \varphi\right) = r$$

Know: $x = r\cos\theta = \rho\cos(\frac{\pi}{2} - \varphi)\cos\theta$

$$x = \rho\sin(\varphi)\cos\theta$$

$$y = r\sin\theta = \rho\sin(\varphi)\sin\theta$$

Relations

$x = \rho \sin(\varphi) \cos\theta$

$y = \rho \sin(\varphi) \sin\theta$

$z = \rho \cos(\varphi)$

$x^2 + y^2 + z^2 = \rho^2$

$\varphi = \cos^{-1}\left(\frac{z}{\rho}\right)$

for θ draw picture as for polar/cylindrical coord.

Note: $\rho \geq 0$

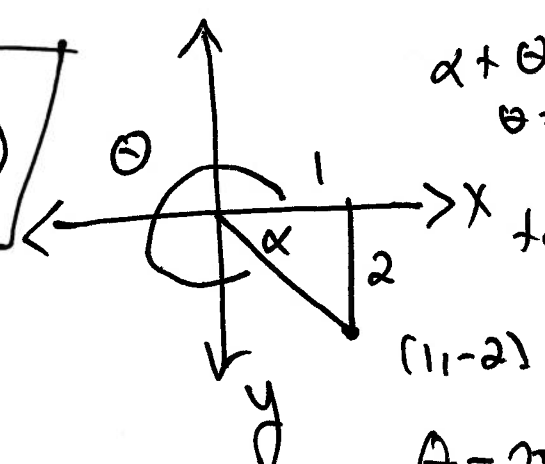
Examples

① Find spherical coordinates of $(1, -2, 2)$

$\rho^2 = 1^2 + (-2)^2 + (2)^2 = 1 + 4 + 4 = 9$, $\rho = 3$

$\varphi = \cos^{-1}\left(\frac{2}{3}\right)$

Answer: $\left(3,\ 2\pi - \tan^{-1}(2),\ \cos^{-1}\left(\frac{2}{3}\right)\right)$



$\alpha + \theta = 2\pi$

$\theta = 2\pi - \alpha$

$\tan\alpha = \frac{2}{1}$

$\alpha = \tan^{-1}(2)$

$\theta = 2\pi - \tan^{-1}(2)$

phi $= \phi = \varphi$

(2) Find rectangular coordinates of $(1, \pi/3, \pi/6)$

$x = \rho \sin\varphi \cos\theta = 1 \cdot \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$y = \rho \sin\varphi \sin\theta = 1 \cdot \sin\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{3}\right) = 1 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$

$z = \rho\cos(\varphi) = 1 \cdot \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

Answer: $\left(\frac{1}{4}, \frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{2}\right)$

Think of cylindrical and spherical coordinates as transformations or mappings from $\mathbb{R}^3$ to $\mathbb{R}^3$.

<u>Cylindrical</u>

$$T(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$$

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

<u>Spherical</u>

$$T(\rho, \theta, \varphi) = (\underset{x}{\underset{\shortparallel}{\rho\sin\varphi\cos\theta}}, \underset{y}{\underset{\shortparallel}{\rho\sin\varphi\sin\theta}}, \underset{z}{\underset{\shortparallel}{\rho\cos\varphi}})$$

Maps are <u>not</u> one-to-one means given (x,y,z) there may be more than one cylindrical or spherical coordinate

both maps are <u>onto</u>: given any (x,y,z) there is a cylindrical or spherical coordinate.

How can we restrict the domains to make the maps one-to-one?

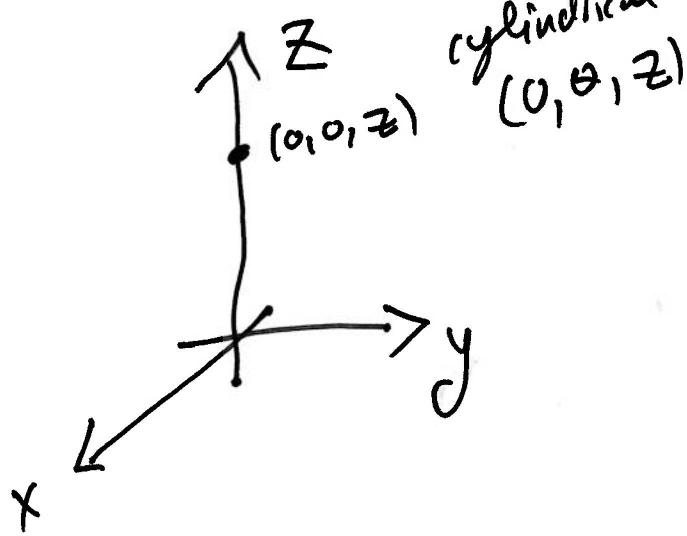
for cylindrical domain would be

$r > 0$

$0 \leq \theta < 2\pi$

$-\infty < z < \infty$

$(0,0,0)$



For spherical think about it.

Given (x,y,z), what's the best choice for (ρ, θ, φ)?