MATH 20B: Lecture 2

01/06/2016 §5.6
In Math 20A, the definite integral $\int_a^b f(x)\,dx$ of a continuous function $f(x)$ on $[a,b]$ was defined using limits.

The number $\int_a^b f(x)\,dx$, which can be positive, negative, or zero, was taken to be the definition of the signed area of the region bounded by the graph of $f(x)$, the horizontal axis, and the vertical lines $x=a$ and $x=b$. 

![Graph of a function with shaded areas representing positive and negative definite integrals.](image-url)
• Our main result concerning the definite integral is FTC1, which says that

\[ \int_{a}^{b} f(x) \, dx = F(b) - F(a) \]

for any antiderivative \( F(x) \) of \( f(x) \) (existence of antiderivatives is guaranteed by FTC2).

• Thus the definite integral makes precise sense out of the intuitive notion of area, and FTC1 gives us a practical means to compute area — provided we’re good at computing antiderivatives.
• There is another way of looking at FTC1 which has many applications in science and engineering.

• Suppose that \( q \) is a quantity that depends on time, i.e. \( q=q(t) \) is a function of the time variable \( t \).

• FTC1 says: \( \int_{t_1}^{t_2} q'(t) \, dt = q(t_2) - q(t_1) \).

• In words, “integrating instantaneous rate of change over an interval is the same as computing net change over that interval.”
Physics Example: Consider a particle initially at rest on a straight wire. Suppose that the velocity of the particle at time $t$ is given by the function $v(t) = t^3 - 10t^2 + 24t$ metres per second. What is the displacement of the particle from its initial position after 6 seconds?

\[ v(t) = t^3 - 10t^2 + 24t \]
Solution: • Let \( q(t) \) denote the position of the particle at time \( t \), relative to its initial position.

• We're trying to compute \( q(6) \).

• By FTC 1,

\[
q(6) = q(6) - q(0) = \int_0^6 q'(t) \, dt.
\]

• But \( q'(t) \) = instantaneous rate of change in position at time \( t \)
  = velocity at time \( t \)
  =\( v(t) \).

• Thus \( q(6) = \int_0^6 v(t) \, dt \).
Now we compute:

\[ q(6) = \int_0^6 v(t) \, dt \]

\[ = \int_0^6 (t^3 - 10t^2 + 24t) \, dt \]

\[ = \int_0^6 t^3 \, dt - 10\int_0^6 t^2 \, dt + 24\int_0^6 t \, dt \]

\[ = \left[ \frac{t^4}{4} \right]_0^6 - 10 \left[ \frac{t^3}{3} \right]_0^6 + 24 \left[ \frac{t^2}{2} \right]_0^6 \]

\[ = \frac{6^4}{4} - 10 \frac{6^3}{3} + 24 \frac{6^2}{2} \]

\[ = 6^2 \left( \frac{6^2}{4} - 10 \frac{6}{3} + 24 \frac{1}{2} \right) \]

\[ = 6^2 \left( \frac{36}{4} - 10 \cdot 2 + 12 \right) \]

\[ = 6^2 \left( 9 - 20 + 12 \right) = 6^2 \cdot 1 = 36. \]

Conclusion: after 6 seconds, the particle is located 36 metres to the right of its initial position.
Physics Example: Consider a particle initially at rest on a straight wire. Suppose that the velocity of the particle at time $t$ is given by the function $v(t) = t^3 - 10t^2 + 24t$ metres per second. What is the total distance travelled by the particle after 6 seconds have elapsed?
Solution: • This is not the same as the previous problem: we’re being asked about total distance travelled in a given time, rather than position after a given time. This is analogous to absolute area vs. signed area.

• Let $d(t)$ be the total distance travelled by the particle after $t$ seconds.

• By FTC 1,

$$d(t) = d(t) - d(0) = \int_0^t d'(t)\,dt.$$ 

• Now, $d''(t) =$ instantaneous rate of change in distance at time $t$
  
  = speed at time $t$
  
  = absolute value of velocity at time $t$

  $$= |t^3 - 10t^2 + 24t|.$$
So, we have to compute

\[ d(6) = \int_{0}^{6} |t^3 - 10t^2 + 24t| \, dt. \]

In order to compute this integral, we have to determine when \( v(t) \) is positive and when it’s negative.

To do this, we factor:

\[ v(t) = t(t^2 - 10t + 24) = t(t-4)(t-6). \]

From the factorization, we see that \( v(t) \) is positive on the open interval \((0, 4)\), and negative on the open interval \((4, 6)\).

Physically, this means that the particle is moving right for the first four seconds, at which time it stops, then starts moving left for the next two seconds.
We now compute

\[
d(6) = \int_0^6 |t^3 - 10t^2 + 24t| \, dt
\]

\[
= \int_0^4 |t^3 - 10t^2 + 24t| \, dt + \int_4^6 |t^3 - 10t^2 + 24t| \, dt
\]

\[
= \int_0^4 (t^3 - 10t^2 + 24t) \, dt + \int_4^6 (t^3 - 10t^2 + 24t) \, dt
\]

\[
= \int_0^4 (t^3 - 10t^2 + 24t) \, dt - \int_4^6 (t^3 - 10t^2 + 24t) \, dt
\]

\[
= \frac{128}{3} - \left( -\frac{20}{3} \right)
\]

\[
= \frac{128}{3} + \frac{20}{3}
\]

\[
= \frac{148}{3} = 49 + \frac{1}{3}.
\]

Conclusion: the total distance travelled by the particle in 6 seconds is 49.33... metres.
Mathematics = The Force

Light Side = Physics
- Displacement = integral of velocity
- Distance = integral of speed

Dark Side = Economics
- Cost = integral of marginal cost
• In economics, one considers the cost $C(x)$ incurred by a manufacturer in producing $x$ units of a product.

• The derivative of the cost function, $C'(x)$, is called the "marginal cost.”

• Economists interpret FTC1,

$$C(b) - C(a) = \int_a^b C'(x)\,dx,$$

as the statement "the cost of increasing production from $a$ units to $b$ units equals the integral of marginal cost from $a$ to $b."
Economics Example: The marginal cost of producing 1000x computer chips is

\[ C'(x) = 300x^2 - 4000x + 40000 \] dollars per thousand chips. Find the cost of increasing production from 10k to 15k chips.

Solution: The increase in cost will be

\[
C(15) - C(10) = \int_{10}^{15} (300x^2 - 4000x + 40000) \, dx \\
= \left(100x^3 - 2000x^2 + 40000x\right)_{10}^{15} \\
= \left(150 \cdot 15^3 - 2000 \cdot 15^2 + 40000 \cdot 15\right) - \left(100 \cdot 10^3 - 2000 \cdot 10^2 + 40000 \cdot 10\right) \\
= 487500 - 300000 \\
= 187500.
\]