Midterm 1 is week from today in lecture.

10/21 is the date of Midterm 1

Covers chapter 5, 1.4, 6.1, 6.2

6.2 Change of Variables

Single variable u-substitution

$$\int_a^b f(x)\,dx \qquad x = h(u) \qquad dx = h'(u)\,du$$

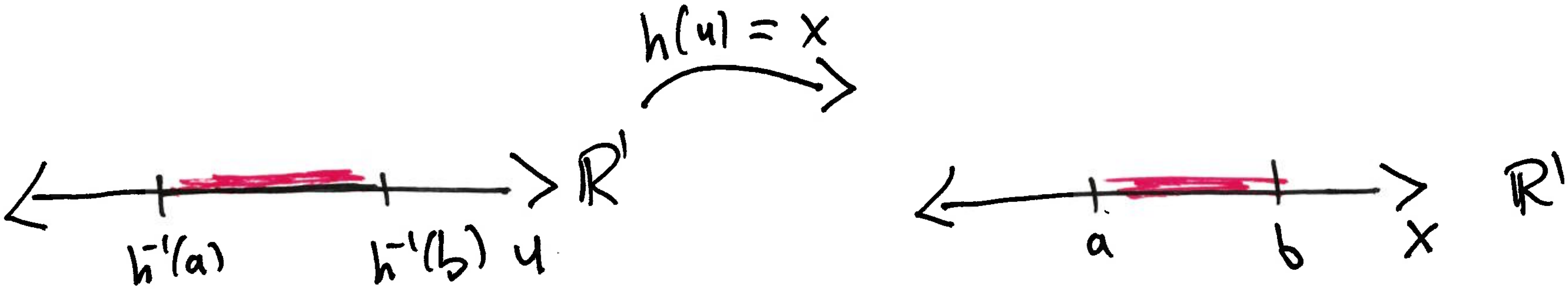
$$u = h^{-1}(x) \qquad x=a, \quad u = h^{-1}(a)$$

$$x=b, \quad u = h^{-1}(b)$$

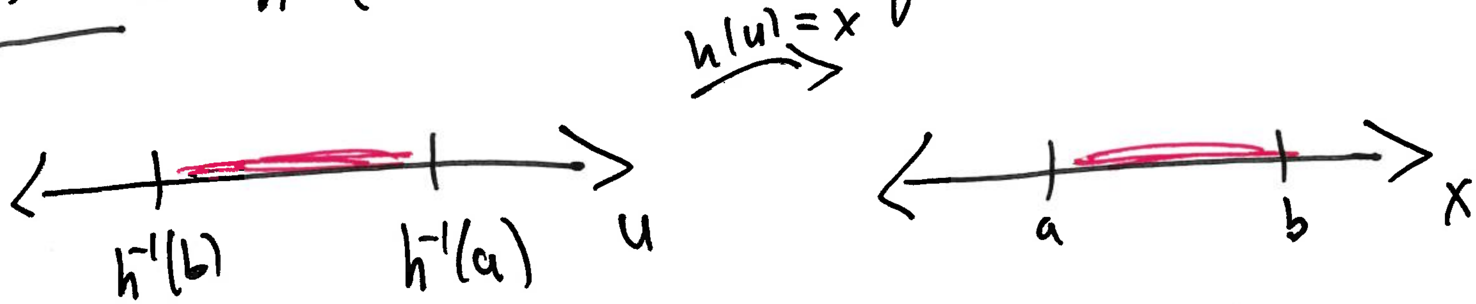
$$\int_a^b f(x)\,dx = \int_{h^{-1}(a)}^{h^{-1}(b)} f(h(u))\,h'(u)\,du$$

Two cases

Case 1: $h^{-1}(a) < h^{-1}(b)$ equivalent to $h'(u) > 0$



<u>Case 2</u>: $h^{-1}(b) < h^{-1}(a)$ equivalent to $h'(u) < 0$



$$\int_{h^{-1}(a)}^{h^{-1}(b)} f(h(u))\,h'(u)\,du = -\int_{h^{-1}(b)}^{h^{-1}(a)} f(h(u))\,h'(u)\,du$$

$$= \int_{h^{-1}(b)}^{h^{-1}(a)} f(h(u))\,|h'(u)|\,du$$

since $h'(u) < 0$
$-h'(u) = |h'(u)|$

<u>Case 1</u>: $h'(u) > 0$ so $|h'(u)| = h'(u)$

In case 1, let $a^* = h^{-1}(a)$, $b^* = h^{-1}(b)$.

In case 2, let $a^* = h^{-1}(b)$, $b^* = h^{-1}(a)$.

Then h sends $[a^*, b^*]$ onto $[a, b]$ and h is one-to-one, ~~and~~ and

$$\int_a^b f(x)\,dx = \int_{a^*}^{b^*} f(h(u))\,|h'(u)|\,du$$

Change of variables for double integrals

$$\iint_D f(x,y)\,dx\,dy$$

$T : D^* \longrightarrow D$

T maps D^* onto D.

$T(u,v) = (x(u,v), y(u,v))$

May guess ~~that~~

$$\cancel{\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(T(u,v))\,du\,dv}$$

NOT CORRECT

Reason not correct: T may not preserve area.

~~$\#$~~ Saying that $\text{Area}(D^*) \neq \text{Area}(D)$.

The derivative of T determines how T changes area!

$$T : D^* \subseteq \mathbb{R}^2_{(u,v)} \longrightarrow D \subseteq \mathbb{R}^2_{(x,y)}$$

$$T(u,v) = (x(u,v), y(u,v))$$

$$DT = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \quad \text{derivative of } T$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Jacobian of T = determinant of the derivative of T

Change of variables formula: T maps D^* onto D and is one-to-one on almost all of D^* (~~app~~ except possibly on finitely many curves and points).

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(T(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Examples

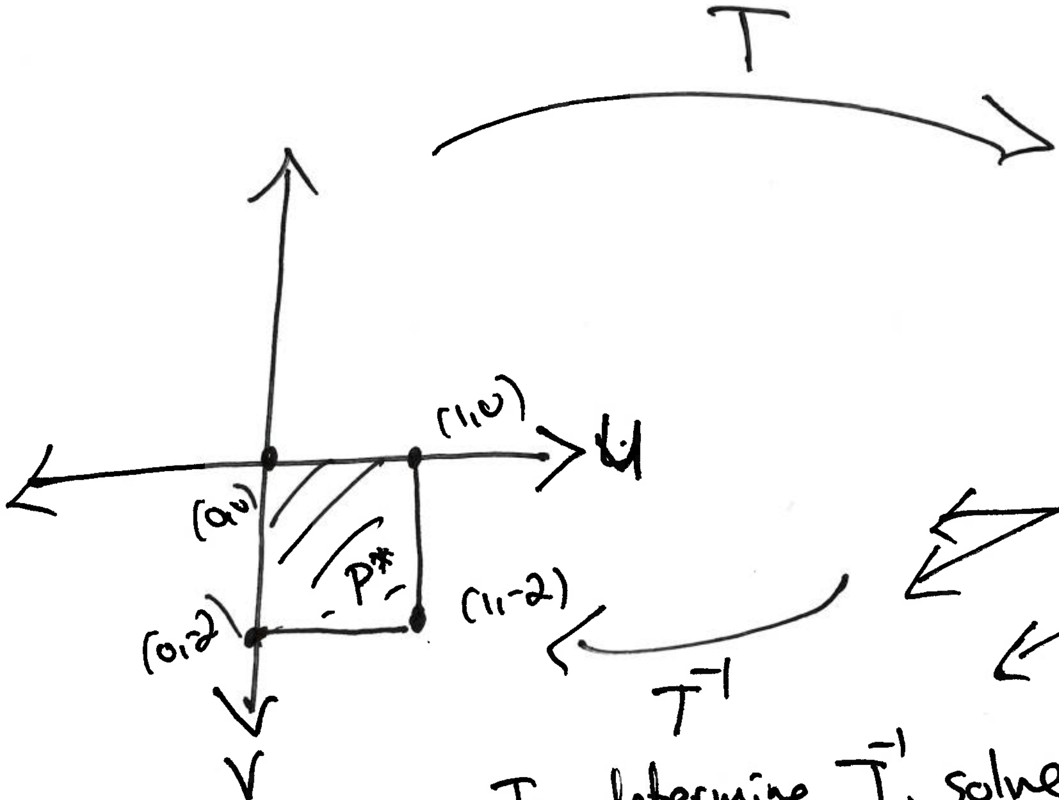
① P = parallelogram bounded by $y=2x$, $y=2x-2$, $y=x$ and $y=x+1$.

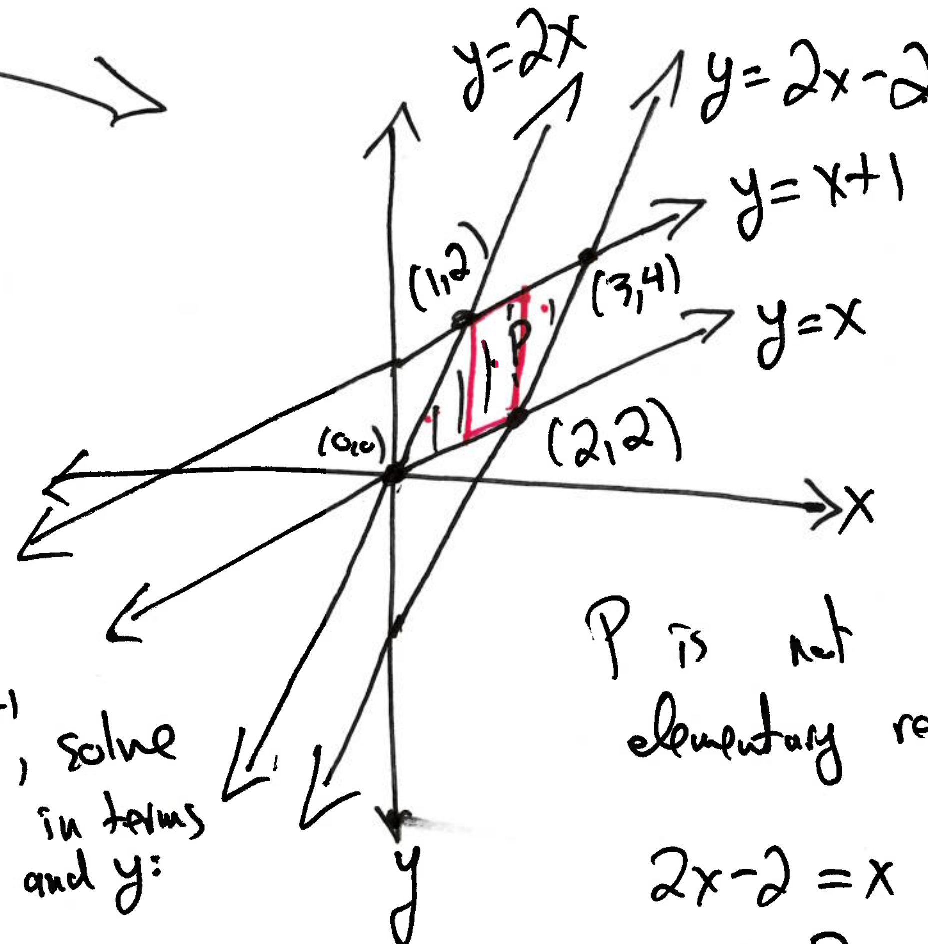
Evaluate $\iint_P xy\,dx\,dy$ by making the change of variables

$$x = u-v$$
$$y = 2u-v$$

$$T(u,v) = (u-v,\ 2u-v)$$

$$\iint_P xy\,dx\,dy$$





P is not elementary region

To determine T^{-1}, solve for u and v in terms of x ~~and~~ and y:

$x = u - v$

$y = 2u - v$

$x - y = -u$

$u = y - x$

$y - 2x = -v + 2v$

$y - 2x = v$

$x + 1 = 2x$

$1 = x$

$2x - 2 = x$

$x = 2$

$2x - 2 = x + 1$

$x = 3$

$T^{-1}(x,y) = (y-x,\ y-2x)$

$T^{-1}(0,0) = (0,0)$, $T^{-1}(2,2) = (0, 2-4) = (0,-2)$

$T^{-1}(3,4) = (4-3,\ 4-6) = (1,-2)$

$T^{-1}(1,2) = (2-1,\ 2-2) = (1,0)$

P^*, $0 \le x \le 1$
$-2 \le y \le 0$

$$\iint_P xy\,dx\,dy = \iint_{P^*} (u-v)(2u-v) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

$T(u,v) = (u-v,\ 2u-v)$

$x = u-v$
$y = 2u-v$

$$DT = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, \quad \frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} = 1\cdot(-1) - (-1)\cdot 2 = 1$$

$$\iint_P xy\,dx\,dy = \int_{-2}^{0} \int_0^1 (2u^2 - uv - 2uv + v^2)\cdot 1\,du\,dv$$

Easy Integral!