(1) Let $\Gamma$ be a graph on $n$ vertices. Prove that if every vertex of $\Gamma$ has degree at least $\frac{1}{2}(n - 1)$, then $\Gamma$ is connected.

(2) Construct a graph $\Gamma$ with edge connectivity $\lambda(\Gamma) = 2016$ and vertex connectivity $\kappa(\Gamma) = 1$. Explain why your graph has the required connectivity values.

(3) Let $\Gamma$ be a graph. Prove that it is possible to colour every vertex of $\Gamma$ either black or white in such a way that, for every vertex $v$, the number of neighbours of $v$ which have the same colour as $v$ does not exceed the number of neighbours of $v$ which do not have the same colour as $v$.

(4) Let $\Gamma$ be a connected graph. Prove that the boundary of $\Gamma$ contains at least two vertices. (Hint: start by proving that the boundary is nonempty).

\footnote{All graphs on this exam are simple graphs.}