Instructions

1. No calculators, tablets, phones, or other electronic devices are allowed during this exam.
2. You may use one handwritten, double-sided page of notes, but no books or other assistance during this exam.
3. Read each question carefully and answer each question completely.
4. Show all of your work. No credit will be given for unsupported answers, even if correct.
5. Write your Name at the top of each page.

0. (1 point) Carefully read and complete the instructions at the top of this exam sheet and any additional instructions written on the chalkboard during the exam.

1. (6 points) Consider the following matrices:

\[ A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 3 \\ 2 & -2 \\ 1 & 0 \end{bmatrix}. \]

State whether each of the following matrix operations makes sense. If so, compute the answer.

(a) \( AB \).

(b) \( AB^\top \).

(c) \( A - B \).

(d) \( A + B^\top \).
2. The matrix $B$ below is the reduced row-echelon form of the matrix $A$ below.

$$A = \begin{bmatrix}
1 & -1 & -2 & 2 & -4 & 0 \\
-2 & 2 & 4 & -4 & 8 & 0 \\
1 & -1 & 2 & 2 & 0 & -8
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & -1 & 0 & 2 & -2 & -4 \\
0 & 0 & 1 & 0 & 1 & -2 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$ 

(a) Find a basis for $\text{Col} A$.

(b) Find a basis for $\text{Row} A$.

(c) Find a basis for $\text{Nul} A$. 
(6 points) 3. Let $A$ be any $16 \times 25$ matrix. Answer the following questions, and explain your answers.

(a) What is the maximum possible rank of $A$ and the minimum possible dimension of the nullspace of $A$?

(b) What is the maximum possible rank of $A^\top$ and the minimum possible dimension of the nullspace of $A^\top$?

(c) Suppose that the set of all $b \in \mathbb{R}^{16}$ for which $Ax = b$ has a solution is a subspace of dimension 10. What is the dimension of the nullspace of $A$?
4. $V$ is a 3-dimensional vector space. Let $B = \{ u_1, u_2, u_3 \}$ be a basis for $V$. Consider 3 vectors $v_1, v_2, v_3 \in V$, whose coordinate vectors in terms of the basis $B$ are

$$[v_1]_B = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad [v_2]_B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad [v_3]_B = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}.$$

(a) Calculate the volume of the parallelepiped determined by the 3 coordinate vectors above.

(b) Use part (a) to show that $C = \{ v_1, v_2, v_3 \}$ is a basis for $V$. Explain your reasoning.

(c) What is the determinant of the change of basis matrix $P_{C \to B}$?