(1) Let $a_1, \ldots, a_n$ be a list of positive integers. Prove that there exists a string of consecutive elements of this list whose sum is divisible by $n$. (Hint: as always, begin by experimenting with small numbers).

(2) The expected (or “average”) maximal length of an increasing subsequence in a random permutation of the numbers $1, 2, \ldots, N$ is defined to be

$$E_N = \frac{1}{N!} \sum_{\pi} \text{(maximal length of an increasing subsequence in } \pi),$$

where the sum is over all permutations of the numbers $1, 2, \ldots, N$. Prove that

$$E_N \geq \frac{1}{2}\sqrt{N-1}.$$ 

(3) Prove that a natural number $n$ has an odd number of divisors if and only if it is a perfect square (i.e. $\sqrt{n}$ is an integer).

(4) Recall that a partition of a positive integer $n$ is a representation of $n$ as a sum of smaller positive integers, without regard for order of the terms; these unordered terms are called “parts.” Prove that the number of partitions of $n$ into distinct parts equals the number of partitions of $n$ into odd parts.