MATH 20B: Lecture 11

29/01/2016

Supplement §2
• We have been discussing the complex number system, which is denoted \( \mathbb{C} \).

• Unlike the real number system \( \mathbb{R} \), which is made up out of points on a line (the "number line"), the complex number system is made up out of points in a plane.

• When referencing a point \( z \) in the plane, we can either give its Cartesian coordinates \( z = (x, y) \), or its polar coordinates \( z = (r, \theta) \).
Here’s how to add two complex numbers $z_1$ and $z_2$.

First, write their Cartesian coordinates, $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$.

To add $z_1$ and $z_2$, use the formula

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

Similarly, to subtract $z_2$ from $z_1$, use the formula

$$z_1 - z_2 = (x_1, y_1) - (x_2, y_2) = (x_1 - x_2, y_1 - y_2).$$
Here’s how to multiply two complex numbers $z_1$ and $z_2$.

First, write their polar coordinates, $z_1 = (r_1, \theta_1)$ and $z_2 = (r_2, \theta_2)$.

To multiply $z_1$ and $z_2$, use the formula

$$z_1z_2 = (r_1, \theta_1)(r_2, \theta_2) = (r_1r_2, \theta_1 + \theta_2).$$

To divide $z_1$ by $z_2$, use the formula

$$\frac{z_1}{z_2} = \left(\frac{r_1}{r_2}, \theta_1 - \theta_2\right).$$
• Addition and multiplication are examples of “binary operations” — they produce a new number from two given numbers.

• There are two important “unary” operations on complex numbers: inversion and conjugation.

• The inverse $z^{-1}$ of a non-zero complex number $z$ is defined geometrically: first reflect $z$ in the horizontal axis, so that its $\Theta$-value is replaced with $\Theta^{-1}$; then replace its $r$ value by $r^{-1}$:

$$z^{-1} = (r^{-1}, -\Theta).$$

• The conjugate of $z$, denoted $\overline{z}$, is kind of a half-hearted inverse: just reflect $z$ in the horizontal axis:

$$\overline{z} = (r, -\Theta).$$
\[
zz^{-1} = (r, \theta)(r^{-1}, -\theta) = (1, 0) = 1
\]

\[
\overline{z} \overline{z} = (r, \theta)(r, -\theta) = (r^2, 0) = r^2.
\]

**Notation:** If \( z = (r, \theta) \), people write \( |z| = r \). So, \( z \overline{z} = |z|^2 \), and \( z^{-1} = \frac{\overline{z}}{|z|^2} \).
• Remember now that if \( z = (x, y) \) is a complex number in Cartesian coordinates, one also writes \( z = x+iy \).

• In this notation, \( x \) is shorthand for the number \((x, 0)\), \( y \) is shorthand for the number \((y, 0)\), and \( i \) is shorthand for the number \((0,1)\) — all in Cartesian coords.

• Remembering that \( i^2 = -1 \), this gives an efficient way to do computations in Cartesian coordinates.

• Multiplication: \( z_1 z_2 = (x_1+i y_1)(x_2+i y_2) = x_1 x_2 + i x_1 y_2 + i x_2 y_1 + i^2 y_1 y_2 = (x_1 x_2 - y_1 y_2) + i (x_1 y_2 + x_2 y_1) \).

• Conjugation: \( \overline{z} = \overline{x+iy} = x-iy \).

• Absolute value / modulus: \( |z| = |x+iy| = \sqrt{x^2+y^2} \).

• Inversion: \( z^{-1} = \frac{\overline{z}}{|z|^2} = \frac{x-iy}{x^2+y^2} = \left( \frac{x}{x^2+y^2} \right) - i \left( \frac{y}{x^2+y^2} \right) \).
• Multiplication of complex numbers explains the trigonometric formulas we’ve been using.

• Take two complex numbers of modulus 1, i.e. two points \( z_1 = (1, \theta_1), \ z_2 = (1, \theta_2) \) and multiply them

\[
z_1z_2 = (1, \theta_1)(1, \theta_2) = (1, \theta_1+\theta_2) = \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2).
\]

• On the other hand, if we do the same multiplication in Cartesian coordinates, we get

\[
z_1z_2 = (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2)
= \cos \theta_1 \cos \theta_2 + i \cos \theta_1 \sin \theta_2 + i \sin \theta_1 \cos \theta_2 + i^2 \sin \theta_1 \sin \theta_2
= (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2).
\]

• Since the Cartesian coordinates of \( z_1z_2 \) are unique, this forces

\[
\cos(\theta_1+\theta_2) = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 \quad \text{and} \quad \sin(\theta_1+\theta_2) = \cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2.
\]
• Remember that if \( z \) is the complex number whose Cartesian coordinates are \((x, y)\), we've taken to writing \( z = x + iy \).

• Here's more strange-but-convenient notation: if \( z \) is the complex number with polar coordinates \((r, \theta)\), we're going to write

\[
z = re^{i\theta}
\]

• This seemingly bizarre equation actually defines complex exponentials: if \( z = x + iy \), then

\[
e^z = e^{x+iy} = e^x e^{iy}
\]

is by definition the complex number with polar coordinates \((e^x, y)\).
• The main reason to set things up in this way is in order to express trig functions as complex exponentials.

• Take the two equations

\[ e^{i\theta} = \cos \theta + i\sin \theta \]
\[ e^{-i\theta} = \cos (-\theta) + i\sin (-\theta) = \cos \theta - i\sin \theta. \]

• Adding these equations yields

\[ e^{i\theta} + e^{-i\theta} = 2\cos \theta \implies \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \]

• Subtracting them yields

\[ e^{i\theta} - e^{-i\theta} = 2i\sin \theta \implies \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \]
Here's our first example of applying complex numbers to make calculus easier.

\[
\int \cos^2 \theta \, d\theta = \int \left( \frac{e^{i\theta} + e^{-i\theta}}{2} \right)^2 \, d\theta
\]

\[
= \frac{1}{4} \int \left( e^{i\theta} + e^{-i\theta} \right)^2 \, d\theta
\]

\[
= \frac{1}{4} \int \left( e^{2i\theta} + 2e^{i\theta}e^{-i\theta} + e^{-2i\theta} \right) \, d\theta
\]

\[
= \frac{1}{4} \int \left( e^{2i\theta} + 2 + e^{-2i\theta} \right) \, d\theta
\]

\[
= \frac{1}{4} \int e^{2i\theta} \, d\theta + \frac{1}{2} \int 1 \, d\theta + \frac{1}{4} \int e^{-2i\theta} \, d\theta
\]

\[
= \frac{1}{4} \frac{1}{2i} e^{2i\theta} + \frac{1}{2} \theta + \frac{1}{4} \left( -\frac{1}{2i} \right) e^{-2i\theta}
\]

\[
= \frac{1}{2} \theta + \frac{1}{4} \frac{e^{2i\theta} - e^{-2i\theta}}{2i}
\]

\[
= \frac{1}{2} \theta + \frac{1}{4} \sin 2\theta
\]

check: \[ \frac{d}{d\theta} \left( \frac{1}{2} \theta + \frac{1}{4} \sin 2\theta \right) \]

\[
= \frac{1}{2} + \frac{1}{4} \cos 2\theta \cdot 2
\]

\[
= \frac{1}{2} + \frac{1}{2} \cos 2\theta
\]

\[
= \frac{1}{2} + \frac{1}{2} \left( \cos^2 \theta - \sin^2 \theta \right)
\]

\[
= \frac{1}{2} + \frac{1}{2} \left( \cos^2 \theta - (1 - \cos^2 \theta) \right)
\]

\[
= \frac{1}{2} + \frac{1}{2} \left( 2 \cos^2 \theta - 1 \right)
\]

\[
= \cos^2 \theta.
\]