**Last Time**

- Proved that, for any finite connected graph $\Gamma = (V, E)$, the geodesic distance on $V$ really is a metric.

- Computed all metric data for several example graphs (5-star, 5-path, 5-cycle, $K_5$).

- Proved the diameter/radius sandwich inequality: $\text{rad}(\Gamma) \leq \text{diam}(\Gamma) \leq 2\text{rad}(\Gamma)$.

**Today**

- Question: What can the centre of a graph look like?
  Answer: anything.

- Question: What can the boundary of a graph look like?
  Answer: Quite specific.
• Start with a finite connected graph \( \Gamma = (V, E) \).

• From this data, construct a larger graph \( \Gamma' = (V', E') \) as follows.

• The vertex set of \( \Gamma' \) is the vertex set of \( \Gamma \) together with four new vertices: \( V' = V \cup \{w, x, y, z\} \).

• The edge set of \( \Gamma' \) is the edge set of \( \Gamma \) together with \( 2|V| + 2 \) new edges:

\[
E' = E \cup \{\{w, x\}, \{y, z\}\} \cup \{\{x, v\} : v \in V\} \cup \{\{y, v\} : v \in V\}.
\]

• Schematically,

\[\text{Schematically,}\]

\[\text{Clearly, the induced subgraph of } \Gamma' \text{ on vertex set } V \text{ just returns } \Gamma. \text{ We claim that } V \text{ is the centre of } \Gamma', \text{ so that "} \Gamma \text{ is the centre of } \Gamma' \text{"} \]
Theorem: $C(\Gamma') = V$.

Proof: • Remember that $C(\Gamma')$ is the set of vertices of $\Gamma$ of minimal eccentricity.

• Let us compute the eccentricity of the "new" vertices. We have:

$$
\begin{align*}
\text{ecc}(w) &= d(w,z) = 4 \\
\text{ecc}(x) &= d(x,z) = 3 \\
\text{ecc}(y) &= d(y,w) = 3 \\
\text{ecc}(z) &= d(z,x) = 4
\end{align*}
$$

• The eccentricity of any "old" vertex $v \in V$ is $\text{ecc}(v) = 2$, because $d(v,\tilde{v}) \leq 2$ for any other "old" vertex $\tilde{v}$, and $d(v,x) = d(v,y) = 1$, and $d(v,w) = d(v,z) = 2$. 

$\square$
• I then stated the characterization of finite graphs which can occur as the boundary of another graph (either all vertices have eccentricity 1, or none do), but didn’t prove it.

• I defined the adjacency matrix of a finite simple graph, and computed the adjacency matrices of a 5-star, a 5-path, a 5-cycle, and $K_5$. 