• So far in Math 20B, we have used the definite integral \( \int_a^b f(x) \, dx \) as a means to both define and compute the area of special shapes.

\[ f(x) \]
\[ a \quad b \]

• We can get some additional mileage out of this by studying improper integrals:
\[ \int_a^\infty f(x) \, dx, \quad \int_0^a f(x) \, dx, \quad \int_{-\infty}^a f(x) \, dx. \]

• Improper integrals can be used to define and compute the area of certain unbounded regions.
• As an example, consider the function \( f(x) = \frac{1}{x^2} \)

\begin{align*}
\int_a^b x^{-2} \, dx &= -x^{-1} \bigg|_a^b \\
&= -\frac{1}{b} - \left(-\frac{1}{a}\right) \\
&= \frac{1}{a} - \frac{1}{b}.
\end{align*}

• For any \( 0 < a < b \), the area of the region \( R(a,b) \) is by definition \( \int_a^b x^{-2} \, dx \), and by FTC this number is
• Observe that \( \lim_{b \to \infty} \int_a^b \frac{1}{x^2} \, dx \) exists and is finite:

\[
\lim_{b \to \infty} \int_a^b \frac{1}{x^2} \, dx = \lim_{b \to \infty} \left( \frac{1}{a} - \frac{1}{b} \right) = \frac{1}{a} - \lim_{b \to \infty} \frac{1}{b} = \frac{1}{a}.
\]

• One writes \( \int_a^\infty \frac{1}{x^2} \, dx = \frac{1}{a} \) for this limit, and says that this improper integral converges.

• The convergent improper integral \( \int_a^\infty \frac{1}{x^2} \, dx = \frac{1}{a} \) may be taken as the definition of the area of the unbounded region \( R(a, \infty) \):

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\begin{tikzpicture}
  \draw[->] (-2,0) -- (2,0) node[right] {a};
  \draw[->] (0,-2) -- (0,2) node[above] {R(a, \infty)};
  \draw (-2,0) .. controls (-1,1) .. (0,0);
\end{tikzpicture}
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If we start with the similar integral
\[ \int_a^b x^{-1} \, dx = \ln(x) \bigg|_a^b = \ln(b) - \ln(a), \]
this argument fails.

Indeed, we have
\[ \lim_{b \to \infty} \int_a^b \frac{1}{x} \, dx = \lim_{b \to \infty} (\ln(b) - \ln(a)) = \lim_{b \to \infty} \ln(b) - \ln(a) = \infty, \]
so \[ \int_a^\infty \frac{1}{x} \, dx = \infty. \]

We say that the improper integral \[ \int_a^\infty \frac{1}{x} \, dx \] is divergent, and that the area of the region \( R(a, \infty) \) below is infinite:
• We have now looked at two improper integrals,

\[ \int_a^\infty \frac{1}{x^2} \, dx \quad \text{and} \quad \int_a^\infty \frac{1}{x} \, dx. \]

• The first is convergent, \( \int_a^\infty \frac{1}{x^2} \, dx = \frac{1}{a} \), while the second is divergent, \( \int_a^\infty \frac{1}{x} \, dx = \infty \).

• Members of the family

\[ \int_a^\infty \frac{1}{x^p} \, dx, \quad p > 0, \]

are known as "p-integrals."
Theorem: The $p$-integral $\int_a^\infty \frac{1}{x^p} \, dx$ is convergent if and only if $p > 1$.

Proof: • Suppose first that $p > 1$. Then, for any $0 < a < b$,

$$\int_a^b x^{-p} \, dx = \left[ \frac{x^{1-p}}{1-p} \right]^b_a = \frac{b^{1-p}}{1-p} - \frac{a^{1-p}}{1-p}.$$

• Hence, $\lim_{b \to \infty} \int_a^b x^{-p} \, dx = \lim_{b \to \infty} \frac{b^{1-p}}{1-p} - \frac{a^{1-p}}{1-p}$

$$= \lim_{b \to \infty} \frac{b^{\text{neg} \#}}{1-p} - \frac{a^{1-p}}{1-p}$$

$$= 0 - \frac{a^{1-p}}{1-p}.$$

• So, $p > 1$ implies

$$\int_a^\infty \frac{1}{x^p} \, dx = \frac{1}{(p-1) a^{p-1}}.$$
• Now we prove the converse: if \( p \leq 1 \), then \( \int_a^\infty \frac{1}{x^p} \, dx \) is divergent.

• If \( p = 1 \), we've already done this, so take \( p < 1 \).

• Then,

\[
\lim_{b \to \infty} \int_a^b x^{-p} \, dx = \lim_{b \to \infty} \frac{b^{1-p}}{1-p} - \frac{a^{1-p}}{1-p} \\
= \lim_{b \to \infty} \frac{b^{\text{pos}}} {1-p} - \frac{a^{1-p}} {1-p} \\
= \infty - \frac{a^{1-p}} {1-p} \\
= \infty.
\]
Example: How much work is required to transport an object from the surface of the Earth to an infinitely far away location?

Solution: • The gravitational force which the Earth exerts on an object of mass \( m \) located at distance \( x \) from its center is

\[
f(x) = \frac{GMm}{x^2},
\]

where \( M \approx 6 \times 10^{24} \) kg is the mass of the Earth, and \( G \) is the universal gravitational constant.

• So the work in question is

\[
W = \int_{R}^{\infty} \frac{GMm}{x^2} \, dx = GMm \int_{R}^{\infty} \frac{1}{x^2} \, dx = \frac{GMm}{R},
\]

where \( R \approx 6 \times 10^6 \) m is the radius of Earth.
Note that an improper integral $\int_a^\infty f(x) \, dx$ can be divergent even if it doesn’t "blow up", i.e. even if there is a number $M$ such that
\[
| \int_a^b f(x) \, dx | \leq M
\]
for all $b > a$.

Indeed, consider the improper integral $\int_0^\infty \cos(x) \, dx$. We have
\[
\int_0^b \cos(x) \, dx = \sin(x) \bigg|_0^b = \sin(b),
\]
which is always a number between $-1$ and $1$.

However, $\lim_{b \to \infty} \sin(b)$ doesn’t exist, so the improper integral $\int_0^\infty \cos(x) \, dx$ is divergent.
• Let’s return to the function \( f(x) = \frac{1}{x^2} \)

![Graph of \( f(x) = \frac{1}{x^2} \) with limits at \( a \) and \( b \)]

\[ R(a, b) \]

• There is another improper integral associated to this function, namely \( \int_{0}^{b} f(x) \, dx \).

• This might not look improper, but it is, because \( f(x) = \frac{1}{x^2} \) is not defined at \( x = 0 \).

• Consequently, \( \int_{0}^{b} \frac{1}{x^2} \, dx \) must be interpreted as the limit \( \lim_{a \to 0^+} \int_{a}^{b} \frac{1}{x^2} \, dx \), which may or may not exist.
• For any \(0 < a < b\), we have
\[
\int_a^b \frac{1}{x^2} \, dx = \int_a^b x^{-2} \, dx = -x^{-1} \bigg|_a^b = \frac{1}{a} - \frac{1}{b}.
\]

• Thus,
\[
\lim_{a \to 0^+} \int_a^b \frac{1}{x^2} \, dx = \lim_{a \to 0^+} \left( \frac{1}{a} - \frac{1}{b} \right) = \lim_{a \to 0^+} \frac{1}{a} - \frac{1}{b} = \infty - \frac{1}{b} = \infty,
\]
so the improper integral \(\int_0^b \frac{1}{x^2} \, dx\) is divergent.

• So, the unbounded region \(R(0,b)\) must be assigned infinite area.
• But now suppose we consider the function \( f(x) = \frac{1}{x^{1/2}} \) and the associated improper integral

\[
\int_a^b \frac{1}{x^{1/2}} \, dx.
\]

• Since \( f(x) = \frac{1}{x^{1/4}} \) is the inverse of \( g(x) = \frac{1}{x^2} \), i.e.

\[
f(g(x)) = \frac{1}{(\frac{1}{x^2})^{1/2}} = x \quad \text{and} \quad g(f(x)) = \frac{1}{(\frac{1}{x^{1/4}})^2} = x,
\]

the graph of \( f(x) \) is just the reflection of the graph of \( g(x) \) in the line \( y = x \).

• This means that

\[
\int_a^b \frac{1}{x^{1/2}} \, dx = \int_b^a \frac{1}{x^2} \, dx = \frac{1}{b},
\]

so \( \int_0^b \frac{1}{x^{1/2}} \, dx \) is convergent and we know its value.
Theorem: The improper $p$-integral $\int_0^b \frac{1}{x^p} \, dx$ is convergent if and only if $p < 1$.

Proof: • If $p \neq 0$, then we can write $p = \frac{1}{q}$, and

$$\int_0^b \frac{1}{x^p} \, dx = \int_0^b \frac{1}{x^{1/q}} \, dx = \int_b^\infty \frac{1}{x^{1/q}} \, dx.$$ 

• But we know that the improper integral $\int_b^\infty \frac{1}{x^{1/q}} \, dx$ converges if and only if $q > 1$, which is true if and only if $p < 1$.

• If $p = 0$, then we're looking at the integral $\int_0^b 1 \, dx$, which is not improper at all:

$$\int_0^b 1 \, dx = b.$$
Example: Is the improper integral \( \int_{1}^{\infty} \frac{1}{x^2 + 1} \, dx \) convergent, or divergent.
Solution 1. • For any \( 1 < b \), we have that

\[
\int_1^b \frac{1}{x^2 + 1} \, dx = \arctan(x) \bigg|_1^b = \arctan(b) - \arctan(1) = \arctan(b) - \frac{\pi}{4}.
\]

• Since

\[
\lim_{b \to \infty} \arctan(b) = \frac{\pi}{2},
\]

we have that

\[
\lim_{b \to \infty} \int_1^b \frac{1}{x^2 + 1} \, dx = \lim_{b \to \infty} \arctan(b) - \frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.
\]

• Conclusion: \( \int_1^\infty \frac{1}{x^2 + 1} \, dx = \frac{\pi}{4} \).
Solution 2: While we're not sure about \( \int_{1}^{\infty} \frac{1}{x^2+1} \, dx \), we've already seen that 
\[
\int_{1}^{\infty} \frac{1}{x^2} \, dx = 1.
\]

- Now, for any \( x > 1 \), it's certainly true that \( \frac{1}{x^2+1} < \frac{1}{x^2} \).

- Thus,
\[
\int_{1}^{b} \frac{1}{1+x^2} \, dx < \int_{1}^{b} \frac{1}{x^2} \, dx
\]
for all \( b > 1 \), whence
\[
\lim_{b \to \infty} \int_{1}^{b} \frac{1}{1+x^2} \, dx < \lim_{b \to \infty} \int_{1}^{b} \frac{1}{x^2} \, dx = \int_{1}^{\infty} \frac{1}{x^2} \, dx = 1.
\]

- This means that \( \int_{1}^{\infty} \frac{1}{1+x^2} \, dx \) converges to some number less than 1.
Theorem (Comparison Test): If $0 \leq f(x) \leq g(x)$ for all $x > a$, and if
$$\int_a^\infty g(x) \, dx$$
converges, then
$$\int_a^\infty f(x) \, dx$$
converges as well.

Pseudoproof:
Example: Does the integral $\int_{1}^{\infty} e^{-x^2} \, dx$ converge?

Solution: • It’s absolutely true that $e^{-x^2} \leq e^{-x}$ for all $x \geq 1$.

• Indeed

\[ x \geq 1 \Rightarrow x^2 \geq x \]
\[ \Rightarrow -x^2 \leq -x \]
\[ \Rightarrow e^{-x^2} \leq e^{-x}. \]

• Now,

\[ \int_{1}^{b} e^{-x} \, dx = -e^{-x} \bigg|_{1}^{b} = e^{-1} - e^{-b} \]

so \[ \int_{1}^{\infty} e^{-x} \, dx = e^{-1}. \]

• Consequently, \[ \int_{1}^{\infty} e^{-x^2} \, dx < \infty, \] by the Comparison Test.