MATH 102: PROBLEM SET 2

DUE AT 16:00 ON FRIDAY, OCTOBER 13

(1) Let $V$ be a two-dimensional vector space, let \{e_1, e_2\} be a basis in $V$, let

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$$

be a $2 \times 2$ matrix, and define a function

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$$

by

$$\langle \xi_1 e_1 + \xi_2 e_2, \eta_1 e_1 + \eta_2 e_2 \rangle := \alpha_{11} \xi_1 \eta_1 + \alpha_{12} \xi_1 \eta_2 + \alpha_{21} \xi_2 \eta_1 + \alpha_{22} \xi_2 \eta_2.$$

(a) Show that $\langle \cdot, \cdot \rangle$ is an inner product if $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

(b) Compute the angle between the vectors $e_1$ and $e_2$ in the above Euclidean space.

(c) Show that $\langle \cdot, \cdot \rangle$ is not an inner product if $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(d) Try to compute the angle between the vectors $e_1$ and $e_2$ in the above non-Euclidean space. What goes wrong?

(2) Let $(V, \langle \cdot, \cdot \rangle)$ be a Euclidean space. Show that $v, w \in V$ are linearly dependent if and only if

$$\langle v, w \rangle \langle v, w \rangle = \langle v, v \rangle \langle w, w \rangle.$$

(3) The first four Legendre polynomials are listed on Page 24 of the textbook. Calculate the fifth Legendre polynomial, and check that it is orthogonal to the first four. Educate yourself on the importance of Legendre polynomials in science and engineering.

(4) Trigonometric polynomials are defined on Page 26 of the textbook.

(a) Verify that the set of trigonometric polynomials of degree 2 forms a vector space $V$ of dimension 5.

(b) Make $V$ into a Euclidean space as on Page 26. Verify that

$$\{1, \cos(t), \cos(2t), \sin(t), \sin(2t)\}$$

is an orthogonal basis of $V$. 