Definition: A metric space is a pair \( M = (X, d) \) consisting of a set \( X \) together with a function \( d: X \times X \to \{ \text{nonnegative real numbers} \} \) such that

\[
\begin{align*}
M1) \quad & d(x, y) = d(y, x) \quad \text{for all } x, y \in X; \\
M2) \quad & d(x, y) = 0 \quad \text{if and only if } x = y; \\
M3) \quad & \text{For any } x, y, z \in X, \quad d(x, z) \leq d(x, y) + d(y, z). \\
\end{align*}
\]

Informally, a metric space is a set of points, \( X \), together with a notion of distance \( d(x, y) \) between points. The axioms insist that \( d \) should have the basic properties associated with our physical notion of distance; the distance from \( x \) to \( y \) is the same as the distance from \( y \) to \( x \) (M1); two points \( x, y \) are distance zero apart if and only if \( x = y \) (M2); the shortest distance between two points is a straight line (M3).
Example: The Euclidean plane is the metric space $E^2 = (\mathbb{R}^2, d)$ with point set $\mathbb{R}^2 = \{(x, y) \in \mathbb{R}^2 : x, y \in \mathbb{R}\}$ and metric

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$  

This is not the only metric which can be defined on $\mathbb{R}^2$; some other frequently encountered metrics are

$$d_1((x_1, y_1), (x_2, y_2)) = |x_2 - x_1| + |y_2 - y_1|$$  

and

$$d_\infty((x_1, y_1), (x_2, y_2)) = \max \{|x_2 - x_1|, |y_2 - y_1|\}.$$  

In each case, one has to check that the proposed metric does in fact obey the axioms (M1), (M2), (M3).
Definition: A metric space \((X, d)\) is said to be bounded if \(\{d(x, y) : x, y \in X\}\) is a bounded set of real numbers, i.e. if there exists a number \(M\) such that

\[
d(x, y) \leq M \quad \text{for all } x, y \in X.
\]

In a bounded metric space, we can discuss the following notions.

The eccentricity of a point \(x \in X\) is the smallest number larger than the distance from \(x\) to any other point:

\[
ecc(x) = \sup \{d(x, y) : y \in X\}.
\]

The radius of \(X\) is the largest number smaller than the eccentricities of all points

\[
\text{rad}(X) = \inf \{\ecc(x) : x \in X\}.
\]

The diameter of \(X\) is the smallest number larger than the eccentricities of all points:

\[
\text{diam}(X) = \sup \{\ecc(x) : x \in X\}.
\]

The centre of \(X\) is the set of all points whose eccentricity realizes the radius of \(X\):

\[
C(X) = \{x \in X : \ecc(x) = \text{rad}(X)\}.
\]

The boundary of \(X\) is the set of all points whose eccentricity realizes the diameter of \(X\):

\[
B(X) = \{x \in X : \ecc(x) = \text{diam}(X)\}.
\]
Example: Let $M$ be the metric space with underlying point set $X = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$, and distance function $d = \text{Euclidean metric}$. Then, $M$ is a bounded metric space. The radius of $M$ is 1, and the diameter of $M$ is 2. The centre of $M$ is $C(M) = \{(0, 0)\}$, and the boundary of $M$ is $B(M) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

Example: let $M$ be as above, but with $X = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. Then, everything stays the same, but the boundary is now the empty set.

Example: Let $M$ be as above, but with $X = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1, \ x, y \neq 0\}$. Radius and diameter stay the same, but now both the centre and boundary are empty.
Let $\Gamma=(V,E)$ be a finite connected graph, and define $d:V \times V \rightarrow \{\text{nonnegative numbers}\}$ by

$$d(v,w) = \begin{cases} 
\text{length of a shortest path from } v \text{ to } w, & \text{if } v \neq w; \\
0, & \text{if } v = w.
\end{cases}$$

**Proposition:** $(V,d)$ is a bounded metric space.