MATH 20B: Lecture 14

05/02/2016

54, 55, 75
• Last lecture we were discussing the integration of rational functions:

\[ \int \frac{P(x)}{Q(x)} \, dx. \]

• We formulated a two-part strategy to compute integrals of this type:

1) Factor the denominator into linear polynomials:

\[ Q(x) = (x-\alpha_1)(x-\alpha_2) \ldots (x-\alpha_N). \]

2) Decompose \( \frac{P(x)}{Q(x)} \) into a sum of rational functions with constant numerators and linear denominators:

\[ \frac{P(x)}{Q(x)} = \frac{A_1}{x-\alpha_1} + \frac{A_2}{x-\alpha_2} + \ldots + \frac{A_n}{x-\alpha_n}. \]
Whenever this two-part strategy can be carried out, we’ll have that

\[ \int \frac{P(x)}{Q(x)} \, dx = \int \frac{P(x)}{(x-a_1)(x-a_2)\ldots(x-a_n)} \, dx \]

\[ = \int \left( \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \ldots + \frac{A_n}{x-a_n} \right) \, dx \]

\[ = \int \frac{A_1}{x-a_1} \, dx + \int \frac{A_2}{x-a_2} \, dx + \ldots + \int \frac{A_n}{x-a_n} \, dx \]

\[ = A_1 \ln |x-a_1| + A_2 \ln |x-a_2| + \ldots + A_n \ln |x-a_n| + C. \]

Unfortunately, there are a couple of things that can go wrong; fortunately, these things are easily fixed.
• Cases where a patch is required:

1) Some of the numbers \( \alpha_1, \alpha_2, ..., \alpha_n \) aren’t distinct;

2) Some of the numbers \( \alpha_1, \alpha_2, ..., \alpha_n \) aren’t real;

3) The degree of \( P(x) \) exceeds the degree of \( Q(x) \);

4) Any combination of the above.
• Patches required:

1) Some of the numbers \(a_1, a_2, \ldots, a_n\) aren't distinct;
   Fix: simple modification of partial fraction expansion.

2) Some of the numbers \(a_1, a_2, \ldots, a_n\) aren't real;
   Fix: simple modification of partial fraction expansion.

3) The degree of \(P(x)\) exceeds the degree of \(Q(x)\);
   Fix: long division of \(Q(x)\) into \(P(x)\).
Example (Repeated Roots): Compute \( \int \frac{3x-9}{(x-1)(x+2)(x+2)} \, dx \).

Solution:
- The factorization of \( Q(x) \) is given in the problem (this will often be the case), and so Step 1 is complete.

- Step 2 is partial fractions; naively, we're looking for numbers \( A, B, C \) such that

\[
\frac{3x-9}{(x-1)(x+2)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x+2}.
\]

- But this can't be correct; the RHS is

\[
\frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x+2} = \frac{A}{x-1} + \frac{B+C}{x+2},
\]
and if we add these fractions using the lowest common denominator, we get

\[
\frac{A}{x-1} + \frac{B+C}{x+2} = \frac{A(x+2) + (B+C)(x-1)}{(x-1)(x+2)},
\]

which is missing a factor of \(x+2\) in the denominator.

• To fix this, we instead look for numbers \(A, B, C\) such that

\[
\frac{3x-9}{(x-1)(x+2)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}.
\]

• If we add the fractions on the RHS by making a lowest common denominator, we get

\[
\frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2} = \frac{A(x+2)^2 + B(x-1)(x+2) + C(x-1)}{(x-1)(x+2)(x+2)}.
\]
• Equating numerators in the equation

\[
\frac{3x-9}{(x-1)(x+2)(x+2)} = \frac{A(x+2)^2 + B(x-1)(x+2) + C(x-1)}{(x-1)(x+2)(x+2)}
\]

yields

\[3x-9 = A(x+2)^2 + B(x-1)(x+2) + C(x-1)\]

• If we set \(x = -2\) in this equation, we get

\[3(-2) - 9 = A(-2+2)^2 + B(-2-1)(-2+2) + C(-2-1)\]

\[\Rightarrow -15 = -3C\]

\[\Rightarrow C = 5\]
• If we set \( x = 1 \), we get

\[
3(1) - 9 = A(1+2)^2 + B(1-1)(1+2) + C(1-1)
\]

\[
\Rightarrow -6 = 9A
\]

\[
\Rightarrow A = -\frac{2}{3}
\]

• It seems that there is no way to isolate \( B \) in the equation

\[
3x - 9 = A(x+2)^2 + B(x-1)(x+2) + C(x-1),
\]

because if we set \( x = 1 \) we lose \( B \), and if we set \( x = -2 \) we again lose \( B \).

• The trick is to set \( x \) equal to anything other than these values, e.g. \( x = 2 \), and use the fact that we’ve already found \( A \) and \( B \):

\[
3(2) - 9 = -\frac{2}{3}(2+3)^2 + B(2-1)(2+2) + 5(2-1)
\]

\[
\Rightarrow B = \frac{2}{3}
\]
We can now complete the evaluation of the integral:

\[
\int \frac{3x-9}{(x-1)(x+2)(x+2)} \, dx = \int \left( \frac{-\frac{2}{3}}{x-1} + \frac{\frac{2}{3}}{x+2} + \frac{5}{(x+2)^2} \right) \, dx
\]

\[
= -\frac{2}{3} \int \frac{1}{x-1} \, dx + \frac{2}{3} \int \frac{1}{x+2} \, dx + 5 \int \frac{1}{(x+2)^2} \, dx
\]

\[
= -\frac{2}{3} \ln |x-1| + \frac{2}{3} \ln |x+2| + 5 \cdot (-1) \cdot \frac{1}{(x+2)} + C
\]

\[
= -\frac{2}{3} \ln |x-1| + \frac{2}{3} \ln |x+2| - 5 \frac{1}{(x+2)} + C.
\]

Exercise: check this by differentiating.
Example (Unreal roots): Compute \( \int \frac{18}{(x+3)(x+3i)(x-3i)} \, dx. \)

Solution: • As discussed last lecture, you can find numbers \( A, B, C \) such that

\[
\frac{18}{(x+3)(x+3i)(x-3i)} = \frac{A}{x+3} + \frac{B}{x+3i} + \frac{C}{x-3i},
\]

but \( B \) and \( C \) won’t be real, and the known integrals you want to rely on are false.

• Therefore, you need to consider the integrand in the form

\[
\frac{18}{(x+3)(x+3i)(x-3i)} = \frac{18}{(x+3)(x^2+9)},
\]

and look for a partial fractions decomposition of this object.
• Your first instinct is to seek numbers $A,B$ such that

$$\frac{18}{(x+3)(x^2+9)} = \frac{A}{x+3} + \frac{B}{x^2+9},$$

but this instinct is wrong; if you put the RHS back together you get

$$\frac{A}{x+3} + \frac{B}{x^2+9} = \frac{A(x^2+9) + B(x+3)}{(x+3)(x^2+9)} = \frac{Ax^2 + Bx + 9A + 3B}{(x+3)(x^2+9)},$$

and there's no way the numerator of this fraction can be equal to 18 (or any other number).
• Instead, you have to look for numbers $A,B,C$ such that

\[
\frac{18}{(x+3)(x^2+9)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+9}.
\]

• Making a common denominator on the RHS, this becomes

\[
\frac{18}{(x+3)(x^2+9)} = \frac{A(x^2+9) + (Bx+C)(x+3)}{(x+3)(x^2+9)}
\]

which forces

\[
18 = A(x^2+9) + (Bx+C)(x+3)
\]

\[
= Ax^2 + 9A + Bx^2 + 3Bx + Cx + 3C
\]

\[
= (A+B)x^2 + (3B+C)x + 9A+3C.
\]
• The identity \( 18 = (A+B)x^2 + (3B+C)x + 9A+3C \) in turn forces three equations:

\[
\begin{align*}
A + B &= 0 \\
3B + C &= 0 \\
9A + 3C &= 18
\end{align*}
\]

• Equation 1 says \( B = -A \); sub this into equation 2 to get:

\[
\begin{align*}
-3A + C &= 0 \\
9A + 3C &= 18
\end{align*}
\]

\( \Rightarrow -9A + 3C = 0 \quad \Rightarrow 6C = 18 \quad \Rightarrow C = 3 \)

• Sub \( C = 3 \) into equation 2 to get \( B = -1 \), then sub this into equation 1 to get \( A = 1 \).
We now return to the integral we want to evaluate:

\[
\int \frac{18}{(x+3)(x^2+9)} \, dx = \int \left( \frac{1}{x+3} - \frac{x-3}{x^2+9} \right) \, dx
\]

\[
= \int \frac{1}{x+3} \, dx - \int \frac{x-3}{x^2+9} \, dx
\]

\[
= \ln |x+3| - \int \frac{x}{x^2+9} \, dx + 3 \int \frac{1}{x^2+9} \, dx
\]

\[
= \ln |x+3| - \frac{1}{2} \int \frac{1}{u+q} \, du + 9 \int \frac{1}{(3v)^2+9} \, dv
\]

\[
= \ln |x+3| - \frac{1}{2} \ln |u+q| + \int \frac{1}{v^2+1} \, dv
\]

\[
= \ln |x+3| - \frac{1}{2} \ln |x^2+9| + \arctan(v) + C
\]

\[
= \ln |x+3| - \frac{1}{2} \ln |x^2+9| + \arctan \left( \frac{x}{3} \right) + C.
\]