MATH 154: Lecture 18

05/06/2016
• Today: proof of Euler's formula.

• Start with a connected planar graph $\Gamma = (V, E)$, and a planar representation thereof.

• It will be helpful to keep a concrete example in mind, e.g. $K_4$:
• Choose a point in each face of the planar representation of $\Gamma$:

• For each of the $\binom{12}{2}$ pairs of "face points", join them by one curve for each edge shared by the faces in which they live:
• You can view this as a representation of a graph $\Gamma^*$, called the planar dual of $\Gamma$. Subtle point: the dual $\Gamma^*$ depends on the planar representation of $\Gamma$; different representations may lead to non-isomorphic duals.

• Because $\Gamma$ is connected, it admits a spanning tree $T$: 

![Diagram](attachment:graph_diagram.png)
Let $T^*$ be the subgraph of $P^*$ whose vertices are the vertices of $P$, and edges are the edges of $P^*$ which don't intersect the edges of $T$.

We claim that $T^*$ is a spanning tree of $P^*$. 
• First, by construction the vertex set of $T^*$ coincides with the vertex set of $\Gamma^*$.

• Second, $T^*$ is acyclic, since if it contained a cycle there would be some vertex of $\Gamma$ not on $T$, which can’t be the case because $T$ is a spanning tree:

\begin{center}
\includegraphics[width=0.1\textwidth]{triangle}
\end{center}

• Third, $T^*$ is connected. Indeed, $T^*$ can’t have an isolated vertex, because this would force a cycle in $T$:

\begin{center}
\includegraphics[width=0.1\textwidth]{triangle}
\end{center}
• Similarly, there can’t be two connected vertices isolated from the rest of \( T^* \), because this would again force a cycle in \( T \):

• Thus \( T^* \) cannot consist of multiple connected components.

• So, \( T^* \) is indeed a spanning tree of \( \Gamma^* \).
• Now observe that, by construction,

$$|V(T)| + |V(T^*)| = |V(\Gamma)| + |V(\Gamma^*)| = |V(\Gamma)| + |F(\Gamma)|.$$  

• On the other hand,

$$|V(T)| + |V(T^*)| = |E(T)| + 1 + |E(T^*)| + 1$$

$$= |E(T)| + |E(T^*)| + 2$$

$$= |E(\Gamma)| + 2.$$  

• Thus  $$|V(\Gamma)| + |F(\Gamma)| = |E(\Gamma)| + 2,$$  which is Euler's formula.
• **Last thing about planarity:** Kuratowski’s Theorem.

• Start with a graph $\Gamma = (V, E)$. Let $\{u, w\}$ be an edge of $\Gamma$. The graph $\Gamma' = (V', E')$ with $V' = V \cup \{v\}$ (i.e. $v$ is a new vertex) and $E' = E - \{\{u, w\}\} + \{\{u, v\}, \{v, w\}\}$ is called an edge subdivision of $\Gamma$:

![Edge subdivision](image)

**Theorem:** $\Gamma$ is planar if and only if it does not contain a subgraph isomorphic to an edge subdivision of $K_{3,3}$ or $K_5$. 