LAST TIME...

- Introduced the general notion of a metric space.
- Looked at some examples of bounded metric spaces.
- Proposed to associate to each finite graph $\Gamma = (V,E)$ the metric space $(V,d)$ whose points are the vertices of $\Gamma$, with metric

$$d(v,w) = \begin{cases} 0, & \text{if } v=w; \\ \text{length of a shortest path from } v \text{ to } w, & \text{if } v \neq w. \end{cases}$$

- Shortest paths are often called "geodesic paths," or simply "geodesics."
- The function $d$ above is called the "geodesic distance" or "graph theory distance."

TODAY...

- Verify that the geodesic distance actually is a bona fide distance.
- Look at some concrete examples.
Let $\Gamma = (V, E)$ be a finite connected graph, and define $d : V \times V \rightarrow \{\text{nonnegative integers}\}$ by

$$d(v, w) = \begin{cases} 
\text{length of a shortest path from } v \text{ to } w, & \text{if } v \neq w; \\
0, & \text{if } v = w.
\end{cases}$$

**Proposition:** $(V, d)$ is a bounded metric space.

**Proof:**
- Obviously $\{d(v, w) : v, w \in V\}$ is bounded - it's a finite set of numbers.
- It remains to check that the geodesic distance has the properties $(M1)$, $(M2)$, $(M3)$ required in order to qualify as a metric.

$(M1)$

$$d(v, w) = d(w, v)$$

- Suppose $d(v, w) = k$. This means that there exists a path $v = v_0, v_1, \ldots, v_k = w$ of length $k$ from $v$ to $w$ in $\Gamma$, and that there does not exist a path from $v$ to $w$ of length $k-1$ or less.

- Observe that $v_k, v_{k-1}, \ldots, v_0$ is a path of length $k$ from $w$ to $v$. Thus, $d(w, v) \leq k$. But if $d(v, w) = j$ for some $j < k$, then there would exist a path $w = u_0, \ldots, u_j = v$ from $w$ to $v$, and reversing this would give a length $j$ path from $v$ to $w$, which is impossible. So, $d(w, v) = k$. 


(M2) \( d(v,w) = 0 \) if and only if \( v=w \).

- The statement \( v=w \Rightarrow d(v,w) = 0 \) is part of the definition of \( d \).

- We need to check that \( d(v,w) = 0 \Rightarrow v=w \) also holds. We can instead check the contrapositive: \( v \neq w \Rightarrow d(v,w) > 0 \). This is clear; if \( v \neq w \), then the length of a geodesic path from \( v \) to \( w \) is at least 1.

(M3) For any \( u,v,w \in V \), \( d(u,w) \leq d(u,v) + d(v,w) \).

- If \( u=v \), the inequality becomes \( d(u,w) = d(u,w) \), by (M2). This is obviously true.

- If \( u=w \), the inequality becomes \( 0 \leq d(w,v) + d(v,w) \), which is obviously true.

- If \( v=w \), the inequality becomes \( d(u,v) = d(u,v) \), by (M2). This is obviously true.

- Now consider the case where \( u, v, w \) are pairwise distinct: \( u \neq v, u \neq w, v \neq w \).

- Suppose that \( d(u,v) = m \) and \( d(v,w) = n \). Then there exist paths
  
  \[ u = x_0, \ldots, x_m = v \quad \text{and} \quad v = y_0, \ldots, y_n = w, \]

  and these paths are geodesics.
• The concatenation of these paths,

\[ U = x_0, \ldots, x_m = v = y_0, \ldots, y_n = w \]

is a walk from \( u \) to \( w \), and this walk has length \( m+n \). Consequently, there exists a path from \( u \) to \( w \) of length \( m+n \) or less, whence

\[ d(u, w) \leq m+n = d(u, v) + d(v, w). \]
• The fact that the metric space \((V, d)\) coming from a finite connected graph \(\Gamma = (V, E)\) is finite means that everything is, in principle, computable.

Example: \(\Gamma\) is a star of order 5:

![Graph diagram]

• The following table gives all values of the geodesic distance \(d\):

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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</tbody>
</table>

• Here's the rest of the metric data for this graph:
  \(\text{rad}(\Gamma) = 1\)
  \(\text{C}(\Gamma) = \{5\}\)
  \(\text{diam}(\Gamma) = 2\)
  \(\text{B}(\Gamma) = \{1, 2, 3, 4\}\)
Example: $\Gamma$ is a path of order 5:

```
1 2 3 4 5
1 0 1 2 3 4
2 1 0 1 2 3
3 2 1 0 1 2
4 3 2 1 0 1
5 4 3 2 1 0
ecc(1) = 4
ecc(2) = 3
ecc(3) = 2
ecc(4) = 3
ecc(5) = 4
```

rad($\Gamma$) = 2  \quad \mathcal{C}(\Gamma) = \{3\}

\text{diam}(\Gamma) = 4  \quad B(\Gamma) = \{1, 5\}
Example: $\Gamma$ is a cycle of order 5:

$$
\begin{array}{cccccc}
   & 1 & 2 & 3 & 4 & 5 \\
1 & 0 & 1 & 2 & 2 & 1 \\
2 & 1 & 0 & 1 & 2 & 2 \\
3 & 2 & 1 & 0 & 1 & 2 \\
4 & 2 & 2 & 1 & 0 & 1 \\
5 & 1 & 2 & 2 & 1 & 0 \\
\end{array}
$$

$\text{ecc}(1) = 2$  
$\text{ecc}(2) = 2$  
$\text{ecc}(3) = 2$  
$\text{ecc}(4) = 2$  
$\text{ecc}(5) = 2$

$\text{rad}(\Gamma) = 2$  
$\text{Cl}(\Gamma) = \{1, 2, 3, 4, 5\}$

$\text{diam}(\Gamma) = 2$  
$\text{B}(\Gamma) = \{1, 2, 3, 4, 5\}$
Remark: For metric spaces which arise from finite graphs, the boundary and center are always nonempty.

Theorem: Let \( \Gamma \) be a finite connected graph equipped with the geodesic distance. We have

\[
\text{rad}(\Gamma) \leq \text{diam}(\Gamma) \leq 2 \text{rad}(\Gamma).
\]

Proof: • The first inequality is a direct consequence of the definition - \( \text{rad}(\Gamma) \) is the minimum of all vertex eccentricities, while \( \text{diam}(\Gamma) \) is the maximum.

• For the second, let \( v \in B(\Gamma) \), \( c \in C(\Gamma) \), and let \( w \in V \) be a vertex such that \( d(v, w) = \text{diam}(\Gamma) \).

• Apply the triangle inequality:

\[
\text{diam}(\Gamma) = d(v, w) \leq d(v, c) + d(c, w) \\
\leq \text{ecc}(c) + \text{ecc}(c) \\
= 2 \text{rad}(\Gamma).
\]