Polar Coordinate Change of Variables

$$T(r,\theta) = (r\cos\theta,\ r\sin\theta) \qquad \begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

$$DT = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r\cos^2\theta + r\sin^2\theta$$

$$= r$$

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|$$

If T maps D^* to D, then

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(r\cos\theta,\ r\sin\theta)\, r\,dr\,d\theta$$

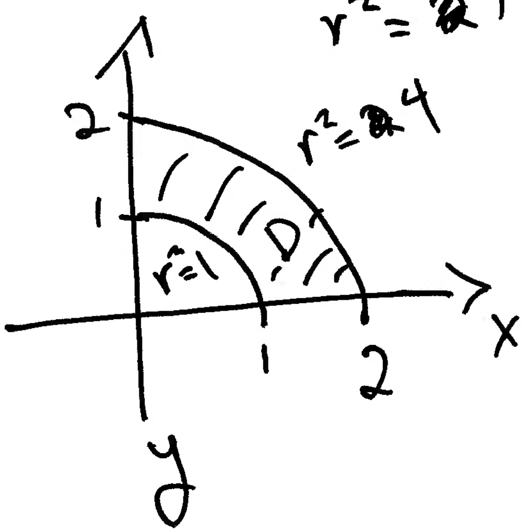
Formula

Example

Evaluate $\iint_D e^{x^2+y^2}\,dx\,dy$ where D is the region in first quadrant between $x^2+y^2=1$ and $x^2+y^2=4$.

$r^2=1$

$r^2 = \cancel{2}\,4$



D^*: $1 \le r \le 2$

$0 \le \theta \le \frac{\pi}{2}$

$$\iint_D e^{x^2+y^2}\,dx\,dy = \int_0^{\frac{\pi}{2}}\int_1^2 e^{r^2}\, r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_1^2 e^{r^2} r\,dr$$

$u=r^2$

$du=2r\,dr$

$r=1,\ u=1$

$r=2,\ u=4$

$$= \frac{\pi}{2} \cdot \frac{1}{2} \int_1^4 e^u \, du$$

$$= \frac{\pi}{4} e^u \Big|_1^4$$

$$= \boxed{\frac{\pi}{4} \left(e^4 - e^1\right)}$$

Change of Variables for Triple Integrals

$T : W^* \longrightarrow \mathbb{R}^3 \qquad T(u,v,w) = \left(x(u,v,w),\ y(u,v,w),\ z(u,v,w)\right)$

$$DT = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}, \qquad \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det(DT)$$

caveats about one-to-oneness ... (see book)

Say T maps W^* to $W = T(W^*)$

$$\iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} f(T(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

Cylindrical Coordinates

$T(r,\theta,z) = (r\cos\theta,\ r\sin\theta,\ z)$

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$$DT = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \cos\theta\,(r\cos\theta\cdot 1 - 0\cdot 0) - \sin\theta\,(-r\sin\theta\cdot 1 - 0\cdot 0) + 0\cdot(0)$$

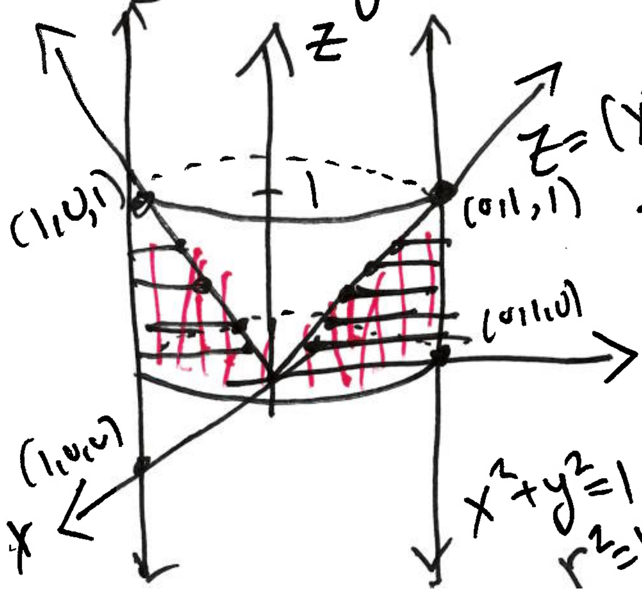
$$= r\cos^2\theta + r\sin^2\theta = r$$

Formula If $T(W^*) = W$, then

$$\iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} f(r\cos\theta, r\sin\theta, z)\, r\,dr\,d\theta\,dz$$

Example

Evaluate $\iiint_B z\,dx\,dy\,dz$ where B is the region within the cylinder $x^2+y^2=1$, above xy-plane, and below $z=(x^2+y^2)^{1/2}$.



$z=(x^2+y^2)^{1/2}$

$z=(r^2)^{1/2}$

$z=r$

$x^2+y^2=1$

$r^2=1$, $r=1$

Rectangular

$-1 \le x \le 1$

$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$

$0 \le z \le (x^2+y^2)^{1/2}$

Cylindrical

$$\left.\begin{array}{l} 0 \le z \le 1 \\ 0 \le \theta \le 2\pi \\ z \le r \le 1 \end{array}\right\} \text{More simple bounds}$$

$$\iiint_B z \, dx\,dy\,dz = \int_0^1 \int_0^{2\pi} \int_z^1 z \, r \, dr \, d\theta \, dz$$

$$= 2\pi \int_0^1 \left(\frac{z r^2}{2} \Big|_{r=z}^{r=1} \right) dz$$

$$= 2\pi \int_0^1 \frac{z}{2} - \frac{z^3}{2} \, dz$$

$$= 2\pi \left(\frac{z^2}{4} - \frac{z^4}{8} \Big|_0^1 \right)$$

$$= 2\pi \left(\frac{1}{4} - \frac{1}{8} \right) = 2\pi \left(\frac{2}{8} - \frac{1}{8} \right)$$

$$= \frac{2\pi}{8} = \boxed{\frac{1}{4}\pi}$$

Spherical Coordinates

$$T(\rho, \theta, \varphi) = (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi)$$

$DT = \ldots$ a complicated 3×3 matrix

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \varphi)} = -\rho^2 \sin\varphi$$

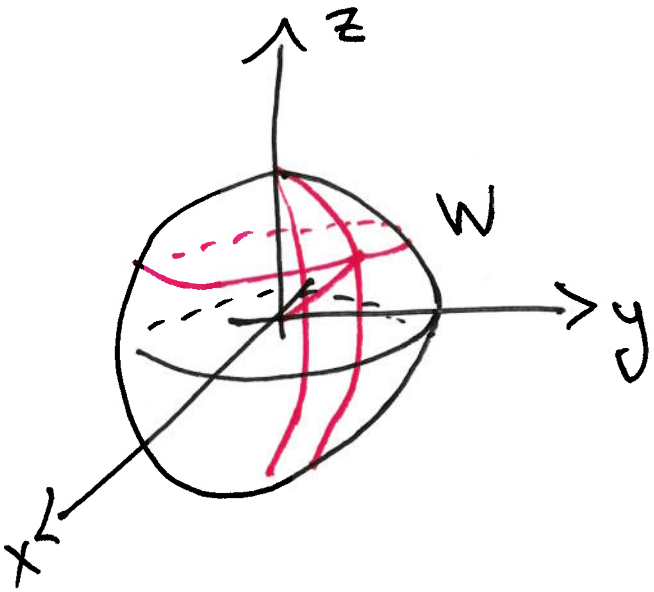
Formula $T(W^*) = W$

$$\iiint_W f(x, y, z)\, dx\, dy\, dz = \iiint_{W^*} f(\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi)\, \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

Example

$$e^{(\rho^2)^{3/2}} = e^{\rho^3}$$

Evaluate $\displaystyle\iiint_W e^{[(x^2+y^2+z^2)^{3/2}]}\,dV \xrightarrow{\text{spherical}} e^{(\rho^2)^{3/2}} = e^{\rho^3}$ where W is unit ball in $\mathbb{R}^3$, $x^2+y^2+z^2 \le 1$



Rectangular

$-1 \le x \le 1$

$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$

$-\sqrt{1-x^2-y^2} \le z \le \sqrt{1-x^2-y^2}$

Spherical

$$\left.\begin{array}{l} 0 \le \rho \le 1 \\ 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi \end{array}\right\} \text{ more simple than}$$

$$\iiint_W e^{[(x^2+y^2+z^2)^{3/2}]} = \int_0^{2\pi}\int_0^1\int_0^{\pi} e^{\rho^3}\rho^2 \sin\varphi \, d\varphi\, d\rho\, d\theta$$

$$= 2\pi \int_0^1 e^{\rho^3} \rho^2 \, d\rho \cdot \int_0^\pi \sin\varphi \, d\varphi$$

$$= 2\pi \int_0^1 e^{\rho^3} \rho^2 \, d\rho \left(-\cos\varphi \Big|_0^\pi \right)$$

$$= 2\pi \int_0^1 e^{\rho^3} \rho^2 \, d\rho \, (1 - -1) \qquad u = \rho^3 \quad du = 3\rho^2$$

$$= \frac{4\pi}{3} \left(\int_0^1 e^u \, du \right) = \frac{4\pi}{3} e^u \Big|_0^1$$

$$= \boxed{\frac{4\pi}{3} (e^1 - 1)}$$