Simple, locally finite dimensional Lie algebras in positive characteristic

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We prove two structure theorems for simple, locally finite dimensional Lie algebras over an algebraically closed field of characteristic p which give sufficient conditions for the algebras to be of the form $[R^{(-)}, R^{(-)}]/(Z(R) \cap [R^{(-)}, R^{(-)}])$ or $[K(R, *), K(R, *)]$ for a simple, locally finite dimensional associative algebra $R$ with involution $\ast$. The first proves that a condition we introduce, known as locally nondegenerate, along with the existence of an ad-nilpotent element suffice. The second proves that an ad-integrable Lie algebra is of this type if the characteristic of the ground field is sufficiently large. Lastly we construct a simple, locally finite dimensional associative algebra $R$ with involution $\ast$ such that $K(R, \ast) \neq [K(R, \ast), K(R, \ast)]$ to demonstrate the necessity of considering the commutator in the first two theorems.

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1. Introduction

An important result which characterizes the simple, finite dimensional Lie algebras over a field of positive characteristic is the Kostrikin–Strade–Benkart Theorem [20,25,7]: Suppose $L$ is a simple, finite dimensional Lie algebra over an algebraically closed field of
characteristic $p > 5$. If $L$ is nondegenerate and there exists a nonzero element $x \in L$ such that $ad(x)^{p-1} = 0$, then $L$ is a Lie algebra of classical type. This theorem was improved by Premet in [23]: Every finite dimensional, nondegenerate simple Lie algebra over an algebraically closed field of characteristic $p > 5$ is classical.

Suppose $L$ is a simple, infinite dimensional Lie algebra over an algebraically closed field of characteristic zero. In [1], Bahturin, Baranov, and Zalesskii prove that $L$ embeds into a locally finite associative algebra if and only if $L$ is isomorphic to $[K(R, *), K(R, *)]$ where $*$ is an involution and $R$ is an involution simple locally finite associative algebra. This utilizes and extends earlier work of Baranov in [5]. In [26], Zalesskii asks for a characterization of such Lie algebras when the ground field is algebraically closed of positive characteristic. The following result extends the Kostrikin–Strade–Benkart Theorem to the infinite dimensional case and also addresses Zalesskii’s question in the spirit of this theorem.

**Theorem 1.** Let $L$ be a simple, infinite dimensional, locally finite Lie algebra over an algebraically closed field $F$ of characteristic $p > 7$ or characteristic zero. Then the following conditions are equivalent:

1. $L$ is locally nondegenerate and there is some nonzero element $x \in L$ such that $ad(x)^{p-1} = 0$.
2. $L \cong [R^\mathbf{(-)}, R^\mathbf{(-)}]/(Z(R) \cap [R^\mathbf{(-)}, R^\mathbf{(-)}])$ where $R$ is a locally finite, simple associative algebra or $L \cong [K(R, *), K(R, *)]$ where $R$ is a locally finite, simple associative algebra with involution $*$.

For the definition of locally nondegenerate, see Section 2.4. We note that over a field of characteristic zero, the condition that $L$ be locally nondegenerate is trivial. Also, when the characteristic of the ground field is zero, the condition that $ad(x)^{p-1} = 0$ for some $x$ simply means that there exists some ad-nilpotent element (the index of nilpotence is inconsequential).

Another result of this kind does not assume $L$ is locally nondegenerate, but imposes a stronger assumption on the characteristic of the ground field. We say $L$ is ad-integrable (or has an algebraic adjoint representation, see [28]) if for each element $a \in L$, $ad(a)$ is a root of some nonzero polynomial $f_a(t) \in F[t]$. Note that a simple, locally finite Lie algebra embeds into a locally finite associative algebra if and only if it is ad-integrable (see for example Theorem 9.1 of [5]). For $a \in L$, let $\mu_a(t)$ be the minimal polynomial of $ad(a)$ and define $d(L) = \min\{\deg \mu_a(t) \mid a \text{ is not ad-nilpotent}\}$.

**Theorem 2.** Let $L$ be a simple, locally finite Lie algebra which is ad-integrable over an algebraically closed field $F$ of characteristic $p > 3$. If $p > 2d(L) - 2$, then $L \cong [R^\mathbf{(-)}, R^\mathbf{(-)}]/(Z(R) \cap [R^\mathbf{(-)}, R^\mathbf{(-)}])$ where $R$ is a locally finite, simple associative algebra or $L \cong [K(R, *), K(R, *)]$ where $R$ is a locally finite, simple associative algebra with involution $*$. 

In the case that $F$ has characteristic zero, the condition $p > 2d(L) - 2$ is not required for the proof of Theorem 2 to hold. Thus, our proof of Theorem 2, together with the result of Theorem 1, provides a new proof of Theorem 1.2 in [1].

We note that Theorem 1 cannot be improved by removing the condition that $L$ contains an ad-nilpotent element, as in the finite dimensional version. This is because in characteristic zero, there exist simple locally finite Lie algebras which are not of this form (see for instance [3] and [8]).

In [14], Herstein asks if there is a simple associative algebra $R$ with involution $\ast$ such that $K(R, \ast) \neq [K(R, \ast), K(R, \ast)]$. An example of a division ring with this property was provided in [22]. However, this example is not locally finite. In the final section, we prove the following theorem.

**Theorem 3.** There exists a simple, locally finite associative algebra $R$ with involution $\ast$ so that $K(R, \ast) \neq [K(R, \ast), K(R, \ast)]$.

This demonstrates the necessity of considering the commutator $[K(R, \ast), K(R, \ast)]$ in Theorem 1 and Theorem 2.

2. Preliminaries

An algebra $A$ is *locally finite dimensional* (locally finite) if any finitely generated subalgebra is finite dimensional. It is straightforward to check that a subalgebra or quotient of a locally finite algebra is also locally finite. An algebra is *locally nilpotent* if every finitely generated subalgebra is a finite dimensional nilpotent algebra. An algebra is *locally solvable* if every finitely generated subalgebra is a finite dimensional solvable algebra. The first statement of the following lemma is well known. For the second statement, the reader is referred to [2].

**Lemma 4.** A locally nilpotent algebra cannot be simple. A locally solvable Lie algebra cannot be simple.

Let $R$ be an associative algebra. We use the notation $R^{(-)}$ to denote the Lie algebra induced by the commutator of the associative product. It is straightforward to check that if $R$ is locally finite as an associative algebra, then $R^{(-)}$ is locally finite as a Lie algebra. In [14], Herstein proves the following: Suppose $R$ is a simple associative algebra with center $Z$ over a field $F$ of characteristic other than 2. Then $[R^{(-)}, R^{(-)}]/(Z \cap [R^{(-)}, R^{(-)}])$ is a simple Lie algebra.

Let $A$ be an arbitrary algebra over a field $F$. An involution $\ast : A \to A$ is a linear transformation such that $(a^*)^* = a$ and $(ab)^* = b^*a^*$ for any $a, b$ in $A$. If $\ast$ is an involution of an associative algebra $R$, the set of skew-symmetric elements $K(R, \ast) = \{x \in R \mid x^* = -x\}$ is a Lie algebra. In [14], the following theorem is also proved: Suppose $R$ is a simple associative algebra over $F$ of characteristic other than 2 with involution $\ast$ and $R$ is more
than 16-dimensional over its center $Z$. If $K = K(R, \ast)$ then $[K, K]/(Z \cap [K, K])$ is a simple Lie algebra.

The following two lemmas are classical results which appear in [6] as Lemmas 2.4 and 2.3, respectively. We do not assume the ground field $F$ is of characteristic zero, as in [6]. However, $F$ is perfect since $F$ is algebraically closed, so we can apply the Wedderburn Principle Theorem in the proof of Lemma 5 and otherwise the proof remains the same as presented in [6].

**Lemma 5.** Suppose $A$ is a finite dimensional, associative algebra with involution $\ast : A \to A$ over an algebraically closed field $F$ of characteristic other than 2. Let $N$ be the radical of $A$, which is the largest nilpotent ideal. Then $A = B \oplus N$, where $B \cong A/N$ is a semisimple subalgebra invariant under $\ast$.

**Lemma 6.** Suppose $B$ is a semisimple, finite dimensional associative algebra over an algebraically closed field $F$ with involution $\ast : B \to B$. Then $K(B, \ast) \cong \bigoplus_{(i)} K(M_{n_i}(F), \ast) \bigoplus_{(i)} M_{n_i}(F)^{(\ast)}$, where the involution $\ast$ restricted to $M_{n_i}(F)$ is transposition or the symplectic involution.

### 2.1. $\mathbb{Z}$-graded algebras

A $\mathbb{Z}$-grading of an algebra $A$ is a decomposition into a sum of subspaces $A = \sum_{i \in \mathbb{Z}} A_i$ such that $A_i A_j \subseteq A_{i+j}$. Such a grading is finite if the set $\{i \in \mathbb{Z} \mid A_i \neq (0)\}$ is finite and the grading is nontrivial if $\sum_{i \neq 0} A_i \neq 0$.

**Lemma 7.** Suppose $R = R_{-n} + \cdots + R_n$ is a simple, associative algebra with nontrivial $\mathbb{Z}$-grading. Then $R$ is generated by $\bigcup_{i \neq 0} R_i$.

We note that the same result holds for a simple, graded Lie algebra.

The following lemma will be used in Section 3 to show that a locally nondegenerate simple Lie algebra $L$ embeds into a locally finite associative algebra. For a set $X$, we utilize the notation Lie$(X)$ to denote the Lie algebra generated by $X$ and Assoc$(X)$ to denote the associative algebra generated by $X$.

**Lemma 8.** Suppose $R$ is an associative algebra over $F$ with a finite nontrivial $\mathbb{Z}$-grading, $R = \sum_{i=-M}^{M} R_i$, where $F$ is of characteristic 0 or $p > M$. Suppose $a_1, \ldots, a_n \in R$ such that $a_i \in R_{\alpha_i}$ where $\alpha_i \neq 0$ for all $i$. If Lie$(a_1, \ldots, a_n)$ is finite dimensional, then Assoc$(a_1, \ldots, a_n)$ is finite dimensional as well.

**Proof.** We will show that there is an $N \geq 1$ such that any associative product $a_{i_1} \cdots a_{i_k}$ can be written as a sum of products of the form $\rho_1 \cdots \rho_h$, where each $\rho_i \in$ Lie$(a_1, \ldots, a_n)$ and $h \leq N$. From this it follows that if dim Lie$(a_1, \ldots, a_n) = d$, then dim Assoc$(a_1, \ldots, a_n) \leq d^{N+1} < \infty$. 


Suppose \( w = a_{i_1} \cdots a_{i_k} \) is written as an associative product of \( k \) generators. Since \( a_i a_j = [a_i, a_j] + a_j a_i \), we can move generators with negative degrees to the left and generators with positive degrees to the right at the expense of introducing commutators, which result in new elements \( w'_i \), each of which is a product of \( k_i < k \) elements from \( \text{Lie}(a_1, \ldots, a_n) \). Thus \( w \) can be written as \( w = w_- w_0 w_+ + \sum w'_i \).

Suppose \( w_- = b_{k_1} \cdots b_{k_u} \) where \( b_{k_i} \in R_{\alpha_{k_i}} \cap \text{Lie}(a_1, \ldots, a_n) \) and \( -M \leq \alpha_{k_i} < 0 \). Then \( w_- \in R_{\alpha_{k_1} + \cdots + \alpha_{k_u}} \) which becomes 0 once \( \alpha_{k_1} + \cdots + \alpha_{k_u} < -M \). Hence \( u \leq M \).

Similarly, if \( w_+ = c_1 c_2 \cdots c_t \) for \( c_{i_1} \in R_{\alpha_{i_1}} \cap \text{Lie}(a_1, \ldots, a_n) \), then \( t \) is also at most \( M \). It remains to consider \( w_0 \).

Since the generators each have nonzero grading, \( w_0 = [y_1, y'_1] \cdots [y_r, y'_r] \) must be a product of \( r \) commutators, where \( |y_i| = -|y'_i| \). We may write each commutator as a sum of left-normed commutators, so \( w_0 \) is a sum of elements of the form \( [y''_{k_1}, a_{j_1}] \cdots [y''_{k_s}, a_{j_s}] \) for \( y''_{k_i} \in \text{Lie}(a_1, \ldots, a_n) \). Then since \( [y''_{k_1}, a_{j_1}][y''_{k_2}, a_{j_2}] = [[y''_{k_1}, a_{j_1}][y''_{k_2}, a_{j_2}]] + [y''_{k_1}, a_{j_2}][y''_{k_2}, a_{j_1}] \), we may group commutators with the same righthand generator together, again at the expense of introducing new elements, each of which is a product of less than \( r \) elements. Thus without loss of generality we may consider only products of this type: \( [x_1, a] \cdots [x_s, a] \) for some generator \( a \) and \( x_i \in \text{Lie}(a_1, \ldots, a_n) \).

By repeatedly using the fact that \( ad(a) \) is a derivation, one can prove that
\[
[x_1 x_2 \cdots x_s, a, a, \ldots, a] = s! [x_1, a] \cdots [x_s, a] + \sum v_i, \quad \text{where each } v_i \text{ is a product of less than } s \text{ elements. Thus, assuming char } F = 0 \text{ or } p > M,
\]
\[
[x_1, a] \cdots [x_s, a] = \frac{1}{s!} \left( [x_1 x_2 \cdots x_s, a, a, \ldots, a] - \sum v_i \right)
\]

If \( |a| = \alpha \), then \( |x_i| = -\alpha \), and \( |x_1 \cdots x_s| = -s\alpha \). Hence, \( x_1 \cdots x_s = 0 \) if \( -s\alpha < -M \) or \( -s\alpha > M \), i.e. if \( s > M \). Therefore, \( s \) is bounded above by \( M \). So \( w_0 \) can be written as a sum of products of the form \( \rho_1 \cdots \rho_h \), where \( \rho_i \in \text{Lie}(a_1, \ldots, a_n) \) and \( h \leq Mn \), where \( n \) is the number of generators. Thus \( w \) can be written as a sum of products of the form \( \rho_1 \cdots \rho_h \), where \( \rho_i \in \text{Lie}(a_1, \ldots, a_n) \) and \( h \leq N = M(n + 2) \). \( \square \)

2.2. The centroid and central polynomials

For basic results on the centroid, see [16]. Let \( A \) be an algebra over a field \( F \) and let \( \text{End}_F(A) \) denote the algebra of all linear transformations on \( A \).

Let \( \psi_a \) (respectively, \( \phi_a \)) denote left (right) multiplication by \( a \). Let \( M(A) \) be the subalgebra of \( \text{End}_F(A) \) generated by \( \{\psi_a, \phi_a \mid a \in A\} \). We define the centroid \( \Gamma \) of \( A \) to be the centralizer of \( M(A) \) in \( \text{End}_F(A) \). That is, the elements of \( \Gamma \) are linear transformations which commute with all right and left multiplications. If \( A \) is a simple algebra, then the centroid \( \Gamma \) of \( A \) is a field.

In the case that \( L \) is a Lie algebra, we use the notation \( \text{Ad}(L) \) for the multiplication algebra.
Lemma 9. Suppose $L$ is a simple, locally finite Lie algebra over $F$. Suppose $\phi \in \Gamma \cap \text{Ad}(L)$. Then $\phi$ is algebraic over $F$.

Proof. Suppose $\phi = \sum_{i=1}^{l} \alpha_i ad(a_{i,1}) \ldots ad(a_{i,n_i})$ for $a_{i,j} \in L$ and $\alpha_i \in F$. Let $0 \neq b \in L$. Then $L_1 := \text{Lie}(b, a_{i,j})$ is a finite dimensional subalgebra of $L$ and $\phi|_{L_1} \in \text{End}_F(L_1)$. Hence, there is a polynomial $f(t) \in F[t]$ such that $f(\phi) \cdot L_1 = 0$. Note that $f(\phi) \in \Gamma$. Since $\Gamma$ is a field and $L$ is an algebra over $\Gamma$, we see that $f(\phi) = 0$. \square

The following proposition will be used at the end of the proof of Theorem 1.

Proposition 10. Suppose $L$ is a simple, locally finite Lie algebra over $F$ which is finite dimensional over its centroid, $\Gamma$. Then $\Gamma$ is an algebraic field extension of $F$.

Proof. Let $R = \text{End}_F(L) \cong M_d(\Gamma)$, where $d$ is the dimension of $L$ over $\Gamma$. By Formanek and Razmyslov (see [12,24]), there is a central polynomial $c_d(x_1, \ldots, x_n)$ of $M_d(\Gamma)$, that is, $c_d(a_1, \ldots, a_n) \in \Gamma$ for all $a_i \in M_d(\Gamma)$ and $c_d(b_1, \ldots, b_n) \neq 0$ for some $b_1, \ldots, b_n \in M_d(\Gamma)$.

By the Jacobson Density Theorem (see [18]), $R = \text{End}_F(L) = \text{Ad}(L)$, that is, $\text{End}_F(L)$ is generated by elements from $\{ad(a) \mid a \in L\}$. Since $c_d \neq 0$, there must be some $w_1, \ldots, w_n \in \text{Ad}(L)$ such that $c := c_d(w_1, \ldots, w_n) \neq 0$. Note that $c \in \Gamma$.

By Lemma 9, $c$ is algebraic over $F$. Let $\psi$ be an arbitrary element of $\Gamma$. Then $\psi \in \Gamma$ and $\psi \in \text{Ad}(L)$ as well, since if $c = \sum_{i=1}^{l} \alpha_i ad(a_{i,1}) \ldots ad(a_{i,n_i})$, then $\psi = \sum_{i=1}^{l} \alpha_i ad(\psi a_{i,1}) \ldots ad(a_{i,n_i})$. Therefore, by Lemma 9, $\psi$ is algebraic over $F$. Therefore, since $c$ and $\psi$ are algebraic, $\psi$ must be algebraic as well. Thus $\Gamma$ is an algebraic field extension of $F$. \square

As a corollary, if $F$ is algebraically closed, then $L$ must be central.

Suppose $R$ is a simple, locally finite associative algebra over an algebraically closed field $F$. The next lemma proves that if $F$ has zero characteristic, then $Z(R) \cap [R^l, R^l] = (0)$ and so $[R^l, R^l]$ is simple. Suppose $A = R \oplus R^{op}$ with involution $(a,b)^* = (b,a)$. Then $A$ is involution simple and $[K(A, *), K(A, *)] \cong [R^l, R^l]$. This explains why one needs only consider $[K(A, *), K(A, *)]$ for $A$ an involution simple, locally finite associative algebra in [1] when the ground field has zero characteristic.

Lemma 11. Suppose $R$ is a simple, locally finite associative algebra over an algebraically closed field $F$ of characteristic zero. Then $Z(R) \cap [R^l, R^l] = (0)$.

Proof. Suppose $a \in Z(R) \cap [R^l, R^l]$. We wish to show that the ideal in $R$ generated by $a$, denoted $id_R(a)$, is locally nilpotent. We can write $a = \sum_i [a_i, b_i]$ for $a_i, b_i \in R$. Let $R_1$ be the finite dimensional subalgebra generated by $\{a_i, b_i\}$ and suppose $R_2$ is a finite dimensional subalgebra containing $R_1$. Then $a \in Z(R_2) \cap [R_2^l, R_2^l]$. Since every element of $R$ is contained in such a subalgebra, $R_2$, it suffices to show that $id_{R_2}(a)$ is nilpotent.
By Lemma 5, $R_2 = B_2 \oplus N_2$ where $N_2$ is a nilpotent ideal and $B_2 \cong R_2$ is semisimple. Consider $\bar{a} \in R_2$. Then $\bar{a} \in Z(R_2) \cap [R_2^{(-)}, R_2^{(-)}]$ and, in particular, $\bar{a} \in Z([R_2^{(-)}, R_2^{(-)}])$. Since $R_2 \cong B_2$ is semisimple, $R_2 \cong \bigoplus_{i \in I} M_{n_i}(F)$ and $[R_2^{(-)}, R_2^{(-)}] \cong \bigoplus_{i \in I} [M_{n_i}(F)^{(-)}, M_{n_i}(F)^{(-)}]$. Since each Lie algebra $[M_{n_i}(F)^{(-)}, M_{n_i}(F)^{(-)}]$ is simple, $Z([R_2^{(-)}, R_2^{(-)}]) = (0)$. Thus $\bar{a} = 0$ and $a \in N_2$ so $id_{R_2}(a)$ is nilpotent.

This implies that $id_{R}(a)$ is locally nilpotent. By Lemma 4, $a$ cannot be nonzero. Thus $Z(R) \cap [R^{(-)}, R^{(-)}] = (0)$. □

Lastly, the following lemma will be used in the proof of Theorem 1.

**Lemma 12.** Let $R$ be a simple, locally finite associative algebra and let $\alpha \subset R$ be a finite subset. Let $R_{\alpha}$ denote the finite dimensional subalgebra generated by $\alpha$ and let $N_{\alpha}$ be its radical. If $f$ is a polynomial identity such that $f(R_{\alpha}/N_{\alpha}) = (0)$ for every finite subset $\alpha \subset R$, then $f(R) = (0)$ as well.

**Proof.** Let $d$ be the degree of $f$. Suppose to the contrary that $f(a_1, \ldots, a_d) \neq 0$ for some set of $a_i \in R$. Let $I$ be the nonzero ideal generated by $f(a_1, \ldots, a_d)$. Since $R$ is simple, $I = R$.

Choose $x_1, \ldots, x_m \in I$. By the definition of $I$, we can write $x_i = \sum_{j} y_{i,j} f(a_1, \ldots, a_d) z_{i,j}$. Let $\alpha$ be the finite set $\{y_{i,j}, z_{i,j}, a_1, \ldots, a_d\}$. By assumption, $f(R_{\alpha}) \subseteq N_{\alpha}$. Since $N_{\alpha}$ is an ideal of $R_{\alpha}$, we have that $x_i \in N_{\alpha}$ for every $i$. Thus, the subalgebra generated by $x_1, \ldots, x_m$ is nilpotent, which shows that $I$ is locally nilpotent.

Thus $I = R$ is locally nilpotent. But this contradicts Lemma 4. Therefore, we must have that $f(a_1, \ldots, a_d) = 0$ for all $a_i \in R$. □

### 2.3. Jordan elements in a Lie algebra

We adopt the notation used in [11]. A nonzero element $x$ of a Lie algebra $L$ is a Jordan element if $ad(x)^3 = 0$. First, by a theorem of Kostrikin [19], a nonzero element which is ad-nilpotent gives rise to a nonzero Jordan element.

**Lemma 13 (Kostrikin’s Descent Lemma).** Suppose $L$ is a Lie algebra over $F$ of characteristic $p \geq 5$ and let $a$ be a nonzero element of $L$ such that $ad(a)^n = 0$ for $4 \leq n \leq p - 1$.

Then for every $b \in L$,

$$(ad[b,a_{\ldots},a])^{n-1} = 0$$

Hence, if there is some $0 \neq y \in L$ such that $ad(y)^{p-1} = 0$, then $L$ contains a nonzero Jordan element, $x$. We first state some useful identities concerning Jordan elements which appear in [19] (see also [11]). We will utilize the notation $X := ad_x$, where $ad_x(y) = [x, y]$. 


Lemma 14. Let $x$ be a Jordan element in a Lie algebra $L$ and let $a$, $b$ be arbitrary elements of $L$.

i. $ad^2_{x^2(a)} = X^2 A^2 X^2$.

ii. $ad^2_{x}(a)$ is a Jordan element.

We now use a Jordan element to create a Jordan algebra associated to $L$ as is done in [11]. Recall that a Jordan algebra, $J$, is an algebra over a field $F$ with a commutative binary product $\circ$ satisfying the Jordan identity: $x^2 \circ (y \circ x) = (x^2 \circ y) \circ x$ for each $x, y \in J$. This product is in general nonassociative. The triple product in a Jordan algebra is $\{x, y, z\} = (x \circ y) \circ z + x \circ (y \circ z) - y \circ (x \circ z)$ and the $U$-operator is the quadratic map $J \rightarrow \text{End}_F(J)$ given by $x \mapsto U_x$ where $U_x(y) = \{x, y, x\}.$

Lemma 15. Let $x$ be a Jordan element of a Lie algebra $L$. Define a new multiplication on $L$ via $a \cdot b := [[a, x], b]$ and denote this nonassociative algebra by $L^{(x)}$.

- $\ker_L(x) = \{a \in L \mid [x, [x, a]] = 0\}$ is an ideal of $L^{(x)}$.
- $L_x := L^{(x)}/\ker_L(x)$ is a Jordan algebra with $U$-operator:

$$U_x b = ad^2_a ad^2_x b$$

Lemma 16. Suppose $L$ is a locally finite Lie algebra and $x \in L$ is a Jordan element. Then $L_x$ is a locally finite Jordan algebra.

Proof. It suffices to show that the algebra $L^{(x)}$ with multiplication $a \cdot b := [[a, x], b]$ is locally finite. Suppose $a_1, \ldots, a_n$ is a finite subset of $L$. By the definition of $\cdot$, the algebra $\langle a_1, \ldots, a_n \rangle$ is contained in the Lie algebra $\text{Lie}(a_1, \ldots, a_n, x)$ which is finite dimensional. \[\square\]

The remaining results in this section can be found in [9].

Definition 1. A nonzero element $e$ in a Lie algebra is called von Neumann regular if:

- $ad^3_e = 0$, and
- $e \in ad^2_e(L)$.

We say that a pair of elements $(e, f)$ is an idempotent in a Lie algebra if $ad^3_e = ad^3_f = 0$ and $(e, [e, f], f)$ is an $\mathfrak{sl}_2$ triple.

The following lemma appears as Proposition 1.18 in [9].

Lemma 17. Suppose $L$ is a Lie algebra over a field $F$ of characteristic zero or $p > 5$. Suppose $e \in L$ is von Neumann regular.
i. For every \( h \in [e, L] \) such that \([h, e] = 2e\), there exists \( f \in L \) such that \([e, f] = h \) and \((e, f)\) is an idempotent.

ii. Let \((e, f)\) be an idempotent and \( h := [e, f] \). Then \( \text{ad}_h \) is semisimple and its action induces a finite \( \mathbb{Z} \)-grading on \( L \):

\[
L = L_{-2} + L_{-1} + L_0 + L_1 + L_2
\]

2.4. Locally nondegenerate Lie algebras

A Lie algebra is nondegenerate if there are no nonzero elements \( x \in L \) such that \( \text{ad}_x^2 = 0 \). It is well known that if \( R \) is a simple ring, then the Lie algebras \([R^{-1}, R^{-1}] / (Z(R) \cap [R^{-1}, R^{-1}])\) and \([K(R, *), K(R, *)] / (Z(R) \cap [K(R, *), K(R, *)])\) are nondegenerate (see for instance Lemmas 4.2 and 4.9 of [9]).

A Jordan algebra \( J \) is nondegenerate if there are no nonzero elements \( x \in J \) such that \( U_x(J) = \{x, J, x\} = (0) \). Nondegeneracy in Jordan algebras is closely related to the notion of an m-sequence. An m-sequence in a Jordan algebra \( J \) is a sequence \( \{a_n\} \) such that \( a_{n+1} = U_{a_n}(b_n) \) for some \( b_n \in J \). We say an m-sequence has length \( k \) if \( a_k \neq 0 \) and \( a_{k+1} = 0 \), and we say an m-sequence terminates if it has finite length. M-sequences characterize the elements of the McCrimmon radical, \( M(J) \), which is defined to be the smallest ideal of \( J \) inducing a nondegenerate quotient. An element \( x \in J \) is in the McCrimmon radical \( M(J) \) if and only if any m-sequence which begins with \( x \) terminates. Thus, a Jordan algebra \( J \) is nondegenerate if and only if there are no nonzero elements \( x \) such that every m-sequence beginning with \( x \) terminates.

There is also a notion of m-sequence in Lie algebras which is defined in [13]: an m-sequence in a Lie algebra \( L \) is a sequence \( \{a_n\} \) such that \( a_{n+1} = [a_n, [a_n, b_n]] \) for some \( b_n \in L \). Recall that for a Lie algebra \( L \), the Kostrikin radical \( K(L) \) is the smallest ideal of \( L \) inducing a nondegenerate quotient [28]. If any m-sequence beginning with an element \( x \) terminates, then \( x \in K(L) \). (The other direction is currently unknown and is the subject of [13].)

We now define an S-sequence in a Lie algebra. For any finite set \( S \) in a Lie algebra \( L \), an S-sequence is a sequence \( \{x_n\} \) in \( L \) such that for each \( n \), \( x_{n+1} = [x_n, [x_n, s_n]] \) for some \( s_n \in S \). Similarly, for a finite set \( S \) of a Jordan algebra \( J \), an S-sequence is a sequence \( \{x_n\} \) in \( J \) such that for each \( n \), \( x_{n+1} = U_{x_n}(s_n) \) for some \( s_n \in S \). We say an S-sequence has length \( k \) if \( x_k \neq 0 \) and \( x_{k+1} = 0 \), and we say an S-sequence terminates if it has finite length.

A Lie algebra \( L \) is locally nondegenerate if there are no nonzero elements \( x \) such that for any finite set \( S \), every S-sequence beginning with \( x \) terminates. We define the local Kostrikin radical, \( K_{loc}(L) \), to be the smallest ideal of \( L \) which induces a locally nondegenerate quotient. The following lemma is straightforward.

**Lemma 18.** If \( L \) is a locally nondegenerate Lie algebra, then \( L \) is nondegenerate.
The following proposition shows that the condition of being locally nondegenerate is trivial when the ground field has characteristic zero.

**Proposition 19.** Let $L$ be a simple, locally finite Lie algebra over a field of characteristic zero. Then $L$ is locally nondegenerate.

**Proof.** Suppose there is some element nonzero $a \in L$ such that for any finite set $S$, any $S$ sequence beginning with $a$ terminates. We will show that the ideal generated by $a$, denoted $id_L(a)$, is nonzero and locally solvable, which will contradict Lemma 4 and complete the proof.

Suppose $L_1$ is a finite dimensional subalgebra of $L$ containing $a$. It suffices to show that $id_{L_1}(a)$ is solvable. Let $L_1 = L_1/Rad(L_1)$. Then $\bar{a}$ is an element of $L_1$ such that every $S$-sequence beginning with $\bar{a}$ terminates. Therefore, without loss of generality, we may assume $L_1$ is semisimple.

Thus $L_1$ is a finite dimensional, semisimple Lie algebra over a field of characteristic zero. Thus $L_1$ is nondegenerate, i.e. there are no nonzero elements $x \in L_1$ such that $ad^2_{x_1}(L_1) = 0$. Let $S$ be a basis for $L_1$. Since $ad^2_a(L_1) \neq (0)$, there is some element $s_1 \in S$ such that $x_1 = ad^2_{s_1}(s_1) \neq 0$. Similarly, $ad^2_{x_1}(L_1) \neq (0)$, so there is some element $s_2 \in S$ such that $x_2 = ad^2_{x_1}(s_2) \neq 0$. Continuing in this way, we can create an $S$-sequence $\{x_n\}$ which does not terminate for a contradiction. This completes the proof. \[\square\]

The proof of the previous lemma also shows that if $L$ is finite dimensional, then $L$ is nondegenerate if and only if $L$ is locally nondegenerate. The next proposition shows that if $R$ is a simple, locally finite associative algebra, then its associated Lie algebras $[R^{(-)}, R^{(-)}]/(Z(R) \cap [R^{(-)}, R^{(-)}])$ and $[K(R, *), K(R, *)]/(Z(R) \cap [K(R, *), K(R, *)])$ are locally nondegenerate. We first prove a lemma.

**Lemma 20.** Suppose $B$ is a semisimple, finite dimensional associative algebra with involution $\ast$. Then $K_{\text{loc}}([K(B, *), K(B, *)]) \subseteq Z([K(B, *), K(B, *)])$.

**Proof.** Suppose $a$ is a nonzero element of $[K(B, *), K(B, *)]$ such that every $S$-sequence beginning with $a$ terminates. By Lemma 6,

$$[K(B, *), K(B, *)] \cong \bigoplus_{i \in \Omega_0} K\{M_{n_i}(F), *\} \bigoplus_{i \in \Omega_1} [M_{n_i}(F)^{(-)}, M_{n_i}(F)^{(-)}]$$

Each summand $K(M_{n_i}(F), *)$ is a simple classical Lie algebra, hence nondegenerate and therefore locally nondegenerate. Therefore, we must have that $a \in \bigoplus_{i \in \Omega_1} [M_{n_i}(F), M_{n_i}(F)]$. Since $[M_{n_i}(F), M_{n_i}(F)]/Z([M_{n_i}(F), M_{n_i}(F)])$ is nondegenerate, this implies that $a \in Z([K(B, *), K(B, *)])$. \[\square\]

Note that if $R$ is a simple, locally finite associative algebra with involution $\ast$ over an algebraically closed field $F$, then $Z(R)$ is either $(0)$ (if $R$ is not unital) or the ground field $F$ (if $R$ is unital). In either case, $Z(R) \cap K(R, *) = (0)$. 


Proposition 21. Suppose $R$ is a simple, locally finite associative algebra with involution $\ast$ which is infinite dimensional over an algebraically closed field $F$. Then the Lie algebras $[K(R, \ast), K(R, \ast)]$ and $[R^(-), R^(-)]/(Z(R) \cap [R^(-), R^(-)])$ are locally nondegenerate.

**Proof.** We provide a proof when $L = [K(R, \ast), K(R, \ast)]$ and then leave the other case to the reader. By Theorem 10 from [14], $L$ is a simple Lie algebra. Suppose $a$ is a nonzero element of $L$ such that for any finite subset $S$, every $S$-sequence beginning with $a$ terminates. We wish to show that the ideal generated by $a$ is locally solvable, which will contradict Lemma 4.

Choose $R_1$ to be a finite dimensional subalgebra of $R$ invariant under $\ast$ such that $a \in [K(R_1, \ast), K(R_1, \ast)]$. By Lemma 5, $R_1 = B_1 \oplus N_1$, where $N_1$ is the radical of $R_1$, and $B_1 \cong R_1/N_1$ is a semisimple subalgebra invariant under $\ast$. Set $a = b + c$, where $b \in B_1$ and $c \in N_1$, and let $S$ be a finite subset of $[K(B_1, \ast), K(B_1, \ast)]$. For any $S$-sequence $\{b_n\}$ of $[K(B_1, \ast), K(B_1, \ast)]$ beginning with $b$, $b_n = a_n - c_n$, where $\{a_n\}$ is an $S$-sequence beginning with $a$ and $\{c_n\} \subset N_1$. Since every $S$-sequence beginning with $a$ terminates and $R_1 = B_1 \oplus N_1$, the $S$-sequence $\{b_n\}$ terminates as well. Thus, by Lemma 20, $b \in Z([K(B_1, \ast), K(B_1, \ast)])$.

Let $I_1$ denote the ideal of $[K(R_1, \ast), K(R_1, \ast)]$ generated by $a = b + c$. It is clear that $I_1 \subset Fb + N_1$ and hence that $[I_1, I_1] \subset N_1$, which shows that $I_1$ is solvable. This implies that the ideal generated by $a$ in $L = [K(R, \ast), K(R, \ast)]$ is locally solvable, hence $K_{loc}(L) = (0)$. □

The following proposition will be used in the proof of Theorem 1.

Proposition 22. Let $x$ be a Jordan element in a Lie algebra $L$. If the Jordan algebra $L_x$ is locally nilpotent, then $x \in K_{loc}(L)$.

**Proof.** Fix $S = \{s_1, \ldots, s_p\}$, a finite set from $L$. Then $\bar{S}$ denotes the finite set $\{\bar{s}_1, \ldots, \bar{s}_p\}$ of $L_x$. Suppose $\{x_n\}$ is an $S$-sequence which begins with $x$. Thus $x_0 = x$ and for each $n \geq 1$, $x_n = [x_{n-1}, [x_{n-1}, s_n]]$ for some $s_n \in S$.

We prove by induction that for every $n \geq 1$, there is some $c_n \in \text{Lie}(S \cup \{x\})$ such that $x_n = [x, [x, c_n]]$. For $n = 1$, set $c_1 = s_1$. Suppose the result holds for some $n$, that is, $x_n = ad_x^2(c_n)$. Then $x_{n+1} = [x_n, [x_n, s_{n+1}]] = ad_x^2x(c_n)s_{n+1} = ad_x ad_x c_n ad_x(s_{n+1})$ by Lemma 14. Setting $c_{n+1} = ad_x c_n ad_x(s_{n+1})$ finishes the argument.

Now consider the sequence $\{\bar{c}_n\}$ in $L_x$. We have that $U_{\bar{c}_n} \bar{s}_{n+1} = \bar{c}_{n+1}$, thus every element of the sequence is contained in the subalgebra of $L_x$ generated by $\bar{S}$, which is finite dimensional and nilpotent since we assumed $L_x$ is locally nilpotent. Thus, the length of $\{\bar{c}_n\}$ must be finite and the sequence must terminate. Thus $\bar{c}_N = \bar{0}$ for some $N$ and $x_n = [x, [x, c_n]] = 0$. Therefore, any $S$ sequence beginning with $x$ terminates, and we conclude that $x \in K_{loc}(L)$. □
3. Proof of Theorem 1

This section is devoted to the proof of the following theorem.

**Theorem.** Let $L$ be a simple, infinite dimensional, locally finite Lie algebra over an algebraically closed field $F$ of characteristic $p > 7$ or characteristic zero. Then the following conditions are equivalent:

1. $L$ is locally nondegenerate and there is some nonzero element $x \in L$ such that $ad(x)^{p-1} = 0$.
2. $L \cong [R(-), R(-)]/(Z(R) \cap [R(-), R(-)])$ where $R$ is a locally finite, simple associative algebra or $L \cong [K(R, *), K(R, *)]$ where $R$ is a locally finite, simple associative algebra with involution $*$.

First we prove the backward direction, (2) implies (1).

Suppose $R$ is a simple, locally finite associative algebra and $L \cong [R(-), R(-)]/(Z(R) \cap [R(-), R(-)])$. By Theorem 4 of [14], $L$ is locally finite and simple. By Proposition 21, $L$ is locally nondegenerate. It remains to show that $L$ contains an ad-nilpotent element.

Let $\alpha$ be a finite subset of $R$ and let $R_\alpha$ denote the finite dimensional, associative algebra generated by $\alpha$. By Lemma 5, $R_\alpha = B_\alpha \oplus N_\alpha$ where $B_\alpha$ is semisimple and $N_\alpha$ is nilpotent. Since $F$ is algebraically closed, $B_\alpha$ is a direct sum of matrix rings $B_\alpha = \bigoplus M_{n_i}(F)$.

If $n_i < 2$ for every $i$ and $\alpha$, then $B_\alpha$ is commutative for every $\alpha$ which implies $R$ is commutative by Lemma 12 and $L = (0)$. Hence, we must have that $n_i \geq 2$ for some $\alpha \subset R$. Then $[R(-), R(-)] \supset [M_n(F)^(-), M_n(F)^(-)] \cong \mathfrak{sl}_n(F)$ for $n \geq 2$.

If $n = 2k$ is even, we set $a := \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$ where $I$ denotes the $k \times k$ identity matrix. If $n = 2k + 1$ is odd, we set $a := \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, where $I$ denotes the $k \times k$ identity matrix. In either case, $a$ is a nonzero element of $R$ such that $a^2 = 0$, implying that $ad(a)^3 = 0$ in $[R(-), R(-)]$. Clearly $a$ is not in the center of $R$, thus we have that $ad(a)^3 = 0$ in $L$.

Now suppose that $L \cong [K(R, *), K(R, *)]$. Since $F$ is algebraically closed, $Z(R) \cap [K(R, *), K(R, *)] = (0)$. Hence, $L$ is locally finite and simple by Theorem 10 from [14]. Furthermore, $L$ must be locally nondegenerate by Proposition 21.

The last step is to produce an ad-nilpotent element. As before, let $R_\alpha$ denote the finite dimensional, associative algebra generated by $\alpha$, a finite subset of $R$. Without loss of generality we may further assume that $R_\alpha$ is invariant under the involution $*$. By Lemma 5, $R_\alpha = B_\alpha \oplus N_\alpha$ where $B_\alpha$ is semisimple and $*$-invariant. As before, we conclude that there is some finite subset $\alpha$ such that $B_\alpha$ is isomorphic to a direct sum of matrix rings $B_\alpha = \bigoplus M_{n_i}(F)$ where $n_i \geq 2$ for some $i$.

By Lemma 6, $K(B_\alpha, *) \cong \bigoplus_{i \in \Omega_0} K(M_{n_i}(F), *) \bigoplus_{i \in \Omega_1} M_{n_i}(F)^(-)$. Therefore, $[K(B_\alpha, *), K(B_\alpha, *)]$ must contain a subalgebra isomorphic to $\mathfrak{o}_n(F)$, $\mathfrak{sp}_n(F)$, or $\mathfrak{sl}_n(F)$.
for some \( n \geq 2 \). In each case we can find a nonzero element \( a \) such that \( a^2 = 0 \), implying that \( ad(a)^3 = 0 \) in \( L \).

This was already done for \( \text{sl}_n(F) \). For \( \text{sp}_n(F) \) for \( n = 2k \), we use the realization of \( \text{sp}_n(F) \) as those matrices which preserve the skew-symmetric form given by \( \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix} \) where \( I_k \) denotes the \( k \times k \) identity matrix. Then \( a := \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \), where \( B \) is any nonzero \( k \times k \) symmetric matrix, will be an element of \( \text{sp}_n(F) \) whose square is zero.

Similarly, we realize \( \mathfrak{o}_n(F) \) as those matrices which preserve the symmetric form given by \( \begin{pmatrix} 0 & I_k \\ I_k & 0 \end{pmatrix} \) if \( n = 2k \) is even or \( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_k \\ 0 & I_k & 0 \end{pmatrix} \) if \( n = 2k + 1 \) is odd. We set \( a := \begin{pmatrix} 0 & C \\ 0 & 0 & 0 \end{pmatrix} \) if \( n = 2k \) is even or \( a := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & C \\ 0 & 0 & 0 \end{pmatrix} \) if \( n = 2k + 1 \) is odd, where \( C \) is any nonzero \( k \times k \) skew-symmetric matrix.

Now, we prove the forwards direction, (1) implies (2).

Suppose \( L \) is a simple, locally finite Lie algebra over \( F \) which is locally nondegenerate and contains some nonzero element \( x \in L \) such that \( ad(x)^{p-1} = 0 \). By the results of Section 2.3, we may assume \( x \) is a Jordan element, that is, \( ad(x)^3 = 0 \), and the Jordan algebra \( J = L_x \) is locally finite.

Our goal is to prove that \( J \) must contain a nonzero idempotent. If we suppose to the contrary that \( J \) does not contain a nonzero idempotent, then by Lemma 1 of 3.7 in [17], every \( a \in J \) must be nilpotent, so \( J \) is nil. The following is a corollary of a theorem of Albert and can be found in [17]: A locally finite, nil Jordan algebra is locally nilpotent.

Therefore, \( J = L_x \) is a locally nilpotent Jordan algebra. By Lemma 22, \( x \) must be contained in the local Kostrikin radical, \( K_{loc}(L) \). But this is a contradiction since we have assumed that \( L \) is locally nondegenerate. Therefore, \( J = L_x \) must contain a nonzero idempotent, \( \bar{e}^2 = \bar{e} \).

Since \( \bar{e}^2 = \bar{e} \), we have

\[
U_\bar{e}\bar{e} = (\bar{e} \bullet \bar{e}) \bullet \bar{e} + \bar{e} \bullet (\bar{e} \bullet \bar{e}) - \bar{e} \bullet (\bar{e} \bullet \bar{e}) = \bar{e}
\]

Hence by (i) of Lemma 14,

\[
X^2(e) = X^2E^2X^2(e) = ad^2_{X^2(e)}(e)
\]

which proves along with (ii) of Lemma 14 that \( e' := X^2(e) = ad^2_x(e) \) is von Neumann regular. Note also that \( e' \neq 0 \) since \( \bar{e} \neq 0 \).

By the results of Section 2.3, we obtain a finite \( \mathbb{Z} \)-grading on \( L \) via the action of \( ad(h) \) for some \( h \in [e', L] \):

\[
L = L_{-2} + L_{-1} + L_0 + L_1 + L_2
\]

Therefore, \( L \) is a simple, finite \( \mathbb{Z} \)-graded Lie algebra over a field of characteristic \( p > 7 \) (or of characteristic zero). Hence, we can utilize Theorem 1 from [29]: Since \( L = \sum_{i=-2}^2 L_i \) is a simple graded Lie algebra over a field of characteristic at least 7 (or of characteristic zero) and \( \sum_{i \neq 0} L_i \neq 0 \), we have that \( L \) is isomorphic to one of the following algebras:
I. \([R^{(-)}, R^{(-)}]/Z\), where \(R = \sum_{i=-2}^{2} R_i\) is a simple associative \(\mathbb{Z}\)-graded algebra.

II. \([K(R, *), K(R, *)]/Z\), where \(R = \sum_{i=-2}^{2} R_i\) is a simple associative \(\mathbb{Z}\)-graded algebra with involution \(* : R \to R\).

III. The Tits–Kantor–Koecher construction of the Jordan algebra of a symmetric bilinear form.

IV. An algebra of one of the types \(G_2, F_4, E_6, E_7, E_8\) or \(D_4\).

In cases I and II, it remains to show that the simple, graded associative algebra \(R\) is locally finite. Suppose we are in case I, that is, \(L \cong [R^{(-)}, R^{(-)}]/(Z(R) \cap [R^{(-)}, R^{(-)}])\). Since \(Z(R) \cap [R^{(-)}, R^{(-)}]\) is an ideal which is locally finite and central, \([R^{(-)}, R^{(-)}]\) is a locally finite Lie algebra. Choose a finite set \(a_1, \ldots, a_n\) from \(R\). Since \(R\) is generated by \(\bigcup_{i \neq 0} R_i\) by Lemma 7, we may assume that \(a_1, \ldots, a_n\) are homogeneous of nonzero degree. Note that for \(i \neq 0\), \(R_i \subseteq [R^{(-)}, R^{(-)}]\) since for any \(a \in R_i\), \([h, a] = ia\). Then since \([R^{(-)}, R^{(-)}]\) is locally finite, \(\text{Lie}(a_1, \ldots, a_n)\) is finite dimensional, and so \(\text{Assoc}(a_1, \ldots, a_n)\) is finite dimensional by Lemma 8.

Similarly, suppose we are in case II, that is, \(L \cong [K(R, *), K(R, *)]\). Again, we may choose a finite set \(a_1, \ldots, a_n\) from \(R\). By Theorem 2.13 of [15], \(L\) generates \(R\) as an associative algebra. Therefore, \(\text{Assoc}(a_1, \ldots, a_n) \subseteq \text{Assoc}(\{x_i\})\) for some finite set \(\{x_i\} \subset L\). By Lemma 7, we can assume that the \(x_i\) each have nonzero degree. Hence, since \(\text{Lie}(\{x_i\})\) is finite dimensional, we can apply Lemma 8 to conclude that \(\text{Assoc}(\{x_i\})\), and in turn \(\text{Assoc}(a_1, \ldots, a_n)\), is finite dimensional.

In case III, \(L = TKK(J)\), where \(J\) is the Jordan algebra defined by a nondegenerate symmetric bilinear form on a vector space of dimension (possibly infinite) greater than 2 over a field \(F\) of characteristic other than 2. We have by 5.11 of [10] that \(L\) is the finitary orthogonal Lie algebra \(\mathfrak{so}(X, q)\), where \(q\) is a nondegenerate bilinear form of a vector space \(X\) over \(F\). Then \(L = K(R, \star)\), where \(R = \mathcal{F}_X(X)\) is the associative algebra of all finite rank linear transformations having an adjoint with respect to \(q\), and \(\star\) is the adjoint involution. The associative algebra \(\mathcal{F}_X(X)\) is simple and locally finite.

In case IV, the Lie algebra \(L\) is finite dimensional over its centroid. By the results of Section 2.2, the centroid of \(L\) is an algebraic field extension. Since \(F\) is algebraically closed, the centroid must be \(F\) itself, in which case \(L\) is a finite dimensional Lie algebra. Hence, since we have assumed that \(L\) is infinite dimensional, case IV can be eliminated.

4. Proof of Theorem 2

In this section we provide the proof of Theorem 2. We begin with some preliminary results. The next lemma will be used several times in the proof of Theorem 2.

**Lemma 23.** Suppose \(L\) is a Lie algebra over a field \(F\) of characteristic \(p\) (or zero) which is generated by a set \(X\). Suppose for all \(x \in X\), \(\text{ad}(x)^dL = (0)\) for some \(d > p\) and \(\exp(\xi \text{ad}(x))\) is a well-defined automorphism of \(L\) for every \(\xi \in F\). If \(V\) is a subspace of \(L\) which is invariant under \(\text{Aut}(L)\), then \(V\) is an ideal of \(L\).
Proof. Fix $x \in X$ and $v \in V$. Since $L$ is generated by $X$, it suffices to show that $[x, v] \in V$. Choose $\xi_1, \ldots, \xi_d$ distinct nonzero elements from $F$. Since $ad(x)^d = 0$ and $exp(\xi_i ad(x)) \in \text{Aut}(L)$, we set $v_i = exp(\xi_i ad(x)) v \in V$ for $i = 1, \ldots, d$. Consider the following system of $d$ linear equations with $d$ unknowns: $v, ad_x v, \frac{1}{2!} ad_x^2 v, \ldots, \frac{1}{(d-1)!} ad_x^{d-1} v$.

\[
v + \xi_1 ad_x v + \xi_1^2 \frac{ad_x^2 v}{2!} + \cdots + \xi_1^{d-1} \frac{ad_x^{d-1} v}{(d-1)!} = v_1
\]

\[
\vdots
\]

\[
v + \xi_d ad_x v + \xi_d^2 \frac{ad_x^2 v}{2!} + \cdots + \xi_d^{d-1} \frac{ad_x^{d-1} v}{(d-1)!} = v_d
\]

The matrix for this linear system is the Vandermonde matrix whose determinant is nonzero since $\xi_i \neq \xi_j$ for $i \neq j$. Therefore the system is invertible, and we conclude that each of the unknowns is an element of $V$, including $ad_x(v) = [x, v]$. \qed

We can now use the previous lemma to prove the following.


Proof. Suppose there is some nonzero element $x \in L$ such that $ad_x^2 L = 0$. It follows from Lemma 23 that $L = \text{span}\{ x \in L \mid ad_x^2 L = 0 \}$. Hence by Theorem 1 of [27], $L$ is locally nilpotent, which is a contradiction. \qed

Recall that $L$ is ad-integrable if for each element $a \in L$, $ad(a)$ is a root of some nonzero polynomial $f_a(t) \in F[t]$. It follows that for each $a \in L$, $ad(a)$ has finitely many eigenvalues. The motivation for the study of such algebras follows from the fact that a simple, locally finite Lie algebra embeds into a locally finite associative algebra if and only if it is ad-integrable.

Suppose $L$ is simple and ad-integrable. For $a \in L$, let $\mu_a(t)$ be the minimal polynomial of $ad(a)$. An element $a \in L$ is ad-nilpotent if $ad(a)^N L = 0$ for some $N$, in which case the only root of $\mu_a(t)$ is 0. Define $d(L) = \min \{ \deg \mu_a(t) \mid a$ is not ad-nilpotent$\}$. Since $L$ is simple, there is some element $a \in L$ which is not ad-nilpotent. Otherwise, by Engel’s Theorem $L$ would be locally nilpotent, which contradicts Lemma 4. Therefore, $d(L)$ is well-defined.

We now prove Theorem 2.

Theorem. Let $L$ be a simple, locally finite Lie algebra which is ad-integrable over an algebraically closed field $F$ of characteristic $p > 3$. If $p > 2d(L) - 2$, then $L \cong [R(-), R(-)]/(Z(R) \cap [R(-), R(-)])$ where $R$ is a locally finite, simple associative algebra or $L \cong [K(R, *), K(R, *)]$ where $R$ is a locally finite, simple associative algebra with involution $*$. 
Proof. Choose \( a \in L \) so that \( \deg \mu_a(t) = d(L) = d \). Suppose \( L = \sum_{\alpha \in \Phi} L_\alpha \) is the decomposition of \( L \) into root spaces with respect to \( a \). We see that \( |\Phi| \leq d(L) \). Set \( \Delta = \{ 0 \neq \alpha \in \Phi \mid L_\alpha \neq (0) \} \), which is nontrivial since \( a \) is not ad-nilpotent. By Lemma 7, \( L \) is generated by \( \{ L_\alpha \mid \alpha \in \Delta \} \).

Choose \( \alpha \in \Delta, \beta \in \Phi, x \in L_\alpha \), and \( y \in L_\beta \). Since \( p > 2d - 2 \), \( p > d \) as well. Thus \( \beta, \beta + \alpha, \ldots, \beta + d\alpha \) are all distinct roots. Since \( |\Delta| \leq d \), the sets \( L_\beta, ad(x_\alpha)L_\beta, ad(x_\alpha)^2L_\beta, \ldots, ad(x_\alpha)^dL_\beta \) cannot all be nonzero, which implies that \( ad(x_\alpha)^dL_\beta = (0) \). Therefore, for every \( x \in L_\alpha \), for \( \alpha \in \Delta \), \( ad(x)^dL = (0) \), and thus \( \exp(ad(x)) \) is a well-defined automorphism of \( L \) for any \( \xi \in F \).

We have shown that for any \( \alpha \in \Delta \) and \( x \in L_\alpha \), \( ad(x)^dL = 0 \). Thus by Lemma 13, there is some \( 0 \neq y \in L \) such that \( ad(y)^3L = 0 \), that is, \( L \) contains a Jordan element. Thus \( V = \text{span}\{ x \in L \mid ad(x)^3L = 0 \} \) is a nonzero subspace of \( L \). Since \( V \) is invariant under \( \text{Aut}(L) \), we can apply Lemma 23 with \( X = \{ L_\alpha \mid \alpha \in \Delta \} \) to show that \( V \) is a nonzero ideal of \( L \). Thus \( V = L \) and \( L \) is spanned, hence generated, by Jordan elements.

Suppose \( 0 \neq x \in L \) is a Jordan element. Then if \( J = L_x \) is not locally nilpotent, then \( J \) contains a nonzero idempotent and we proceed as in the proof of Theorem 1. Suppose to the contrary that for every Jordan element \( x \in L \), the Jordan algebra \( L_x \) is locally nilpotent.

Suppose \( L_1 = \text{Lie}(a_1, \ldots, a_m) \) is a finite dimensional subalgebra generated by Jordan elements \( a_1, \ldots, a_m \) which is nonsolvable. Such an algebra must exist by Lemma 4. Furthermore, suppose \( L_1 \) is such an algebra of minimal dimension.

We will prove that \( [L_1, L_1] = L_1 \). Suppose not, that is, \( [L_1, L_1] \neq L_1 \). Let \( V = \text{span}\{ x \in [L_1, L_1] \mid ad(x)^3L = 0 \} \). Since \( V \) is invariant under \( \text{Aut}(L_1) \), \( V \) is an ideal of \( L_1 \), hence \( V \) is a subalgebra of \( L_1 \) generated by Jordan elements. By minimality of \( L_1 \), \( V \) must be solvable. Thus we have shown that all Jordan elements from \( [L_1, L_1] \) lie in the radical of \( L_1 \).

Let \( R \) denote the solvable radical of \( L_1 \) and \( \overline{L_1} = L_1/R \). For each Jordan element \( a \) and for each \( b \in L_1 \), \( ad^2_a(b) \in [L_1, L_1] \) is a Jordan element by Lemma 14. Thus, \( ad^2_a \overline{L_1} = (0) \), so \( \overline{L_1} \) is a finite dimensional Lie algebra generated by sandwich elements, that is, by elements \( a \in \overline{L_1} \) such that \( ad(a)^2\overline{L_1} = (0) \). By [21], \( \overline{L_1} \) is nilpotent, thus \( L_1 \) is solvable, which is a contradiction.

We have proved that \( L_1 = [L_1, L_1] \). Choose a maximal ideal \( I \triangleleft L_1 \). Then \( L'_1 = L_1/I \) has no proper ideals. Since \( L_1 = [L_1, L_1], L'_1 \) is not abelian. Therefore, \( L'_1 \) is a simple finite dimensional Lie algebra. Since \( L'_1 \) is generated by elements \( a \) satisfying \( ad(a)^3(L'_1) = (0) \), by Lemma 24 \( L'_1 \) must be nilpotent.

Recall that for each \( a \in L_1 \), we have assumed that the Jordan algebra \( L_a \) is locally nilpotent. Thus \( (L_1)_a \) and \( (L'_1)_a \) are nilpotent. Fix \( i \) such that \( 0 \neq \overline{a_i} \in L'_1 \), which is possible since \( I \neq L_1 \). Since \( (L'_1)_a \) is nilpotent, by Proposition 22, \( 0 \neq \overline{a_i} \) is in \( K_{loc}(L'_1) \), hence \( L'_1 \) cannot be nondegenerate. This contradiction completes the proof. \( \square \)

We remark that the reason we impose the condition \( p > 2d(L) - 2 \) is to ensure that \( \exp(ad(x)) \) is a well-defined automorphism of \( L \) when \( ad(x)^d = 0 \). When the ground
field has zero characteristic, this is always true. Thus, the proofs of Theorems 1 and 2 give a new proof of the main result of [1]: Suppose $F$ is an algebraically closed field of characteristic zero and suppose $L$ is a simple, locally finite Lie algebra over $F$ which embeds into a locally finite associative algebra. Then $L$ is ad-integrable, and the proof of Theorem 2 shows that $L$ contains an ad-nilpotent element. Then the proof of Theorem 1 implies that $L \cong [R(-), R(-)]/(Z(R) \cap [R(-), R(-)])$ where $R$ is a locally finite, simple associative algebra or $L \cong [K(R, *), K(R, *)]$, where $R$ is a locally finite, simple associative algebra with involution *. In fact, both of these conclusions imply that $L \cong [K(R, *), K(R, *)]/(Z(R) \cap [K(R, *), K(R, *)])$ for a locally finite, involution simple associative algebra $R$. Since $(Z(R) \cap [K(R, *), K(R, *)]) = (0)$ when the characteristic of the ground field is zero, we obtain the result of Bahturin, Baranov, and Zalesskii.

5. Proof of Theorem 3

The purpose of this section is to address the following question: If $R$ is a simple, locally finite associative algebra with involution, is it true that $K(R, *) = [K(R, *), K(R, *)]$? Certainly if $R$ is locally simple, that is, $R = \bigcup_i R_i$ where each $R_i$ is simple, then $K(R, *) = [K(R, *), K(R, *)]$, as when $R = M_\infty(F)$. In this section we show this does not hold in general for a locally finite associative algebra $R$. Our construction works over an algebraically closed field of arbitrary characteristic.

**Theorem.** There exists a simple, locally finite associative algebra $R$ with involution * so that $K(R, *) \neq [K(R, *), K(R, *)]$.

**Proof.** Let $F$ be an algebraically closed field, and let $p$ be an odd prime, $p = 2k + 1$. Let $n_i = p^i$ for $i \geq 1$ and $R_i = M_{n_i}(F) \oplus M_{n_i}(F)$ with involution * given by $(A, B)^* = (B^T, A^T)$. Let $\phi_i : R_i \hookrightarrow R_{i+1}$ be an embedding of signature $(k + 1, k, 0)$, that is,

$$\phi_i(M, N) = \left(\text{diag}\left(M, \ldots, M, N, \ldots, N\right), \text{diag}\left(N, \ldots, N, M, \ldots, M\right)\right)_{k+1 \times k}$$

It is easy to check that each $\phi_i$ is a map of involutive algebras. Define $R$ as the direct limit of the $R_i$ under each embedding, so $R = \varinjlim_{i} R_i$. Then $R$ is a locally finite, locally semisimple involutive algebra.

We will show that $R$ is simple. We identify $R_i$ with its image in $R_{i+1}$. Suppose $I$ is a nonzero ideal of $R$ and $I_i = R_i \cap I$, so $I_i = I_{i+1} \cap R_i$ for all $i$. Since $I \neq (0)$, there is some $0 \neq (M, N) \in I_l$ for some $l$. Then $\phi_k(M, N) = (M', N')$ where $M' \neq 0$ and $N' \neq 0$. Since $M_{n_{l+1}}(F)$ is a simple ring, $id_{R_{l+1}}(M', N') = id_{R_{l+1}}(M', 0) + id_{R_{l+1}}(0, N') = M_{n_{l+1}}(F) \oplus (0) + (0) \oplus M_{n_{l+1}}(F) = R_{l+1}$. This shows that $R_{l+1} = I_{l+1}$ so $I_l = R_{l+1} \cap R_l = R_{l+1} \cap R_l$ for all $i \leq l$. The above argument also demonstrates that $R_i = I_i$ for all $i \geq l$ as well, thus $I = R$. Therefore, $R$ must be simple.

We now show that $K(R, *) \neq [K(R, *), K(R, *)]$. Note that $K(R_i, *) = \{(M, -M^T)\} \cong M_{n_i}(F)^{-}$. Choose $(M, -M^T) \in K(R_i, *)$ such that the trace of $M$ is nonzero. Then
\[ \phi_i(M, -M^T) = (M_1, -M_1^T) \text{ and } \text{Tr}(M_1) = (k + 1 - k) \text{Tr}(M) = \text{Tr}(M) \neq 0. \] Therefore in any subalgebra of \( R \), \( (M, -M^T) \) is always mapped to an element \( (M', -M'^T) \) such that \( M' \) has nonzero trace.

Suppose \((M, -M^T) \in [K(R, *), K(R, *)]\), so we have that

\[
(M, -M^T) = \sum_j [(a_j, -a_j^T), (b_j, -b_j^T)] = \left( \sum_j [a_j, b_j], \sum_j [-a_j^T, -b_j^T] \right)
\]

in some subalgebra \( R_t \), where \((M, -M^T)\) is identified with its image in \( R_t \). But \( M \) has nonzero trace, whereas \( \sum_j [a_j, b_j] \) must be of trace zero. This is a contradiction, therefore \((M, -M^T) \) cannot be written as a sum of commutators. Therefore, \( K(R, *) \neq [K(R, *), K(R, *)] \).

Lastly, we note that the algebra \( R \) we constructed in the previous proof has symmetry type “two-sided weakly non-symmetric” according to [4]. Therefore, by Theorem 5.2 of [4], \( R \) cannot be isomorphic to a direct limit of involution simple algebras of orthogonal or symplectic type, so \( R \) is not locally simple. This is consistent with our result that \( K(R, *) \neq [K(R, *), K(R, *)] \).

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References


