MATH 20B: Lecture 5

01/13/2016

§6.3
Last time, we computed the volumes of three solids:

- We did this by applying the general principle:

  \[ \text{Volume} = \text{Integral of Cross-Sectional Area}. \]
• IQ test: which of the following objects does not belong?
• **Answer:**

![Diagram of a pyramid, a cone, and a sphere with the pyramid marked with an X]

• **Reason:** The cross sections of a pyramid aren't circles.
• Equally valid answer:

• Reason: the sphere doesn't have a sharp point.

• Conclusion: IQ tests are meaningless.
• The cone and the sphere belong to a large class of solid objects known as “solids of revolution.”

• To make a solid of revolution, start with a function $f(x)$ which is defined on a closed interval $[a,b]$, and satisfies $f(x) \geq 0$ for all $x \in [a,b]$.

• Now shade in the region under the graph of $f(x)$, and rotate the whole thing $360^\circ$ around the $x$-axis.

• This produces a solid object called the “solid of revolution” generated by $f(x)$. 
Example: Let $h$ and $r$ be positive numbers. The function $f(x) = \frac{r}{h}x$ is defined and positive on the interval $[0,h]$. The graph of this function is a segment of the line through the origin with slope $\frac{r}{h}$. The corresponding solid of revolution is a cone of height $h$ and base radius $r$. 

\[ f(h) = r \]
\[ f(x) = \frac{r}{h}x. \]
Example: Let $r$ be a positive number. The function $f(x) = \sqrt{r^2 - x^2}$ is defined on the interval $[-r, r]$. It satisfies $f(x) \geq 0$ for all $x \in [-r, r]$. The corresponding solid of revolution is a sphere of radius $r$. 

\[ f(x) = \sqrt{r^2 - x^2} \]
Example: Let $h$ and $r$ be positive numbers. The function $f(x) = \frac{r}{h^2}x^2$ is defined and nonnegative on $[0, h]$. The corresponding solid of revolution is a “curvy cone” of height $h$ whose circular base has radius $r$. 
Main Point: If $S$ is the solid of revolution generated by a nonnegative function $f(x)$ on an interval $[a,b]$, then

$$\text{Vol}(S) = \pi \int_a^b f(x)^2 \, dx.$$ 

This formula follows from our general principle (volume is the integral of cross sectional area) together with the fact that, by construction, the cross sections of $S$ are circles of radii $f(x)$. 
Example: Let $h$ and $r$ be positive numbers. The function $f(x) = \frac{r}{h}x$ is defined and positive on the interval $[0, h]$. The graph of this function is a segment of the line through the origin with slope $\frac{r}{h}$. The corresponding solid of revolution is a cone of height $h$ and base radius $r$.

$$\text{Vol}(S) = \pi \int_0^h f(x)^2 \, dx = \pi \int_0^h \left(\frac{r}{h}x\right)^2 \, dx = \pi \frac{r^2}{h^2} \left(\frac{1}{3}x^3\right)_0^h = \frac{1}{3} \pi r^2 h.$$
Example: Let $r$ be a positive number. The function $f(x) = \sqrt{r^2 - x^2}$ is defined on the interval $[-r, r]$. It satisfies $f(x) \geq 0$ for all $x \in [-r, r]$. The corresponding solid of revolution is a sphere of radius $r$.

\[
V\text{ol}(S) = \pi \int_{-r}^{r} f(x)^2 dx = \pi \int_{-r}^{r} (r^2 - x^2) dx = \pi r^2 \int_{-r}^{r} 1 dx - \pi \int_{-r}^{r} x^2 dx = \pi r^2 x \bigg|_{-r}^{r} - \pi \frac{x^3}{3} \bigg|_{-r}^{r} = \pi r^2 (r + r) - \pi \left( \frac{1}{3} r^3 + \frac{1}{3} r^3 \right) = 2\pi r^3 - \frac{2}{3} \pi r^3 = \frac{4}{3} \pi r^3.
\]
Example: Let $h$ and $r$ be positive numbers. The function $f(x) = \frac{r^2}{h^2}x^2$ is defined and nonnegative on $[0, h]$. The corresponding solid of revolution is a "curvy cone" of height $h$ whose circular base has radius $r$.

\[
V_{01}(S) = \pi \int_0^h f(x)^2 \, dx = \pi \int_0^h \frac{r^2}{h^4} x^4 \, dx = \pi \frac{r^2}{h^4} \left[ \frac{x^5}{5} \right]_0^h = \frac{1}{5} \pi r^2 h.
\]

- This is smaller than an actual (non-curvy) cone with the same dimensions by a factor of $\frac{3}{5}$. 

\[
\text{Vol}(S) = \pi \int_0^h x^2 \, dx = \pi \frac{r^2}{h} \int_0^1 x^4 \, dx = \frac{1}{5} \pi r^2 h.
\]
Example: When plotted on the same axes, the graphs of the functions $f(x) = \frac{r}{h} x$ and $g(x) = \frac{r}{h^2} x^2$ look like this:

Indeed, since

$$\frac{g(x)}{f(x)} = \frac{\frac{r}{h^2} x^2}{\frac{r}{h} x} = \frac{x}{h},$$

we can be sure that $g(x) \leq f(x)$ for $x \in [0, h]$. 
There are two natural questions:

(1) What is the area of the region between the graphs of $f(x)$ and $g(x)$?

(2) What is the volume of the solid of revolution corresponding to the region from (1)?
• The answer to (1) is just the area under \( f(x) \) from 0 to \( h \) less the area under \( g(x) \) from 0 to \( h \):

\[
A = \int_0^h f(x) \, dx - \int_0^h g(x) \, dx = \frac{r}{h} \int_0^h x \, dx - \frac{r}{h^2} \int_0^h x^2 \, dx = \left. \frac{r}{h} \frac{x^2}{2} \right|_0^h - \left. \frac{r}{h^2} \frac{x^3}{3} \right|_0^h = \frac{rh}{2} - \frac{rh}{3} = \frac{rh}{6}.
\]

• The answer to (2) is the volume of the surface of revolution generated by \( f(x) \) over \([0, h]\) less the volume of the surface of revolution generated by \( g(x) \) over \([0, h]\):

\[
V = \pi \int_0^h f(x)^2 \, dx - \pi \int_0^h g(x)^2 \, dx = \frac{1}{3} \pi r^2 h - \frac{1}{5} \pi r^2 h = \frac{2}{15} \pi r^2 h.
\]