A MOMENT METHOD FOR INVARIANT ENSEMBLES

SHO MATSUMOTO AND JONATHAN NOVAK

Abstract. We introduce a new moment method in Random Matrix Theory specifically tailored to the spectral analysis of invariant ensembles. Our method produces a classification of invariant ensembles which exhibit a spectral Law of Large Numbers, and yields an explicit description of the limiting eigenvalue distribution when it exists. We discuss the future development of this new moment method.

1. Introduction

Random Matrix Theory (RMT) is one of the most active research topics in contemporary probability theory. The goal of the subject is natural and compelling: given a random matrix, describe the statistical behavior of its eigenvalues. In addition to its intrinsic mathematical appeal, interest in RMT has been spurred by the scientific hypothesis that spectra of large random matrices yield models for complex systems comprised of many highly correlated components. Such systems are ubiquitous in mathematics and nature — particular examples include zeros of $L$-functions [9], energy levels of atomic nuclei [23], and arrival times of New York City subway trains [12] — but are not within the purview of classical scalar and vector-valued probability, whose limit theorems describe systems built from weakly correlated components.

The universe of random matrix models is too large to be studied all at once, and in practice it is parceled out into various paradigms. Among the most prominent of these is the invariant paradigm, which is populated by statistical ensembles of three types. A real invariant ensemble is a sequence

$$X^{(N)} = \begin{bmatrix} \cdots & X_{ij}^{(N)} & \cdots \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots \\ \end{bmatrix}^{N}, \quad N = 1, 2, 3, \ldots,$$

of random real selfadjoint matrices such that, for each $N \in \mathbb{N}$ and any $N \times N$ orthogonal matrix $O$, the distribution of $X^{(N)}$ coincides with that of $OX^{(N)}O^{-1}$. Similarly, a complex invariant ensemble is a sequence of random complex selfadjoint matrices each of which has distribution invariant under conjugation by unitary matrices, and a quaternionic invariant ensemble is a sequence of random quaternionic selfadjoint matrices each of which has distribution invariant under conjugation by symplectic matrices. Conjugation invariance is a physically natural assumption (all coordinate systems are equivalent) which also has a natural probabilistic interpretation: it means that the distribution of $X^{(N)}$ is uniform conditional on its eigenvalues.
$E_1^{(N)} \geq \cdots \geq E_N^{(N)}$.

Invariant ensembles of random matrices have been the focus of intense study for decades; see [7, 8, 10, 13] and references therein. However, these investigations have almost exclusively focused on a special class of invariant ensembles: those in which the distribution of $X^{(N)}$ is absolutely continuous with respect to Lebesgue measure, with density proportional to a function of the form $\exp(-\beta^2 N \text{Tr} V(X))$. Here $\text{Tr}$ denotes the matrix trace, $V$ is a sufficiently well-behaved real-valued function, and $\beta$ is the Dyson index, which is equal to 1, 2, or 4 according to whether the ensemble is real, complex, or quaternionic. The fixation on this special class of invariant ensembles stems from the fact that the joint density of eigenvalues is known explicitly, being proportional to $\exp(-\frac{2}{\beta^2} N^2 H)$, where $H$ is the Hamiltonian

$$H(E_1, \ldots, E_N) = \frac{1}{N} \sum_{i=1}^{N} V(E_i) - \frac{1}{N^2} \sum_{i \neq j} \log |E_i - E_j|.$$

This formula furnishes a physical interpretation of the spectrum as a system of two-dimensional electrostatic charges living on a wire, with $V$ playing the role of a confining potential. This system may be analyzed directly using a variety of powerful analytic techniques, leading to a detailed understanding of its macroscopic and microscopic statistics in the large $N$ limit; see [8, 9] for the state of the art.

The Dyson-type ensembles form a small island in the vast sea of all invariant ensembles. In this note, we suggest a new approach to the spectral analysis of general invariant ensembles, the implementation of which will both broaden and deepen current understanding of this important paradigm. This new approach is based on a simple observation: the distribution of any conjugation-invariant random selfadjoint matrix is completely determined by the joint distribution of its diagonal matrix elements. This fundamental consequence of invariance seems to have been overlooked; certainly, it has never been exploited. Nevertheless, it is easily seen from the Fourier transform,

$$A \mapsto \mathbb{E}[e^{i \text{Tr} AX^{(N)}}].$$

Indeed, diagonalizing the selfadjoint matrix $A$, cyclic invariance of the trace and conjugation invariance of $X^{(N)}$ immediately imply that

$$\mathbb{E}[e^{i \text{Tr} AX^{(N)}}] = \mathbb{E}[e^{i(h(a_1 X_{11}^{(N)} + \cdots + a_N X_{NN}^{(N)})},$$

where $a_1, \ldots, a_N$ is any enumeration of the eigenvalues of $A$. Thus, everything one could hope to know about $X^{(N)}$ is encoded in the joint distribution of the real random variables $X_{11}^{(N)}, \ldots, X_{NN}^{(N)}$, which are identically distributed and exchangeable. This reduces the spectral analysis of invariant ensembles to extracting eigenvalue statistics from the joint distribution of diagonal matrix elements.

2. Statement of Results

For a general selfadjoint ensemble $X^{(N)}$, in the absence of an analytical description of the eigenvalues, one must interact with the spectrum via an appropriate system of observables. The random variables
which generate the algebra of symmetric polynomial functions of the eigenvalues, are a natural choice. Indeed, they are precisely the moments of the empirical eigenvalue distribution of $X^{(N)}$. In RMT, the term moment method refers to any technique which relates the distribution of spectral moments to the joint distribution of matrix elements. The prototypical moment method was introduced by Wigner in the 1950s, and is still widely used today; see [26]. Wigner’s moment method applies to ensembles of random selfadjoint matrices whose matrix elements enjoy a high degree of independence; these Wigner ensembles constitute a random matrix paradigm which is, in a sense, orthogonal to the invariant paradigm. Wigner used his moment method to show that, under appropriate hypotheses, the spectral moments $p^{(N)}_d$ of a Wigner ensemble $X^{(N)}$ tend to deterministic limits $p^{(\infty)}_d$ as $N \to \infty$. This is the random matrix version of the Law of Large Numbers.

In this note, we present a new moment method specifically tuned to the invariant paradigm. As explained above, conjugation invariance means that we need only consider the joint distribution of diagonal matrix elements. We will say that an invariant ensemble $X^{(N)}$ is smooth if $X^{(N)}_{11}, \ldots, X^{(N)}_{NN}$ admit joint moments of all orders. In this case, thanks to exchangeability, it suffices to consider joint moments indexed by Young diagrams. More precisely, to a given Young diagram $\lambda$ with $|\lambda| = d$ cells and $\ell(\lambda) = r$ rows, we associate the degree $d$ joint moment of $X^{(N)}_{11}, \ldots, X^{(N)}_{rr}$ defined by

$$m_d(X^{(N)}_{11}, \ldots, X^{(N)}_{11}, \ldots, X^{(N)}_{rr}, \ldots, X^{(N)}_{rr}) = \mathbb{E} \left[ \prod_{i=1}^{r} (X^{(N)}_{ii})^{\lambda_i} \right].$$

We denote this joint moment by $m^{(N)}_{\lambda}$. For example, if $\lambda$ is the diagram

$$\begin{array}{cccc}
\ast & & & \\
& \ast & & \\
& & \ast & \\
& & & \ast
\end{array},$$

then

$$m^{(N)}_{\lambda} = \mathbb{E} \left[ (X^{(N)}_{11})^{5} (X^{(N)}_{22})^{3} (X^{(N)}_{33})^{2} \right].$$

Each diagonal matrix element of $X^{(N)}$ can be decomposed as

$$X^{(N)}_{ii} = \sum_{j=1}^{N} U^{(N)}_{ij} E^{(N)}_{j} \overline{U}^{(N)}_{ij},$$

where $U^{(N)} = [U^{(N)}_{ij}]_{i,j=1}^{N}$ is a random matrix whose distribution is the Haar probability measure on $O(N)$, $U(N)$, or $Sp(N)$ according to whether $X^{(N)}$ is real, complex, or quaternionic. This decomposition of $X^{(N)}_{ii}$ can be substituted into the definition of $m^{(N)}_{\lambda}$, and the eigenvector information can be “integrated out” using the Weingarten Calculus, a unified set of tools for the evaluation of polynomial
integrals on compact topological groups; see [3, 5, 4]. What remains is a presentation of $m^{(N)}_\lambda$ as the expectation of a certain polynomial in the spectral moments of $X^{(N)}$. This polynomial becomes more tractable if one applies a basic statistical principle, which may be traced to the nineteenth century astronomer Thorvald N. Thiele [17]: cumulants package the same information as moments in a more useful way. We thus trade $m^{(N)}_\lambda$ for the corresponding joint cumulant $c^{(N)}_\lambda$. For example, with $\lambda$ as in the previous example, $c^{(N)}_\lambda$ is the coefficient of
\[ a_5^5 a_3^3 a_2^2 a_1 \]
in the logarithm of the characteristic function of the random vector $(X^{(N)}_{11}, X^{(N)}_{22}, X^{(N)}_{33})$.

Execution of this strategy leads to the following result, which gives necessary and sufficient conditions for the emergence of a spectral Law of Large Numbers within the class of smooth invariant ensembles.

**Theorem 2.1.** For any smooth invariant ensemble $X^{(N)}$, the following are equivalent:

1. For each positive integer $d$, the random variable $p^{(N)}_d$ converges, in probability, to a constant $p^{(\infty)}_d$;
2. For each Young diagram $\lambda$, the number $N^{1|\lambda|-1} c^{(N)}_\lambda$ converges to a limit $c^{(\infty)}_\lambda$, and this limit vanishes if $\ell(\lambda) > 1$.

Recalling that vanishing of mixed cumulants characterizes independence, Theorem 2.1 says that the moments of the empirical eigenvalue distribution of $X^{(N)}$ converge in probability to deterministic limits precisely when each diagonal element converges rapidly to a constant, and distinct diagonal elements rapidly decouple.

A detailed proof of Theorem 2.1 implementing the strategy described above will appear in [18]. This proof yields additional information, namely a precise relationship between the numerical sequences $p^{(\infty)}_1, p^{(\infty)}_2, p^{(\infty)}_3, \ldots$ and $c^{(\infty)}_1, c^{(\infty)}_2, c^{(\infty)}_3, \ldots$.

It turns out that this relationship may be concisely described in the language of Free Probability Theory, a highly noncommutative probability theory developed by Voiculescu to address a famous unsolved problem in the theory of von Neumann algebras; see [25]. A fundamental tool in Free Probability is a bijective transform on sequences which plays the same role as the moment-to-cumulant transform in classical probability. Applying this $R$-transform to any sequence produces a corresponding unique sequence of free cumulants; see [19, 20, 21].

**Theorem 2.2.** Let $X^{(N)}$ be a smooth invariant ensemble. Then

\[ R(p^{(\infty)}_1, p^{(\infty)}_2, p^{(\infty)}_3, \ldots) = \gamma^0 0! c^{(\infty)}_1, \gamma^1 1! c^{(\infty)}_2, \gamma^2 2! c^{(\infty)}_3, \ldots, \]

where $\gamma = \beta/2$ is one half the Dyson index.

### 3. Applications

Theorems 2.1 and 2.2 together form a “skeleton key” result which can be used to recover many seemingly disparate theorems in RMT — and even some which seem not to be part of RMT at all — in a unified way. We illustrate this via several examples.
3.1. Gaussian ensembles. Let $X^{(N)}$ be an invariant ensemble such that $X_{11}^{(N)}, \ldots, X_{NN}^{(N)}$ are independent Gaussians of mean $c_1$ and variance $c_2(N)^{-1}$, with $c_1, c_2$ constants. By Gaussianity, all pure cumulants of order higher than two vanish. Moreover, mixed cumulants are identically zero by independence. We thus have existence of the limits $c_{\lambda}^{(\infty)}$ for all Young diagrams $\lambda$. More precisely, we have

$$c_1^{(\infty)} = c_1, \quad c_2^{(\infty)} = c_2,$$

and all other $c_{\lambda}^{(\infty)}$ are zero. Theorem 2.1 thus implies that each spectral moment $p_d^{(N)}$ of $X^{(N)}$ converges in probability to a deterministic limit $p_d^{(\infty)}$. Theorem 2.2 yields the $R$-transform of the limiting moment sequence

$$R(p_1^{(\infty)}, p_2^{(\infty)}, p_3^{(\infty)}, \ldots) = (c_1, c_2, 0, 0, \ldots).$$

There is a unique probability measure $\mu^{(\infty)}$ with this $R$-transform: the Wigner semicircle distribution with mean $c_1$ and variance $c_2$. 

3.2. Wishart ensembles. For each $N$, let $Z^{(N)}$ be a $p \times N$ random matrix whose matrix entries are iid Gaussians with mean 0 and variance $\alpha(N)^{-1}$, with $\alpha$ constant. One then has an affiliated invariant ensemble, known as a Wishart ensemble, whose $N$th member is $X^{(N)} = (Z^{(N)})^*(Z^{(N)})$. Mixed cumulants of the diagonal matrix elements $X_{11}^{(N)}, \ldots, X_{NN}^{(N)}$ are identically zero by independence. Moreover, it is easy to compute the pure cumulants of a single diagonal element: using the Wick formula, one obtains

$$c_d^{(N)} = p\gamma^{1-d} \left( \frac{\alpha}{N} \right)^d (d-1)!, \quad d \in \mathbb{N}.$$ 

Suppose now that $p = p_N$ grows with $N$ in such a way that the limit

$$c = \lim_{N \to \infty} \frac{p}{N}$$

exists. Then, we have

$$c_d^{(\infty)} = \lim_{N \to \infty} N^{d-1}c_d^{(N)} = c\gamma^{1-d}\alpha^d(d-1)!.$$ 

Theorem 2.1 thus implies that each spectral moment $p_d^{(N)}$ of $X^{(N)}$ converges in probability to a deterministic limit $p_d^{(\infty)}$. Theorem 2.2 yields the $R$-transform of the limiting moment sequence:

$$R(p_1^{(\infty)}, p_2^{(\infty)}, p_3^{(\infty)}, \ldots) = (c\alpha, c\alpha^2, c\alpha^3, \ldots).$$

There is a unique probability measure $\mu^{(\infty)}$ with this $R$-transform: the Marchenko-Pastur distribution with rate $c$ and jump size $\alpha$.

3.3. Sum of independent ensembles. Let $X^{(N)}$ and $Y^{(N)}$ be independent smooth invariant ensembles, and suppose it is known that the spectral moments of each converge in probability to deterministic limits:

$$p_d(X^{(N)}) \to x_d \quad \text{and} \quad p_d(Y^{(N)}) \to y_d.$$
From this data, we obtain a new smooth invariant ensemble defined by setting $Z^{(N)} := X^{(N)} + Y^{(N)}$ for each $N \in \mathbb{N}$. Given a Young diagram $\lambda$, the relationship between the corresponding joint cumulants of the diagonal matrix elements of $Z^{(N)}, X^{(N)}, Y^{(N)}$ is, by independence, simply

$$c_\lambda(Z^{(N)}) = c_\lambda(X^{(N)}) + c_\lambda(Y^{(N)}).$$

Now, since the spectral moments of $X^{(N)}$ and $Y^{(N)}$ are known to converge to deterministic limits, Theorem 2.1 implies existence of the limits

$$c_\lambda(X) := \lim_{N \to \infty} N^{\lambda|\lambda|-1} c_\lambda(X^{(N)}),$$
$$c_\lambda(Y) := \lim_{N \to \infty} N^{\lambda|\lambda|-1} c_\lambda(Y^{(N)}),$$

with these limits vanishing if $\ell(\lambda) > 1$. We thus obtain

$$c_\lambda(Z) = \lim_{N \to \infty} N^{\lambda|\lambda|-1} c_\lambda(Z^{(N)}) = c_\lambda(X) + c_\lambda(Y),$$

so that by Theorem 2.1 we have convergence in probability of the spectral moments $p_d(Z^{(N)})$ to deterministic limits $z_d$. Moreover, by Theorem 2.2, the limit

$$R(z_1, z_2, z_3, \ldots) = R(x_1, x_2, x_3, \ldots) + R(y_1, y_2, y_3, \ldots).$$

3.4. Compressed ensembles. Let $X^{(N)}$ be a smooth invariant ensemble, and suppose it is known that the spectral moments $p_d^{(N)}$ of $X^{(N)}$ converge in probability to deterministic limits $p_d^{(\infty)}$. For any choice of $t \in (0, 1)$, one obtains a new smooth invariant ensemble whose $N$th member $X^{(N)}_{[tN]}$ is the $[tN] \times [tN]$ principal submatrix of $X^{(N)}$. Let $p_d^{(N)}$ denote the spectral moments of $X^{(N)}_{[tN]}$. For any Young diagram $\lambda$, the corresponding joint cumulants $c^{(N)}_\lambda$ and $c^{(N)}_{\lambda,t}$ of the diagonal matrix elements of $X^{(N)}$ and $X^{(N)}_{[tN]}$ are well-defined and equal for $\lfloor tN \rfloor \geq \ell(\lambda)$. By Theorem 2.1, the limit

$$c^{(\infty)}_\lambda = \lim_{N \to \infty} N^{\lambda|\lambda|-1} c^{(N)}_\lambda$$

exists and vanishes if $\ell(\lambda) > 1$. Moreover,

$$c^{(\infty)}_{\lambda,t} = \lim_{N \to \infty} ([tN])^{\lambda|\lambda|-1} c^{(N)}_{\lambda,t}$$
$$= t^{\lambda|\lambda|-1} \lim_{N \to \infty} N^{\lambda|\lambda|-1} c^{(N)}_\lambda$$
$$= t^{\lambda|\lambda|-1} c^{(\infty)}_\lambda.$$ 

Thus, Theorem 2.1 implies that each $p_d^{(N)}$ converges to a deterministic limit $p_d^{(\infty)}$, and Theorem 2.2 yields

$$R(p_{1,t}^{(\infty)}, p_{2,t}^{(\infty)}, p_{3,t}^{(\infty)}, \ldots) = \left(\frac{\gamma t}{0!}c_1^{(\infty)}, \frac{\gamma t}{1!}c_2^{(\infty)}, \frac{(\gamma t)^2}{2!}c_3^{(\infty)}, \ldots\right).$$
3.5. **Random lozenge tilings.** Let

\[
\begin{array}{ccc}
\text{ }& b_1^{(1)} & \text{ } \\
\text{ }& b_1^{(2)} & b_2^{(2)} \\
\text{ }& b_1^{(3)} & b_2^{(3)} & b_3^{(3)} \\
\vdots & \vdots & \vdots & \ddots
\end{array}
\]

be a random triangular array of integers whose elements are strictly decreasing along rows. This random data gives rise to a sequence $\Omega^{(N)}$ of random planar domains via the following construction. Fix a coordinate system in the plane whose axes meet at a 120° angle. We specify $\Omega^{(N)}$ by constructing its boundary, which consists of two piecewise linear components, one deterministic and one random. The deterministic component of $\partial \Omega^{(N)}$ is simply the horizontal axis in the plane. The random component is built in three steps. First, construct the line parallel to the lower boundary passing through the point $(0,N)$. Second, affix $N$ outward-facing unit triangles to this line such that the midpoints of their bases have horizontal coordinates $b_1^{(N)} > \cdots > b_k^{(N)}$. Finally, erase the bases of these triangles. We will refer to $\Omega^{(N)}$ as the *sawtooth domain* of rank $N$ with boundary conditions $(b_1^{(N)}, \ldots, b_N^{(N)})$.

A *lozenge* is a unit rhombus in the plane whose sides are parallel to one of the coordinate axes, or to the line bisecting the obtuse angle between them. Lozenges are thus divided into three classes: left-leaning, right-leaning, and vertical. Each instance of the random domain $\Omega^{(N)}$ can be tiled with lozenges in finitely many ways, an example being given in the figure below. Consequently, we may consider a random tiling $T^{(N)}$ of $\Omega^{(N)}$ whose distribution is uniform conditional on its boundary conditions. For each instance of $T^{(N)}$, and any integer $1 \leq k \leq N$, the horizontal line through $(0,k)$ passes through exactly $k$ vertical tiles, as in the figure below. Moreover, the entire tiling can be reconstructed given only the locations of the vertical tiles. The positions of the vertical tiles on adjacent lines interlace, a feature which is reminiscent of Cauchy interlacing for the eigenvalues of a selfadjoint matrix and its principal submatrices.

Let us associate to the random tiling $T^{(N)}$ a finite sequence of unitarily invariant random Hermitian matrices defined by

\[
X_k^{(N)} = U_k \begin{bmatrix} b_{k1}^{(N)} \\ \vdots \\ b_{kk}^{(N)} \end{bmatrix} U_k^{-1}, \quad 1 \leq k \leq N,
\]

where $U_k$ is a Haar-distributed random $k \times k$ unitary matrix and $b_{k1}^{(N)} > \cdots > b_{kk}^{(N)}$ are the horizontal coordinates of the centroids of the vertical tiles on the horizontal line through $(0,k)$. It is tempting to hope that the distribution of $X_k^{(N)}$ coincides with the distribution of the $k \times k$ principal submatrix of $X_N^{(N)}$. In this case, the joint distribution of the diagonal matrix elements $(X_k^{(N)})_{11}, \ldots, (X_k^{(N)})_{kk}$ of $X_k^{(N)}$ would coincide with the joint distribution of the first $k$ diagonal matrix elements $(X_N^{(N)})_{11}, \ldots, (X_N^{(N)})_{kk}$ of $X_N^{(N)}$ and we would be in the setting of the
previous example. This is not quite the case — the $k$-dimensional random vector \((X_k^{(N)})_{11}, \ldots, (X_k^{(N)})_{kk}\) has the same distribution as

\[
((X_N^{(N)})_{11}, \ldots, (X_N^{(N)})_{kk}) - \sum_{i=1}^{N-k} (Z_{i1}, \ldots, Z_{ik}),
\]

where

\[
Z_k^{(N)} = \begin{bmatrix}
Z_{11} & \cdots & Z_{1k} \\
\vdots & \ddots & \vdots \\
Z_{N-k,1} & \cdots & Z_{N-k,k}
\end{bmatrix}
\]
is an \((N-k) \times k\) random matrix independent of \(T^{(N)}\) whose entries \(Z_{ij}\) are iid uniformly random samples from the unit interval \([0, 1]\). For a proof of this, see [22].

Suppose it is known that the spectral moments \(p_{d}^{(N)}\) of \(N^{-1}X_{N}^{(N)}\) converge in probability to deterministic limits \(p_{d}^{(\infty)}\), these numbers being the moments of the “limit profile” of \(N^{-1}\Omega^{(N)}\). Fix \(t \in (0, 1)\), and let \(p_{d,t}^{(N)}\) denote the spectral moments of \(N^{-1}X_{\lfloor tN \rfloor}^{(N)}\). For any Young diagram \(\lambda\), the corresponding joint cumulants \(c_{\lambda}^{(N)}\) and \(c_{\lambda, t}^{(N)}\) of \(N^{-1}X_{N}^{(N)}\) and \(N^{-1}X_{\lfloor tN \rfloor}^{(N)}\) are well-defined for \(\lfloor tN \rfloor \geq \ell(\lambda)\). By the above, the relation between these joint cumulants is

\[
c_{\lambda, t}^{(N)} = c_{\lambda}^{(N)} - \delta_{1, t(\lambda)} (N - \lfloor tN \rfloor) c_{\lambda} |(N^{-1}Z)|
\]

where \(Z\) is a single uniformly random sample from the unit interval. By Theorem 2.1 the limit

\[
c_{\lambda}^{(\infty)} = \lim_{N \to \infty} N^{|\lambda|-1} c_{\lambda}^{(N)}
\]
exists and vanishes if \(\ell(\lambda) > 1\). We thus obtain

\[
c_{\lambda, t}^{(\infty)} = \lim_{N \to \infty} \lfloor tN \rfloor^{|\lambda|-1} c_{\lambda, t}^{(N)}
= \lfloor tN \rfloor^{|\lambda|-1} c_{\lambda}^{(\infty)} - \delta_{1, t(\lambda)} (1 - t) n^{|\lambda|-1} c_{\lambda} (Z),
\]
so that \(p_{d,t}^{(N)}\) converges in probability to a deterministic limit \(p_{d,t}^{(\infty)}\), by Theorem 2.1. The numbers
$p_{d,t}^{(\infty)}$, $d \in \mathbb{N}$, $t \in (0,1)$, are the moments of the “limit shape” of $N^{-1}T^{(N)}$. From Theorem 2.2 we obtain

$$R(p_{1,t}^{(\infty)}, p_{2,t}^{(\infty)}, p_{3,t}^{(\infty)}, \ldots) = \left( \frac{t^0}{0!} c_1^{(\infty)}, \frac{t^1}{1!} c_2^{(\infty)}, \frac{t^2}{2!} c_3^{(\infty)}, \ldots \right)$$

$$- (1-t) \left( \frac{t^0}{0!} u_1, \frac{t^1}{1!} u_2, \frac{t^2}{2!} u_3, \ldots \right),$$

where $u_1, u_2, u_3, \ldots$ is the cumulant sequence of $Z$ (these numbers are Bernoulli numbers). This provides a description of the limit shape of $N^{-1}T^{(N)}$ in terms of the limit profile of $N^{-1}\Omega^{(N)}$.

4. Conclusion

We have outlined a new moment method in Random Matrix Theory specifically tailored to the invariant ensembles. The method is based on the observation that, if the distribution of a random selfadjoint matrix is invariant under conjugation, then it is completely determined by the joint distribution of its diagonal matrix elements, which form a family of real, identically distributed, exchangeable random variables. When these random variables admit joint moments of all orders, the Weingarten calculus may be used to recognize them as statistics of spectral moments. This leads to a characterization of smooth invariant ensembles which exhibit a spectral Law of Large Numbers, and a formula for the limiting spectral moments when they exist. The utility of these results was illustrated via a number of significant examples from RMT, as well as an example from 2D statistical physics which is not a priori related to random matrices. There is much more that can be done with this idea: for example, one can address fluctuations of the eigenvalues of smooth invariant ensembles, thereby producing a Central Limit Theorem to accompany Theorem 2.1 or carry out a moment method analysis of the largest eigenvalue in smooth invariant ensembles analogous to Soshnikov’s analysis of the spectral edge in Wigner ensembles; see [24]. Another extremely interesting direction is the generalization of this moment method to the multimatrix setting. These directions will be the subject of future work.

Let us conclude with a discussion of how our moment method for invariant ensembles fits into the existing literature. First, in ergodic theory, a beautiful paper of Olshanski and Vershik analyzed infinite random selfadjoint matrices with conjugation invariant law; see [23]. It was shown there that the distribution of any such matrix is completely determined by the distribution of a single diagonal matrix element, with distinct diagonal matrix elements being independent. This is the $N = \infty$ version of Theorem 2.1. In the Free Probability community, Collins studied the distribution of the single random variable $X_{11}^{(N)}$ in complex invariant ensembles with deterministic eigenvalues using an early version of Weingarten Calculus, and found a connection with free cumulants and the $R$-transform; see [3]. Building on Collins’ work, the sum of two independent real or complex invariant ensembles was analyzed by Guionnet and Maida, who used large deviation methods to obtain results related to our third example; see [11]. In the Integrable Probability community, Bufetov and Gorin obtained a counterpart of (one direction of) Theorem 2.1 for a certain class of discrete particle systems; see [2]. Their results, which were inspired by work...
of Borodin, Bufetov, and Olshanski in asymptotic representation theory \cite{1}, have in turn been a source of inspiration to us. The work of Bufetov and Gorin is not based in RMT; instead, its technical backbone is the asymptotic analysis of certain families of symmetric functions via determinantal formulas and steepest descent. Using their technology, Bufetov and Gorin obtained a Law of Large Numbers for the random lozenge tilings model discussed above. This lozenge tiling model is a special case of the celebrated dimer model, for which a comprehensive limit theory has been developed by Kenyon, Okounkov, and Sheffield, see \cite{14,15,16} and references therein.

References


Department of Mathematics, Kagoshima University, Kagoshima, Japan  
*E-mail address: shom@sci.kagoshima-u.ac.jp*

Department of Mathematics, University of California, San Diego, USA  
*E-mail address: jinovak@ucsd.edu*