• We’re slowly expanding the types of functions we can integrate - last time we saw how to compute

$$\int \cos^m(x) \sin^n(x) \, dx$$

for any nonnegative integers $m$ and $n$ using complex exponentials, i.e. using the fact that

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \quad \text{and} \quad \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}.$$ 

• The next class of integrals we want to look at are integrals of rational functions, i.e. integrals of the form

$$\int \frac{P(x)}{Q(x)} \, dx$$

with $P(x)$ and $Q(x)$ polynomials.
Example: Compute \( \int \frac{1}{x^2-1} \, dx \).

Solution: • If there wasn’t an exponent of 2 on the \( x \), the integral would be easy:

\[
\int \frac{1}{x-1} \, dx = \ln |x-1| + C.
\]

• You can get rid of the exponent of 2 by observing that

\[
\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1},
\]

so that

\[
\int \frac{1}{x^2-1} \, dx = \frac{1}{2} \int \frac{1}{x-1} \, dx - \frac{1}{2} \int \frac{1}{x+1} \, dx = \frac{1}{2} \ln |x-1| - \frac{1}{2} \ln |x+1| + C.
\]
The previous example, although very simple, already illustrates the general strategy for computing \( \int \frac{P(x)}{Q(x)} \, dx \):

**Step 1:** Factor the denominator to get something like

\[ Q(x) = (x-a_1)(x-a_2) \ldots (x-a_n). \]

**Step 2:** Use the factorization of \( Q(x) \) to simplify the integral:

\[
\int \frac{P(x)}{(x-a_1) \ldots (x-a_n)} \, dx = \int \frac{A_1}{x-a_1} \, dx + \int \frac{A_2}{x-a_2} \, dx + \ldots + \int \frac{A_n}{x-a_n} \, dx.
\]
In order to develop this strategy, we need to do two things:

(1) Discuss how to factor a given polynomial \( Q(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \)
into an expression of the form

\[
Q(x) = (x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n);
\]

(2) Develop techniques for using the factorization from (1) to write rational functions
in the form

\[
\frac{P(x)}{Q(x)} = \frac{A_1}{x-\alpha_1} + \frac{A_2}{x-\alpha_2} + \ldots + \frac{A_n}{x-\alpha_n}.
\]
• Let's begin by reviewing factoring.

• Suppose that \( Q(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \) can be factored:

\[
Q(x) = (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n).
\]

• Then, it's clear that \( Q(\alpha_i) = 0 \) for any of the "roots" \( \alpha_1, \ldots, \alpha_n \).

• Indeed,

\[
Q(\alpha_1) = (\alpha_1 - \alpha_1)(\alpha_1 - \alpha_2) \ldots (\alpha_1 - \alpha_n) = 0(\alpha_1 - \alpha_2) \ldots (\alpha_1 - \alpha_n) = 0 \\
Q(\alpha_2) = (\alpha_2 - \alpha_1)(\alpha_2 - \alpha_2) \ldots (\alpha_2 - \alpha_n) = (\alpha_1 - \alpha_2)0 \ldots (\alpha_2 - \alpha_n) = 0 \\
\vdots \\
Q(\alpha_n) = (\alpha_n - \alpha_1)(\alpha_n - \alpha_2) \ldots (\alpha_n - \alpha_n) = (\alpha_n - \alpha_1)(\alpha_n - \alpha_2) \ldots 0 = 0.
\]
• The converse statement is also true: if $Q(\alpha) = 0$, then the polynomial $x - \alpha$ divides $Q(x)$, i.e. we have

$$Q(x) = (x - \alpha)q(x)$$

for some "smaller" polynomial $q(x)$, where "smaller" means that the highest power of $x$ which can appear in $q(x)$ is $n-1$.

• Indeed, we can always use long division of polynomials to write

$$Q(x) = (x - \alpha)q(x) + R$$

for some number $R$, subbing in $x = \alpha$ we get

$$Q(\alpha) = (\alpha - \alpha)q(\alpha) + R \implies 0 = 0 + R \implies R = 0.$$
• This means that factoring

\[ x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = (x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n) \]

is exactly the same thing as solving the equation

\[ x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = 0. \]

• But what if the equation \( x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = 0 \) doesn’t have any solutions?

Then the polynomial \( x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \) can’t be factored.

Example: The equation \( x^2 + 1 = 0 \) doesn’t have solutions, so the polynomial \( x^2 + 1 \) can’t be factored — it’s "irreducible."
• Of course, we now know how to fix that — the equation \[ x^2 + 1 = 0 \] has two solutions, the imaginary numbers \( i \) and \( -i \), and hence factors as

\[ x^2 + 1 = (x - i)(x + i). \]

**Fundamental Theorem of Algebra:** For any polynomial \( x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \), there are complex numbers \( \alpha_1, \ldots, \alpha_n \) such that

\[ x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = (x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_n). \]

• Note also that the numbers \( \alpha_1, \alpha_2, \ldots, \alpha_n \) guaranteed above are unique. Indeed, if \( \beta_1, \ldots, \beta_n \) were some other numbers with this property, then we'd have

\[ (x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_n) = (x - \beta_1)(x - \beta_2)\ldots(x - \beta_n), \]

and subbing \( x = \beta_1 \) would produce the equation

\[ (\beta_1 - \alpha_1)(\beta_1 - \alpha_2)\ldots(\beta_1 - \alpha_n) = 0 \Rightarrow \beta_1 \in \{\alpha_1, \ldots, \alpha_n\}. \]
• The FTA looks like it solves our calculus problem—unfortunately, it's not quite that simple, and the FTA is not always compatible with the FTC.

Example: Compute \( \int \frac{1}{x^2+1} \, dx \).

Solution: • We factor \( x^2+1 = (x-i)(x+i) \), and write

\[
\frac{1}{x^2+1} = \frac{1}{2i} \cdot \frac{1}{x-i} - \frac{1}{2i} \cdot \frac{1}{x+i}.
\]

• Now integrate:

\[
\int \frac{1}{x^2+1} \, dx = \frac{1}{2i} \int \frac{1}{x-i} \, dx - \frac{1}{2i} \int \frac{1}{x+i} \, dx = \frac{1}{2i} \ln |x-i| - \frac{1}{2i} \ln |x+i|.
\]

• This is nonsense—we already know that \( \int \frac{1}{x^2+1} \, dx = \arctan(x) \), and the equation

\[
\arctan(x) = \frac{1}{2i} \ln |x-i| - \frac{1}{2i} \ln |x+i|
\]

is absurd.
• The basic fact is that, in order for calculus to work the way it should, we need to be able to factor

\[ x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n) \]

with all the numbers \( \alpha_1, \ldots, \alpha_n \) \underline{real}.

• Unfortunately, this isn’t always possible, as the example \( Q(x) = x^2 + 1 \) shows: we have \( x^2 + 1 = (x - i)(x + i) \), and there cannot exist any alternative factorization of this polynomial in which \( i \) and \(-i\) are replaced by real numbers.

**Fundamental Thm of Algebra, Real Version**

Every real polynomial \( Q(x) = x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \) can be factored (uniquely) into a product of real linear and quadratic polynomials.
• The real version of FTA is an easy version of FTA together with the following fact: if \( \alpha \) is a root of \( Q(\alpha) \), then so is \( \overline{\alpha} \).

• This fact has a simple geometric explanation: saying that \( Q(\alpha) = 0 \) means

\[
\alpha^n + a_{n-1}\alpha^{n-1} + \ldots + a_1\alpha + a_0 = 0,
\]

which is the same thing as a bunch of vectors adding up to zero:

![Diagram of vectors adding up to zero](image)

• The equation \( \overline{\alpha^n} + a_{n-1}\overline{\alpha^{n-1}} + \ldots + a_1\overline{\alpha} + a_0 = 0 \) is just the reflection of this picture in the horizontal (real) axis.
This observation means that, for every root $\alpha$ of $Q(x)$ which is not real, $\bar{\alpha}$ is also a root, hence

\[
Q(x) = \ldots (x-\alpha) \ldots (x-\bar{\alpha})
\]

\[
= \ldots (x-\alpha)(x-\bar{\alpha}) \ldots
\]

\[
= \ldots (x^2 - (\alpha + \bar{\alpha})x + \alpha\bar{\alpha}) \ldots
\]

\[
\uparrow
\]

a real quadratic
• That’s it for theory of polynomials.

• We now come back to the practical problem of computing

\[ \int \frac{P(x)}{Q(x)} \, dx \]

assuming we have access to a factorization of \( Q(x) \) into real linear and quadratic polynomials.

• This involves an algorithmic technique known as partial fractions, which is best learned by looking at examples.
Example: Compute $\int \frac{1}{x^2-3x+2} \, dx$.

Solution: We need to factor the denominator, and this can be done by inspection:

$$x^2 - 3x + 2 = (x-1)(x-2).$$

This is the best case scenario for partial fractions: the denominator splits into a product of distinct linear factors.

In this situation, we can write

$$\frac{1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

for some constants $A$ and $B$ to be determined.
• To find A and B, put the fraction back together on the RHS:

\[
\frac{1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} = \frac{A(x-2) + B(x-1)}{(x-1)(x-2)} = \frac{(A+B)x - 2A - B}{(x-1)(x-2)}.
\]

• Since the numerators must be equal, we have

\[1 = (A+B)x - 2A - B,\]

which forces

\[A + B = 0\]
\[-2A - B = 1.\]

• We thus have two equations in two unknowns. The first equation says \(B = -A\), and plugging this into the second equation yields

\[-2A - (-A) = 1 \implies -2A + A = 1 \implies \boxed{A = -1}\]

• Subbing back into the first equation, we solve for B: \(B = -A = -(-1) = 1.\)
• We conclude that

\[
\frac{1}{(x-1)(x-2)} = \frac{-1}{x-1} + \frac{1}{x-2} = \frac{1}{x-2} - \frac{1}{x-1}.
\]

• We can now compute the integral:

\[
\int \frac{1}{x^2-3x+2} \, dx = \int \left( \frac{1}{x-2} - \frac{1}{x-1} \right) \, dx = \int \frac{1}{x-2} \, dx - \int \frac{1}{x-1} \, dx = \ln|x-2| - \ln|x-1| + C.
\]

• As always, check your answer:

\[
\frac{d}{dx} \ln|x-2| - \frac{d}{dx} \ln|x-1| + \frac{d}{dx} C = \frac{1}{x-2} - \frac{1}{x-1} + 0
\]

\[
= \frac{(x-1)-(x-2)}{(x-1)(x-2)}
\]

\[
= \frac{1}{x^2-3x+2}
\]
Example: Compute \( \int \frac{1}{(x-1)(x-2)(x-3)} \, dx \).

Solution: If we can find numbers \( A, B, C \) such that

\[
\frac{1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3},
\]

then

\[
\int \frac{1}{(x-1)(x-2)(x-3)} \, dx = \int \frac{A}{x-1} \, dx + \int \frac{B}{x-2} \, dx + \int \frac{C}{x-3} \, dx = A \ln |x-1| + B \ln |x-2| + C \ln |x-3| + D,
\]

and we're done.

To find \( A, B, C \), we can either solve the three linear equations for \( A, B, C \) arising from

\[
1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2),
\]

or use the following trick: substitute clever values for \( x \).
• Setting $x=1$ determines $A$:

$$| = A(1-2)(1-3) + B(1-1)(1-3) + C(1-1)(1-2)$$

$$\Rightarrow | = A(-1)(-2)$$

$$\Rightarrow [A = \frac{1}{2}]$$

• Setting $x=2$ determines $B$:

$$| = A(2-2)(2-3) + B(2-1)(2-3) + C(2-1)(2-2)$$

$$\Rightarrow | = B(1)(-1)$$

$$\Rightarrow [B = -1]$$

• Setting $x=3$ determines $C$:

$$| = A(3-2)(3-3) + B(3-1)(3-3) + C(3-1)(3-2)$$

$$\Rightarrow | = C(2)(1) \Rightarrow [C = \frac{1}{2}]$$