On rank one log Del Pezzo surfaces in characteristic different from two, three and five

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1 Introduction

The canonical divisor plays a central role in the classification of algebraic varieties. For example, the Minimal Model Program predicts that every smooth projective variety can be “built” from the following three families:

1. Fano varieties, on which the canonical class is anti-ample.
2. Varieties with canonical class that is trivial (for example Calabi-Yau or abelian varieties).
3. Canonical models, on which the canonical class is ample.

The aim of this paper is to classify all klt Fano surfaces (called log Del Pezzo surfaces) of Picard number one over algebraically closed fields of characteristic different from two, three and five. More precisely:

Theorem 1.1 (Classification of rank one log Del Pezzo surfaces). Let \( S \) be a rank one log Del Pezzo surface defined over an algebraically closed field of characteristic different from two, three and five. If \( S \) is smooth then \( S = \mathbb{P}^2 \); if \( S \) is Gorenstein, then \( S \) is one of the surfaces described in Theorem 2.7; otherwise, it belongs to one of the families LDP1 to LDP17 described in Section 6.

Theorem 1.1 has several immediate consequences. For instance, it is well-known that rank one log Del Pezzo surfaces defined over the complex numbers have at most four singular points (see [KM99, Corollary 1.8.1] and [Bel09]). This result, known as Bogomolov bound, fails in characteristic two, as Keel and McKernan exhibited examples of rank one log Del Pezzo surfaces with arbitrarily many singular points ([KM99, Chapter 9]). Nevertheless, in light of Theorem 1.1 we have the following:

Corollary 1.2 (Bogomolov bound). Let \( S \) be a rank one log Del Pezzo surface defined over an algebraically closed field of characteristic different from two, three and five. Then \( S \) has at most four singular points.
Slightly modifying Keel and McKernan’s example, in characteristic three one easily gets a rank one log Del Pezzo surface with seven singularities (see Example 7.5). By carefully examining the classification of extremal rational elliptic surfaces, we were also able to find a counter-example to Bogomolov’s bound in characteristic five.

**Theorem 1.3.** (Example 7.7) There exists a rank one log Del Pezzo surface $S$ defined over any algebraically closed field of characteristic five and a curve $C \subseteq S$ such that

1. $-(K_S + C)$ is ample.
2. $K_S + C$ is dlt.
3. $S$ has five singular points.

There has been recently a great deal of interest in extending the classical results of the Minimal Model Program to algebraically closed fields of positive characteristic (see for instance [HX15]). It is therefore natural to ask which properties of complex log Del Pezzo surfaces carry to positive characteristic. Cascini, Tanaka and Witaszek [CTW17] have proved that in large characteristic all log Del Pezzo surfaces either lift to characteristic zero with smooth base, or are globally F-regular. Our desire to determine the exact characteristic has been one of the main motivations behind Theorem 1.1.

**Theorem 1.4** (Lifting to characteristic zero). Let $S$ be a rank one log Del Pezzo surface defined over an algebraically closed field of characteristic $p > 5$. Then $S$ lifts to characteristic zero with smooth base.

Notice that Theorem 1.4 is sharp, in light of Theorem 1.3 and of the Bogomolov bound. Furthermore, Theorem 1.4 implies that Kodaira’s vanishing theorem holds for log Del Pezzo surfaces in characteristic $p > 5$ (see [CTW17, Lemma 6.1]). This behavior in dimension two should be contrasted with the failure of Kodaira’s vanishing theorem in higher dimension, even for smooth Fano varieties (see [Tot17]).

We give now a brief sketch of the proof of Theorem 1.1. Let $S$ be a log Del Pezzo surface of Picard number one. A natural approach is to try to “simplify” the singularities of $S$ by extracting an exceptional divisor $E_1$ of its minimal resolution. Let $f : T \to S$ be the extraction of $E_1$. Since $S$ has Picard number one, $T$ has Picard number two, and therefore the closed cone of curves of $T$ is generated by two rays. One of the two rays is generated by the class of $E_1$. The main idea is to realize that we may then play the two-ray game on $T$ by contracting the other ray. Let $\pi : T \to S_1$ be the contraction. There is a priori no reason why $S_1$ would be any “simpler” that $S$, and in fact this is not the case for most choices of $E_1$. The truly remarkable fact, which makes the classification possible at all, is that by choosing $E_1$ to be the divisor with the worst singularity (as measured by the discrepancy), then $S_1$ is indeed simpler than $S$, and one may even classify the possibilities for the contraction $\pi_1$. 

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The idea is then to continue this sequence of extractions and contractions, which produces a sequence of increasingly simpler surfaces $S_1$, $S_2$ and so on. This process, which was first introduced by Keel and M\textsuperscript{c}Kernan [KM99], is called the hunt. The hunt is very efficient and usually terminates within three steps, yielding either a Gorenstein log Del Pezzo surface of Picard number one, a cone over the rational normal curve of degree $n$, or a Mori fiber space. One may therefore recover $S$ by classifying all such surfaces and the contractions $\pi_i$ that appear during the hunt.

Keel and M\textsuperscript{c}Kernan introduced the hunt in order to prove that the smooth locus of log Del Pezzo surfaces of rank one is uniruled. Their proof is divided in two cases, based on the following notion:

**Definition 1.5.** Let $(X, \Delta)$ be a $\mathbb{Q}$-factorial projective log pair. A special tiger for $K_X + \Delta$ is an effective $\mathbb{Q}$-divisor $\alpha$ such that $K_X + \Delta + \alpha$ is numerically trivial, but not klt. If there is a special tiger, a tiger is any divisor $E$ with discrepancy at most $-1$.

For surfaces that admit a tiger they presented a short proof by using deformation theory. To complete their argument, however, they classified all complex log Del Pezzo surfaces with no tigers (more precisely, they constructed a family of surfaces that contains all those that have no tigers). By pushing these methods a bit further, they actually classified all simply connected rank one log Del Pezzo surfaces, with the exception of a bounded family.

In this paper we use Keel and M\textsuperscript{c}Kernan’s ideas to complete the classification over the complex numbers, and to extend it to algebraically closed fields of characteristic $p > 5$. In positive characteristic a whole set of additional difficulties appear. For starters, one cannot use topological arguments in order to simplify the classification, such as reducing to the simply connected case. Furthermore, one cannot use the Bogomolov bound as we do not a priori know whether it holds in characteristic $p > 5$ or not. Other issues are that the classification of rank one Gorenstein log Del Pezzo surfaces was not available in positive characteristic (to the best of author’s knowledge), and that the proof of [KM99, Lemma 22.2] on the existence of complements does not a priori carry through in positive characteristic, as it uses the Kawamata-Viehweg vanishing theorem.

The rest of the paper is organized as follows. We start by classifying rank one Gorenstein log Del Pezzo surfaces in Section 2. In Section 3 we recall the main results concerning the hunt from [KM99], in an effort to make the presentation as self-contained as possible. In Section 4 we start carrying the hunt for log Del Pezzo surfaces that do not have tigers, and in Section 5 we deal instead with the case in which there are tigers. As a byproduct, we classify all pairs $(S, C)$ such that $S$ is a rank one log Del Pezzo surface and $C \subseteq S$ is a curve such that $K_S + C$ is anti-nef. In Section 6 we summarize our findings and list all the families of rank one log Del Pezzo surfaces. Finally, we conclude by considering liftability to characteristic zero issues in Section 7.
2 Gorenstein log Del Pezzo surfaces

In this section we classify the singularities of Gorenstein log Del Pezzo surfaces of Picard number one over algebraically closed fields of characteristic different from two, three and five. This was first done over the complex numbers by Furushima [Fur86]. The analogous result in positive characteristic can easily be derived from existing papers, but we were unable to find it explicitly stated in the literature. In the following treatment, we take the approach of Ye in [Ye02], who reduces the study of Gorenstein log Del Pezzo surfaces to the study of extremal rational elliptic surfaces. These were classified over the complex numbers by Miranda and Persson in [MP86], and over algebraically closed fields of positive characteristic by Lang in [Lan91] and [Lan94].

2.1 Extremal rational elliptic surfaces

Definition 2.1. An elliptic surface is a smooth relatively minimal surface $X$ over a curve $C$, such that the general fiber is a smooth curve of genus one.

Let $f : X \rightarrow C$ be a fibration such that the general fiber is irreducible. Over the complex numbers we can deduce that the general fiber is in fact smooth by generic smoothness. In positive characteristic, however, generic smoothness no longer holds and some care is needed. The following lemma, taken from [Mir89], shows that we recover it in a special case.

Lemma 2.2. Let $f : X \rightarrow \mathbb{P}^1$ be a smooth relatively minimal surface with section such the general fiber is irreducible and of arithmetic genus one. Suppose also that $X$ is rational, the image of the section is a $(-1)$ curve and $\text{char}(k) \neq 2,3$. Then $X$ is obtained by blowing up $\mathbb{P}^2$ nine times at the base locus of a pencil of generically smooth cubic curves which induces the fibration $f$. In particular $X$ is an elliptic surface.

Proof. For the general fiber $F$ we have that $K_X \cdot F = 0$ by adjunction, hence $K_X \equiv nF$ for some integer $n$. Consider the image $S$ of the section of $f$, which is a $(-1)$ curve by assumption. Again by adjunction we get that $K_X \cdot S = -1$, hence $K_X \equiv -F$. It follows that $-K_X$ is and that every rational curve has self intersection at least $-2$. Let $g : X \rightarrow M$ be a blowdown to a Mori fiber space. From the above considerations, $M$ can only be $\mathbb{F}_0, \mathbb{F}_2$ or $\mathbb{P}^2$. In each of these cases $X$ dominates $\mathbb{P}^2$ and therefore we get a contraction $h : X \rightarrow \mathbb{P}^2$. Pushing forward $|F|$ in $\mathbb{P}^2$ we obtain a pencil of curves that are numerically equivalent to the push forward of sections of $-K_X$. It follows then that $f_*|F|$ is a pencil in $-K_{\mathbb{P}^2}$, and it is therefore a pencil of cubics. Finally note that a pencil of cubics in characteristic different from two or three is always generically smooth by [Mir89, Lemma 1.5.2] and the comment after it.

Definition 2.3. Let $f : X \rightarrow C$ be an elliptic surface with section $\sigma$. The section $\sigma$ naturally gives all the fibers the structure of elliptic curves. The set of all sections of $f$ is then a group, where the multiplication is done fiber by
fiber and σ is the identity element. This group is called the Mordell-Weil group of \( X \).

**Definition 2.4.** Let \( X \) be a rational elliptic surface with section \( S \). We say that \( X \) is extremal if the Mordell-Weil group of \( X \) is finite and \( \text{NS}(X)_\mathbb{Q} \) is generated by the classes of \( S \) and of the vertical components.

**Theorem 2.5.** The classification of the singular fibers of extremal rational elliptic surfaces over algebraically closed fields with \( \text{char}(k) \neq 2, 3, 5 \) is the same as over \( \mathbb{C} \). The configurations are listed in the following table using Kodaira’s notation:

| Singular fibers | \( |MW(X)| \) |
|-----------------|--------------|
| \( II, II^* \)   | 1            |
| \( III, III^* \) | 2            |
| \( IV, IV^* \)   | 3            |
| \( I^*_0, I^*_0 \) | 4            |
| \( II^*, I_1, I_1 \) | 1          |
| \( III^*, I_2, I_1 \) | 2          |
| \( IV^*, I_3, I_1 \) | 3          |
| \( I^*_1, I_4, I_1 \) | 2          |
| \( I^*_2, I_2, I_2 \) | 4          |
| \( I_9, I_1, I_1, I_1 \) | 3          |
| \( I_8, I_2, I_1, I_1 \) | 4          |
| \( I_6, I_2, I_1, I_1 \) | 6          |
| \( I_5, I_5, I_1, I_1 \) | 5          |
| \( I_4, I_1, I_2, I_2 \) | 8          |
| \( I_3, I_5, I_3, I_3 \) | 9          |

*Proof.* See [Lan91, Theorem 2.1], [Lan94, Theorem 4.1] and [MP86, Theorem 4.1]. \( \square \)

### 2.2 Reduction to elliptic surfaces

We recall here the relation between Gorenstein log Del Pezzo surfaces and extremal rational elliptic surfaces. Let \( V \) be a Gorenstein log del Pezzo surface of rank one such that \( K_V^2 = 1 \). Let \( U \) be its minimal resolution, with exceptional locus \( D \), consisting of eight \((-2)\) curves by Lemma 3.19. The general member of the pencil \( |-K_U| \) is reduced and irreducible by [Dem80, Chapter III, Théorème 1, (b')] (see the remark in the introduction to Chapter IV), and of arithmetic genus one by adjunction. This pencil has only one base point \( p \) by [Dem80, Chapter III, Proposition 2]. After blowing it up, we are in the hypothesis of Lemma 2.2, where we may take the exceptional curve of the blow up as a section. Therefore \( X = \text{Bl}_p(U) \) is an elliptic surface. Notice that \( p \) is not contained in the support of \( D \) because \( K_U \cdot D = 0 \). Hence the strict transform of \( D \) in \( X \) is contained in the union of the singular fibers and does not meet the section.
By Lemma 3.19 we have that $\rho(U) = 10 - K_U^2 = 9$, and therefore $\rho(X) = 10$. By the Shioda-Tate formula ([SS10, Corollary 6.13]), however, we have that $\rho(X) = \sharp D + \text{rank}(MW(X)) + 2$, where $\sharp D$ indicates the number of irreducible components of the support of $D$. It follows then that the Mordell-Weil group of $X$ is finite and therefore $X$ is an extremal rational elliptic surface.

From this discussion we see that we may obtain every rank one Gorenstein log Del Pezzo surface $V$ with $K_V^2 = 1$ by starting with an extremal rational elliptic surface of Picard number ten, selecting a section, contracting all the $(-2)$ curves not meeting the section and then blowing down the section.

Suppose now that $V$ is a Gorenstein log Del Pezzo surface such that $2 \leq K_V^2 \leq 7$ and let $U$ be its minimal resolution. Let $C$ be any $(-1)$ curve on $U$. Every divisor $A$ in the linear system $|-K_U|$ meets $C$ exactly once, and by taking $A$ to be general, we may assume that $C$ is the only $(-1)$ curve passing through $p = A \cap C$. Blowing up $p$ and then contracting all the $(-2)$ curves, we get a rank one log Del Pezzo surface $V'$ such that $K_V^2 = K_U^2 - 1$. By repeating this process we may reduce our analysis to the case when $K_V^2 = 1$, which is the content of Lemma [Ye02, Lemma 3.2].

We summarize our conclusions in the following result:

**Theorem 2.6.** Let $V$ be a rank one Gorenstein log Del Pezzo surface with $K_V^2 \leq 7$ and let $U \to V$ be the minimal resolution. Then there exists an extremal rational elliptic surface $Y$ and a morphism $f : Y \to U$ that is a composition of blow downs of some $(-1)$-curves.

**Proof.** Immediate from the previous description. For more detail see [Ye02, Theorem 3.4].

Thanks to Theorem 2.6 and Theorem 2.5, we can now classify the singularities of Gorenstein log Del Pezzo surfaces. More precisely:

**Theorem 2.7.** Let $V$ be a rank one Gorenstein log Del Pezzo surface over an algebraically closed field with char$(k) \neq 2, 3, 5$. The singularity types on $V$ are listed in the following table. Furthermore, $V$ is uniquely determined by its singularities, with the exception of the cases $E_8, A_1 + E_7, A_2 + E_6$, which have two classes of isomorphism each, and the case $2D_4$, which has infinitely many classes of isomorphism.

<table>
<thead>
<tr>
<th>$A_1$</th>
<th>$A_1 + A_2$</th>
<th>$A_4$</th>
<th>$2A_1 + A_3$</th>
<th>$D_5$</th>
<th>$A_1 + A_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3A_2$</td>
<td>$E_6$</td>
<td>$3A_1 + D_4$</td>
<td>$A_7$</td>
<td>$A_1 + D_6$</td>
<td>$E_7$</td>
</tr>
<tr>
<td>$A_1 + 2A_3$</td>
<td>$A_2 + A_5$</td>
<td>$D_8$</td>
<td>$2A_1 + D_6$</td>
<td>$E_8$</td>
<td>$A_1 + E_7$</td>
</tr>
<tr>
<td>$A_1 + A_7$</td>
<td>$2A_4$</td>
<td>$A_8$</td>
<td>$A_1 + A_2 + A_5$</td>
<td>$A_2 + E_6$</td>
<td>$A_3 + D_5$</td>
</tr>
<tr>
<td>$4A_2$</td>
<td>$2A_1 + 2A_3$</td>
<td>$2D_4$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Proof.** The same proof as in [Ye02, Theorem 1.2] applies, using Theorem 2.5.

**Notation 2.8.** We will denote any Gorenstein log Del Pezzo surface by the corresponding singularity. For example, $S(A_1)$ denotes the unique Gorenstein log Del Pezzo surface with only one singularity, of type $A_1$. 
3 The hunt

Let $S$ be a rank one log Del Pezzo surface. The hunt is a process of $K$-positive extractions and $K$-negative contractions that simplify $S$. The idea is that it is possible to classify both the end result of the hunt and the transformations it involves to get information on $S$. This will be the most important tool in the rest of the paper, and in this section we recall the relevant facts and definitions from [KM99].

3.1 Setting

Definition 3.1. Let $X$ be a normal variety and $\Delta$ an effective Weil $\mathbb{Q}$-divisor. Let $f : Y \to X$ be any birational morphism. Let $\Gamma$ be a $\mathbb{Q}$-divisor such that $K_Y + \Gamma = f^*(K_X + \Delta)$. Then $\Gamma$ is called log pullback of $\Delta$. If $E$ is an exceptional divisor for $f$, the coefficient of $E$ is the coefficient with which it appears in $\Gamma$, and is denoted by $e(E, K_X + \Delta)$. The coefficient $e(X, \Delta)$ of the pair $(X, \Delta)$ is the largest coefficient of any divisor.

In this paper we will use a slightly different and more restrictive notion of tiger. All the relevant results carry on with this definition.

Definition 3.2. Let $(X, \Delta)$ be a projective pair, where $X$ is a surface. A special tiger for $K_X + \Delta$ is an effective $\mathbb{Q}$-divisor $\alpha$ such that $K_X + \Delta + \alpha$ is numerically trivial and there is an exceptional divisor $E$ on the minimal resolution of $X$ with coefficient at least one for $K_X + \Delta + \alpha$. We call any such $E$ tiger.

Definition 3.3. Suppose that $D$ is a divisor, the pair $(S, D)$ is log terminal and $p \in D$ is a singular point of $S$. Then $p$ is cyclic of type $(-E_1^1, E_1^2, \cdots, E_1^n)$, where $D$ touches $E_1$. We say that $p$ has spectral value $k$ if the coefficient of $E_1$ for $K_S$ has the form $k/r$, where $r$ is the index of $p$.

Definition 3.4. Let $X$ be a projective variety and $\Delta = \sum a_i D_i$ be a boundary. We say that the pair $(X, \Delta)$ is flush (respectively level) if for every exceptional divisor $E$ we have $e(E, K_X + \Delta) < m$ (respectively $e(E, K_X + \Delta) \leq m$), where $m$ is the minimum of the coefficients of $\Delta$.

Flushness is a property that is (almost) preserved under the hunt and is what makes the classification of its steps possible by controlling the geometry of the singularities of the pair $(X, \Delta)$. As an example, here we remark the most important consequences of flushness.

Proposition 3.5. Let $S$ be a log terminal surface, $\pi : \tilde{S} \to S$ its minimal resolution and $\Delta$ a divisor with support $D$. If $p$ is singular, and $(S, \Delta)$ is flush at every $\pi$-exceptional divisor over $p$, then $(X, D)$ is log terminal at $p$.

Proof. This is [KM99, Lemma 8.0.4]. \hfill \Box

Proposition 3.6. Suppose $p$ is smooth and the pair $(S, \Delta)$ is flush, where $\Delta = \sum a_i D_i$. If $M_i$ is the multiplicity of $D_i$ at $p$, then $\sum a_i M_i - 1 < m$. In particular $m < 1/(M - 1)$, where $M$ is the multiplicity of $D$. 
Proof. This is [KM99, Lemma 8.3.7].

The following geometric situations will be common, and we name them according to [KM99].

**Definition 3.7.** Let $A$ and $B$ be two rational curves on $S$ such that $K_S + A + B$ is log terminal at any singular point of $S$. We say that $(S, A + B)$ is

1. banana, if $A$ and $B$ meet in exactly two points, and there normally.
2. fence, if $A$ and $B$ meet at exactly one point, and there normally.
3. tacnode, if $A$ and $B$ meet at most at two points, there is one point $q \in A \cap B$ such that $A + B$ has a node of genus $g \geq 2$ at $q$, and if there is a second point of intersection, then $A$ and $B$ meet there normally.

Now let’s describe the hunt.

**Lemma 3.8.** Start with a pair $(S, \Delta)$ of a rank one log Del Pezzo surface, and an effective $Q$-divisor $\Delta$. Let $f : T \to S$ be an extraction of relative Picard number one, of an irreducible divisor $E$ of the minimal resolution. The cone of curves of $T$ has two edges, one generated by the class of $E$, and let the other be generated by the class of $R$. Let $x = f(E)$, $\Gamma$ such that $K_T + \Gamma = f^*(K_S + \Delta)$ and $\Gamma_\epsilon = \Gamma + \epsilon E$, where $0 < \epsilon \ll 1$. Assume that $-(K_S + \Delta)$ is ample, then

1. $R$ is $K_T$-negative and contractible, hence there is a rational curve $\Sigma$ that generates the same ray. Let $\pi$ be the contraction morphism.
2. $K_T + \Gamma_\epsilon$ is anti-ample.
3. $\Gamma_\epsilon$ is $E$ negative.
4. There is a unique rational number $\lambda$ such that with $\Gamma' = \lambda \Gamma_\epsilon$, $K_T + \Gamma'$ is $R$ trivial, $\lambda > 1$.
5. $K_T + \Gamma'$ is $E$ negative.
6. $\pi$ is either birational or a $\mathbb{P}^1$-fibration (called ”net”).
7. If $\pi : T \to S_1$ is birational, and $\Delta_1 = \pi(\Gamma')$, then $K_{S_1} + \Delta_1$ is anti-ample and $S_1$ is a rank one log Del Pezzo.
8. If $K_S + \Delta$ does not have a special tiger, then $K_S + \Delta$, $K_T + \Gamma'$, and $K_{S_1} + \Delta_1$ are all klt and none of the three has a special tiger.

Proof. See [KM99, Definition-Lemma 8.2.5].

**Definition 3.9.** We call the above transformations $f, \pi$ a next hunt step for $(S, \Delta)$ if $e(E, K_S + \Delta)$ is maximal among exceptional divisors of the minimal resolution. There might be multiple choices. If $x$ is a chain singularity we allow any choice $E$ that is not a $(-2)$ curve (this is always possible). If $x$ is a
non chain singularity we require \( E \) to be the central curve (which has maximal coefficient by [KM99, Lemma 8.3.9]). If two points have the same coefficient, we can pick one at our choice, but unless stated otherwise our choice will be a chain singularity that allows us to extract the most negative curve.

We will start with \((S_0, \emptyset)\) and then apply the hunt process. Let’s fix the following notation for the rest of the paper.

**Notation 3.10.** We start from a surface without boundary, so \( \Delta_0 = \emptyset \). Let \( f_i, \pi_{i+1} \) define the next hunt step for \((S_i, \Delta_i)\). Define \( x_i = f(E_{i+1}) \in S_i \) and \( q_{i+1} = \pi_{i+1}(\Sigma_{i+1}) \in S_{i+1} \). The divisor \( \Gamma_{i+1} \) is defined by \( K_{T_{i+1}} + \Gamma_{i+1} = f_i^*(K_{S_i} + \Delta_i) \). Call \( \Delta_{i+1} = \pi_{i+1}(\Gamma_{i+1}) \); it satisfies \( K_{T_{i+1}} + \Gamma_{i+1} = \pi_{i+1}(K_{S_{i+1}} + \Delta_{i+1}) \). Define \( A_1 = \pi_1(E_1) \subset S_1 \), \( B_2 = \pi_2(E_2) \subset S_2 \). Call \( A_2 \) the strict transform of \( A_1 \) on \( S_2 \). Let \( a_1, b_2 \) be the coefficients of \( A_1, B_2 \) in \( \Delta_1, \Delta_2 \) (which are also the coefficients of \( E_1, E_2 \) in \( \Gamma_1', \Gamma_2' \)) and \( a_2 \) the coefficient of \( A_2 \) in \( \Delta_2 \). We note that \( a_1, b_2 < a_2 \) by the flush condition and the previous scaling. Let \( e_i \) be the coefficient of \( E_{i+1} \) in \((S_i, \Delta_i)\). This is also the coefficient of the pair \((S_i, \Delta_i)\). Finally, we indicate by \( \Sigma_i \) the image of \( \Sigma_i \) in \( S_0 \) or \( S_1 \), depending on the context.

The following result describes the first two hunt steps.

**Proposition 3.11.** For the first hunt step:

1. \( T_1 \) is a net.
2. Otherwise \( K_{S_1} + a_1 A_1 \) is flush and one of the following holds
   1. \( g(A_1) > 1 \).
   2. \( g(A_1) = 1 \) and \( A_1 \) has an ordinary node at \( q = q_1 \).
   3. \( g(A_1) = 1 \) and \( A_1 \) has ordinary cusp at \( q = q_1 \).
   4. \( g(A_1) = 0 \) and \( K_{S_1} + A_1 \) is log terminal.

For the second hunt step one of the following holds.

1. \( T_2 \) is a net.
2. \( A_1 \) is contracted by \( \pi_2 \), \( K_{T_2} + \Gamma_2' \) is flush, \( K_{S_1} + A_1 \) is log terminal, \( q_2 \) is a smooth point of \( S_2 \), \( B_2 \) is singular at \( q_2 \) with a unibranch singularity, and \( K_{S_2} + \Delta_2 \) is flush away from \( q_2 \), but is not level at \( q_2 \). \( \Sigma_2 \) is the only exceptional divisor at which \( K_{S_2} + \Delta_2 \) fails to be flush.
3. \( \Delta_2 \) has two components.

Suppose that in this last case \( a_2 + b_2 \geq 1 \). Then \( \Sigma_2 \cap \text{Sing}(A_1) = \emptyset \), \( K_{T_3} + \Gamma_2' \) is flush away from \( \text{Sing}(A_1) \). \( K_{S_1} + \Delta_2 \) is flush away from \( \pi_2(\text{Sing}(A_1)) \), and at least one of \(-(K_{S_2} + A_2)\) or \(-(K_{S_2} + B_2)\) is ample. Also, one of the following holds:
9. \((S_2, A_2 + B_2)\) is a fence.

10. \((S_2, A_2 + B_2)\) is a banana, \(K_{S_2} + B_2\) is plt, and \(x_1 \in A\).

11. \((S_2, A_2 + B_2)\) is a tacnode, with tacnode at \(q_2\). \(K_{S_2} + B_2\) is plt. If \(x_1 \in A_1\), \(A_2 \cap B_2 = \{x_1, q_2\}\). If \(x_1 \notin A_1\) then \(A_2 \cap B_2 = \{q_2\}\).

Proof. See [KM99, Proposition 8.4.7].

3.2 Classification of the contractions

Let \(\pi : T \to S\) be a proper birational contraction of a \(K_T\) extremal ray and denote by \(\Sigma\) the exceptional divisor. Let \(q\) be the image of \(\Sigma\). We will only be concerned with the local étale description of \(T\) about \(\Sigma\). Let \(W \subset T\) be a curve with smooth components crossing normally. Assume \(W\) has at most two irreducible components \(X, Y\) and that \(K_T + cX + dY\) is \(\pi\) trivial and flush, with \(0 < c, d < 1\). Assume also that \(\pi|_W\) is finite. Thus \(K_S + cX + dY\) is flush by [KM99, Lemma 8.3.1]. Let \(D\) be the image of \(W\) in \(S\) and let \(h : \tilde{T} \to \tilde{S}\) be the induced map between the minimal resolutions.

If \(D\) has multiplicity two at \(q\), then \(q\) is smooth and we have the following classification.

**Lemma 3.12.** Suppose \(D\) has a node of order \(g\). One of the following holds:

1. \(T\) has type \(I\) or \(0\), and \(c + d = 1\).
2. \(g \geq 2\), \(T\) has type \(II\) and \(gd + (g + 1)c = g + 1\).
3. \(T\) has type \((II, x_r^{-1})\) with \(r \geq 1\), there is a unique singularity, an \(A_r\) point, \(X\) meets \(\Sigma\) at a smooth point, \(Y\) meets \(\Sigma\) at the \(A_r\) point, which has type \((2, \ldots, 2')\) and \(c + \frac{d}{r+1} = 1\)
4. \(\Sigma\) meets \(X\) and \(Y\) each at a singular point of \(T\), and those are the only singularities along \(\Sigma\).

Proof. See [KM99, Lemma 11.1.1].

**Lemma 3.13.** Suppose \(D\) has a cusp of order \(g\). Then either

1. \(T\) has type \(I\) and \(c = 1/2\).
2. \(T\) has type \(II\) and \(c = (g + 1)/(2g + 1)\).
3. \(T\) has type \(III\) and \(c = (g + 1)/(2g + 1)\).
4. \(g = 1\), \(T\) has type \(u\) and \(c = 3/4\) or \(g = 2\) and \(c = 9/14\).
5. \(g = 1\), \(T\) has type \(v\) and \(c = 5/7\) or \(g = 2\) and \(c = 7/11\).
6. $g = 1, T$ has type $w$ and $c = 7/9$.

7. $g = 1, T$ has type $(u; n)$ and $c = 11/14$.

8. $g = 1, T$ has type $(v; f)$ and $c = 10/13$.

9. $g = 1, T$ has type $(v; f^2)$ and $c = 15/19$.

10. $g = 1, T$ has type $(v; n)$ and $c = 3/4$.

11. $g = 1, T$ has type $(v; n^2)$ and $c = 7/9$.

Proof. See [KM99, Lemma 11.2.1].

In case $D$ has multiplicity three, we have only following two cases.

Lemma 3.14. Suppose that $D$ has multiplicity three. If $D$ has two branches at $q$, then one branch has a double point, a simple cusp, and the other is smooth. If $X$ is the branch with the cusp, then $\Sigma$ meets $X$ normally at one smooth point, $\Sigma$ contains two singularities (2) and (3), and $Y$ meets the $-2$ curve, and is disjoint from $\Sigma$ on $T$.

Proof. See [KM99, Lemma 11.3.2].

Lemma 3.15. Suppose that $D$ has multiplicity three. If $D$ is unibranch, then either

1. $\Sigma$ has singularities $(3, 2')$, (3) and meets $X$ normally at a smooth point

2. $\Sigma$ has singularities $(3)$, (2) and on the minimal resolution $X$ meets $\Sigma$ normally at the intersection of $\Sigma$ and the $-2$ curve.

Proof. See [KM99, Lemma 11.3.3].

The last type of contraction we are interested in is one that gives rise to a fibration, which we call net. Let $\pi : T \to C$ be a $\mathbb{P}^1$-fibration of relative Picard number one, with $T$ a normal surface. Let $\hat{\pi}$ be the composition $\hat{\pi} : \tilde{T} \to T \to C$, where $\tilde{T}$ is the minimal resolution of $T$. We describe the fiber $F$ of $\tilde{T}$ above $p \in C$ as the sequence $k(-a) + l(-b) + \cdots + m(-c)$, where we mean a chain of curves of self intersection $-a$, $-b$ and so on, with multiplicities $k, l$ and so on.

Definition 3.16. A non Du Val log terminal singularity with coefficient strictly less than $1/2$ is called almost Du Val. They all are of the form $(3, A_k)$ for some $k$.

Lemma 3.17. Assume $T$ is log terminal, $G$ is a multiple fiber of $\pi$ of multiplicity $m$, and $G$ contains a cyclic singularity, either Du Val or almost Du Val. If $e(T) < 2/3$ then $G$ is one of the following:

If $K_T + G$ is not log terminal at any singular point

1. $(2, 2', 2), m = 2$. 

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2. \((3, 2', 2, 2), m = 3\).
   If \(K_T + G\) is log terminal at one singular point, but not log terminal
3. \((2', z), m = 4\). \(z\) is a non chain singularity, with center -2 and branches
   \((2), (2)\) and \((2, \cdots, 3')\) (or \((3')\)).
4. \((2, 3', 2; 2'), m = 4\).
   If \(K_T + G\) is log terminal
5. \((A_k; (k + 1)'), k \leq 4, m = k + 1\). The fiber is \(-(k + 1) + [k + 1](-1) + k(-2) + [k - 1](-2) + \cdots + (-2)\).
6. \((2, 3'; 2', 3), m = 5\). The fiber is \((-2) + 2(-3) + 5(-1) + 3(-2) + (-3)\).
7. \((3, 2, 2'; 4', 2), m = 7\). The fiber is \((-3) + 3(-2) + 5(-2) + 7(-1) + 2(-4) + (-2)\).
8. \((4, 2'; 3', 2, 2), m = 7\). The fiber is \((-4) + 4(-2) + 3(-1) + 7(-3) + 2(-2) + (-2)\).

Proof. See [KM99, Lemma 11.5.9].

Lemma 3.18. If \(G\) is a multiple fiber of multiplicity three and the coefficient
\(e(T) < 2/3\), then \(G\) is one of the fibers of the above classification.

Proof. See [KM99, Lemma 11.5.13].

3.3 Useful facts

Here we collect some results that are somehow unrelated to the previous discussion, but that will be useful later on.

Lemma 3.19. Let \(S\) be a rank one log Del Pezzo surface and \(\tilde{S}\) its minimal
resolution. Then \(K^2_{\tilde{S}} + \rho(\tilde{S}) = 10\).

Proof. Run the minimal model program on \(\tilde{S}\). The end result \(S_{min}\) is a Mori fiber space because \(\tilde{S}\) is birational to \(S\). Then \(S_{min}\) is either \(\mathbb{P}^2\) or a ruled surface. In this last cast it’s a Hirzebruch surface because it’s rational. In any case, it follows that \(K^2_{S_{min}} + \rho(S_{min}) = 10\) and a sequence of smooth blow ups does not change the equality.

Lemma 3.20. Let \(S\) be a rank one log Del Pezzo surface. Let \(n\) be the number
of exceptional components in the minimal resolution. Then
\[K^2_S = 9 - n + \sum_i e_i(2 - E^2_i)\]
where $e_i$ is the coefficient of the divisor $E_i$. If $e(S) < 1/2$, $u$ is the number of exceptional components coming from Du Val singularities, and $n_r$ is the number of points of type $(3, A_r)$, then the above formula takes the form

$$K_S^2 = 9 - u - \sum_r n_r (r + 1)(1 - \frac{1}{2r + 3})$$

**Proof.** Obvious.

**Lemma 3.21.** Let $S$ be a rank one log Del Pezzo surface and $\tilde{S}$ its minimal resolution. Then $\rho(\tilde{S}) \geq 11$ if $S$ has no tigers.

**Proof.** This follows from [KM99, (10.3)].

**Lemma 3.22.** Let $p$ be a log terminal singularity with $n$ components and coefficient $e < 3/5$. Then $n \geq \sum e_i (2 - E_i^2)$ and equality holds if and only if $p$ is of type $(4)$ or $(3, 3)$.

**Proof.** Obvious by [KM99, Proposition 10.1].

**Lemma 3.23.** Let $D$ be a divisor and $(S, D)$ a log terminal pair. If $p$ is a singular point of spectral value $k$ and index $r$, and $E$ is an exceptional divisor above $p$, then

$$e(E, K + \lambda D) = \frac{(k/r) + \lambda(r - 1 - k)/r = \lambda(r - 1)/r + (1 - \lambda)(k/r)}$$

**Proof.** This is [KM99, Lemma 8.3.8].

### 4 Surfaces without tigers

We are now ready to go through the hunt steps. We will assume throughout this section that $S_0$ has no tigers, and will classify all such surfaces.

**4.1 $T_1$ is a net**

Here we assume that $T_1$ is a net and derive a contradiction. This is the content of proposition 4.3.

**Lemma 4.1.** $E_1$ is not fibral. If $d$ is its degree then $d \geq 3$ and $e_0 < a_1 = 2/d$.

**Proof.** [KM99, Lemma 14.2].

**Lemma 4.2.** If $\text{char}(k) \neq 3$, then $e(S_0) \geq 1/2$.

**Proof.** Suppose that we have $e_0 < 1/2$ instead. By the classification of log terminal singularities of low coefficient, all singularities are either Du Val or of the form $(3, A_r)$, for some $r \geq 0$. Suppose there are $n$ non Du Val singularities. If $n \leq 2$ then $S_0$ has a tiger by [KM99, Lemma 10.4], so $n \geq 3$. In particular there are at least two non Du Val points on $T_1$, $S_0$ is not Gorenstein and $e_0 \geq 1/3$, etc.
again by the classification of log terminal singularities. Also, by lemma 4.1, $d \leq 5$. The first hunt step extracts the $(-3)$ curve $E$ from a point with maximal $r$.

If $r = 0$, then on $T_1$ we would have only singularities of type (3) or Du Val. By the classification in lemma 3.17 the only possible fiber through a (3) point has singularities (3) and $A_2$, and multiplicity three. Since $E_1$ is in the smooth locus of $T_1$, $m$ divides $d$, hence $d = 3$. Since there are at least two (3) points, applying Riemann-Hurwitz to the map $\pi_1|_{E_1} \to \mathbb{P}^1$ we see that there are exactly two multiple fibers, and hence the Picard number of $T_1$ is eight. But then $S_0$ has Picard number seven and hence has tigers by lemma 3.21.

Hence $r \geq 1$, and on the exceptional divisor $E_1$ we have only one singularity, an $A_r$ point. Also, $e_0 \geq 2/5$, hence $d \leq 4$. However, a fiber can contain at most two singular points by lemma 3.17, so there is at least one non Du Val point which is not in the same fiber as the $A_r$ point. Call this point $p$ and the respective fiber $F$. The fiber $F$ meets $E_1$ only at smooth points, because $A_r$ is the only singular point on $E$. This tells us that the multiplicity of $F$ can’t be more than four, but, again by lemma 3.17 and the fact that $e_0 < 1/2$, the only non Du Val fibers with $m \leq 4$ have multiplicity exactly three. Hence $d$ is a multiple of three, and so $d = 3$.

Now we see that $F$ has multiplicity three and has either just one singularity, of type $(3, 2, 2, 2)$, or two singularities, an $A_2$ point and a (3) point. This means that $p$ is either of type $(3, 2, 2, 2)$ or (3), and is the only non Du Val point on $F$.

By the above there is at least another non Du Val point on $T_1$, call it $t$. Also, let $G$ be the fiber through the $A_r$ point of $E$.

Suppose first that $t$ lies on $G$. By lemma 3.17, the singularities on $G$ are $A_2$ and (3) and its multiplicity is three. However, then the intersection of $E_1$ with $G$ at $t$ is either one or two, depending whether they meet at opposite ends of the $A_2$ chain or at the same end. This means that they necessarily intersect at another point, but since all other points on $E_1$ are smooth we get a contradiction.

Hence $t$ lies on a fiber distinct from $F$ and $G$, call it $H$, which has multiplicity three. Notice now that one of the points of intersection of $E_1$ and $G$ is ramified for $\pi_1|_{E_1}$, for either they only meet at the $A_r$ point, which then is ramified, or they meet at another point, which is ramified since it is a smooth point and $G$ is a multiple fiber. But then applying Riemann-Hurwitz we get a contradiction, since there would be at least three ramified points, two of which ramified of order three.

Proposition 4.3. If $\text{char}(k) \neq 3$ then $T_1$ is not a net.

Proof. Suppose $T_1$ is a net. By lemma 4.2, $d = 3$ and $1/2 < e(S_0) < 2/3$. There are four possible cases: $T_1$ has zero, one, two or three singularities along $E_1$. In any case, the spectral value of each singularity along $E_1$ is at most one by [KM99, Lemma 8.0.7]. The classification of lemma 3.17 applies: either the multiple fiber meets $E_1$ at smooth points, and then we can use lemma 3.18 because $m = 3$, or in meets it in singular points of spectral value less or equal to one, which are cyclic Du Val or almost Du Val by [KM99, lemma 8.0.8].

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Suppose first that $T_1$ is smooth along $E_1$. Then every multiple fiber has multiplicity three, and either contains just one singularity $(3, 2, 2, 2)$ or two singularities, a $(3)$ point and an $A_2$ point. In any case, they contribute at most four to the Picard number of the minimal resolution. By Riemann-Hurwitz applied to $\pi_1|E_1$, there are exactly two singular fibers, and hence the Picard number of the minimal resolution is at most ten, which contradicts lemma 3.21.

Suppose now that there is just one singular point, $p$, along $E_1$ and call $F$ the fiber through it. First we make the further assumption that the multiplicity of $F$ is at least three. Then $p$ is ramified of order three and there can be at most one other singular fiber, necessarily of multiplicity three. As we saw above, in this other fiber the contribution to the Picard number of the minimal resolution is at most four. So the only cases for the first singular fiber are those with at least five exceptional curves. By proposition 3.17 the list of possibilities is $(A_4; 5)$, $m = 5$ or $(3, 2, 2; 4, 2)$, $m = 7$ or $(2; z)$, $m = 4$, with $z$ a non chain singularity. However the first case does not occur: since the spectral value of $p$ is at most one, $p$ must be the $A_4$ point. The intersection of the fiber with $E_1$ is either one or more than four, so it can’t be three. In the second case $p$ must be the $(3, 2, 2)$ singularity. The fiber can touch either end of the singularity. A computation shows that the only case where $E_1 \cdot F = 3$ is when both the strict transforms of $E_1$ and $F$ touch the $(-3)$ curve, and do not meet on it. But then the singularity of $p$ has to be of the form $(k, 3, 2, 2)$, with $k \geq 5$. Hence $e_0 > 2/3$, contradiction. In the third case the fiber can’t meet $E_1$ on any other point since $m = 4$, but then the intersection of $E_1$ and $F$ is either two or six, hence not three.

We conclude that the multiplicity of $F$ is two, and has either just one $A_3$ singularity or two $A_1$ singularities. Notice that $F$ can’t meet $E_1$ three times, so there is at least one ramified point on it. Any other fiber has multiplicity three and contributes at most four to the Picard number of the minimal resolution. There must be at least two of them by lemma 3.21, but then this contradicts Riemann-Hurwitz.

Suppose there are two singular points on $E_1$. If they lie on the same fiber we have that exactly one is ramified, of order two. Since both these singularities are Du Val or almost Du Val, their fiber contributes at most four to the Picard number of the minimal resolution. Then there are at least two more singular fibers, each of multiplicity three, contradicting Riemann-Hurwitz. The two singular points of $E_1$ lie on distinct fibers. First suppose there is another multiple fiber, necessarily of multiplicity three. Then the first two fibers have multiplicity two by Riemann-Hurwitz. Again by lemma 3.21, at least one of the singularities on $E_1$ is $A_3$. Its fiber $F$ intersects $E_1$ on a smooth point, which is ramified of order two. Note that $x_0$ can’t have type $(2, 2, 2, k, 2, 2, 2)$ with $k \geq 3$ because $e(S_0) < 2/3$, hence the other fiber meeting $E_1$ on a singular point, $G$, contains two $A_1$ singularities. The non Du Val fiber passes through a $(3, 2, 2, 2)$ point by considerations on the Picard number. Note that $G$ is disjoint from $E_1$ on the minimal resolution above the singular point, since otherwise the $A_1$ point would ramify of order three, contradicting Riemann-Hurwitz. It follows that $G$ meets $E_1$ on a smooth point, ramified of order two. Finally $E_1$ is a $(-3)$ curve because $e_0 < 2/3$. Contracting all the $(-1)$ curves from the minimal
resolution of $T_1$ we perform nine blowdowns on $E_1$. That means that we reach a Hirzebruch surface, the image of $E_1$ has self intersection six and is a double section. But there are no such surfaces, hence this case is impossible too.

If instead there are only two singular fibers, one of them needs to contribute at least five to the Picard number of the minimal resolution. But this kind of fiber was ruled out before.

As the last case, suppose there are three singularities along $E_1$. Since $x_0$ is a log terminal singularity with coefficient less than $2/3$, the singularities on $E_1$ are $(2), (2)$ and $(A_j, 3)$ for some $j$. The two $A_1$ points can’t lie on the same fiber, otherwise $E_1 \cdot F \neq 3$. By lemma 3.17 an $A_1$ point and $(A_j, 3)$ can’t lie on the same fiber either. Hence all these singular points are on different fibers. The fibers through the $A_1$ points can’t contain a non chain singularity, for otherwise $m = 4$ and there is no way to get $E_1 \cdot F = 3$. Since the $(A_j, 3)$ point can lie only in fibers of multiplicity at least three, it’s ramified of order three. So the only possibility for the $A_1$ fibers is that they contain another $A_1$ point, and are disjoint from $E_1$ on the minimal resolution. Call the fiber through the $(A_j, 3)$ point $G$. Its contribution to the Picard number of the minimal resolution must be at least five, since there are no more multiple fibers by Riemann-Hurwitz. Then $j = 2$ and $G$ contains a $(4, 2)$ point. The coefficient of $(4, 2)$ is $4/7 > 1/2$, which contradicts the choice of $x_0$ in the hunt.

4.2 $g(A_1) > 1$

In the previous subsection we saw that $T_1$ can’t be a net, so we get a birational contraction. Here we prove

Proposition 4.4. If $g(A_1) > 1$ and $\text{char}(k) \neq 2$, then $S_0$ is the surface of $[\text{KM99}, \text{Lemma 15.2}].$

Our strategy will be to compare the information we get by the local geometry of the hunt contraction, and the global information we get from lemma 3.20 to obtain numerical obstructions.

Note that $S_0$ is not Gorenstein, for otherwise it would have a tiger by lemma [KM99, Lemma 10.4]. Hence $1/3 \leq e_0 < a_1$, and so by lemma 3.6 $A_1$ has either multiplicity three or a double point.

Lemma 4.5. $A_1$ cannot have multiplicity three.

Proof. Suppose that $A_1$ has multiplicity three. Then $a_1 < 1/2$ by lemma 3.6 and all the singularities in $S_0$ are Du Val or almost Du Val. The possible configurations for the contraction are given by lemma 3.14 and 3.15.

Assume that $\pi_1$ is given by lemma 3.14. Then $x_0$ has type $(3, 2)$. Say that on $S_0$ we have $n$ singularities of type $(3)$, $m$ of type $(3, 2)$ and possibly some Du Val ones. We compute

$$K_{S_0}^2 = \frac{(K_{S_0} \cdot \Sigma_1)^2}{\Sigma_1^2} = \frac{1}{15 \cdot 11}$$
From the push-pull formula we get that \( K_{S_0}^2 = K_S^2 + n/3 + 2m/5 \), but this can’t happen because there is a factor eleven in the denominator and \( K_S^2 \) is an integer.

Now assume that \( \pi_1 \) is given by the first case of lemma 3.15 (the second case is numerically the same as above). Say \( x_0 \) has type \((3, A_r)\), with \( r \geq 1 \). We have that \( K_{S_0} \cdot \Sigma_1 = -7/15 + (r + 1)/(2r + 3) \). By adjunction \( \Sigma_1^2 = -1/15 + (r + 1)/(2r + 3) \). Then, as above,

\[
K_{S_0}^2 = \frac{(r - 6)^2}{15(2r + 3)(13r + 12)}
\]

Since \( K_{S_0} \cdot \Sigma_1 < 0 \), we have that \( e_0 < 7/15 \), hence the non Du Val singularities can only be of type \((3, A_j)\) with \( j \leq 5 \). One checks that for any \( r \leq 5 \) lemma 3.20 cannot hold. Suppose for example that \( r = 1 \). Then \( K_{S_0}^2 = 1/75 \), hence there is a factor twenty five in the denominator, contradiction. All the other cases are similar, except when \( r = 5 \). In this case \( K_{S_0}^2 = 1/(3 \cdot 5 \cdot 7 \cdot 11 \cdot 13) \). For this to be possible, there should be a singularity of type \((3, A_j)\) for \( j = 0, 1, 2, 4, 5 \). But then, again by lemma 3.20 and lemma 3.22, \( K_{S_0}^2 < 0 \), contradiction. \( \square \)

Now we consider double points of genus \( g \geq 2 \). Note that configuration 0 does not arise since \( E_1 \) is smooth. Also note that configurations I and II are numerically the same in the node and cusp cases. Let’s start with configuration I, where we allow also \( g = 1 \) because it simplifies some work later on.

**Lemma 4.6.** Suppose \( A_1 \) has a double point and \( g(A_1) \geq 1 \). Then configuration I does not arise.

*Proof.* By lemma 3.12 \( a_1 = 1/2 \), and so \( 1/3 \leq e_0 < 1/2 \). Say \( x_0 \) is of type \((3, A_r)\). The local configuration tells us that

\[
K_{S_0}^2 = \frac{(K_{S_0} \cdot \Sigma_1)^2}{\Sigma_1^2} = \frac{(-1 + 2e_0)^2}{4e_0 - 1/g}
\]

Lemma 3.20 has finitely many possible configurations thanks to lemma 3.22. By running a computer program one sees that the only solutions for lemma 3.20 are \( r = 0, g = 1 \) and \( r = 1, g = 1 \).

In the case \( r = 0, g = 1 \), \( A_1 \) is in the smooth locus and is a tiger by adjunction. So assume \( r = 1, g = 1 \). There are three solutions: in the notation of lemma 3.20 we have \( u = 6, n_0 = 2, n_1 = 1 \), \( u = 4, n_0 = 5, n_1 = 1 \) and \( u = 2, n_0 = 8, n_1 = 1 \). Now we will apply the results of [KM99, Chapter 12] to our situation. By [KM99, Lemma 12.1] we have that \( F = \emptyset \) and \( M \in |K_{S_1} + A_1| \) is an irreducible curve that passes through the only singularity along \( A_1 \), a (2) point. \( M \) also passes through all the non Du Val points of \( S_1 \), which are all (3) points by the above list. Since \( (K_{S_1} + M) \cdot M < 0 \) and \( M \) passes through a (2) point and two (3) points, we have that \( K_{S_1} + M \) is log terminal and these are all the singularities through \( M \). Hence we are in the case \( u = 6, n_0 = 2, n_1 = 1 \). From the fact that \( M \equiv K_{S_1} + A_1 \), we have that \( M \cdot A_1 = 1/2 \), and hence
$M$ does not pass through the cusp of $A_1$. That means that the pullback of $M$
under $\pi_1$ is equal to its strict transform. Again by $M \equiv K_{S_1} + A_1$, we have
that $\pi_1^*(M) \equiv 2/3\Sigma_1 + 1/3E_1$ on $T_1$. So the pushforward on $S_0$ is numerically
equivalent to $2/3\Sigma_1$, which means that there must be an effective curve $C$ in $S_0$ such that $3C \equiv \Sigma_1$. But since $K_{S_0} \cdot \Sigma_1 = -1/5$ and $\Sigma_1^2 = 3/5$, we have that
$(K_{S_0} + C) \cdot C = 0$, hence $C$ is a tiger. \hfill \Box

**Lemma 4.7.** Suppose $A_1$ has a double point and $g(A_1) \geq 2$. If $\text{char}(k) \neq 2$
then configuration II does not arise, unless $S_0$ is the surface described in [KM99, Lemma 15.2].

*Proof.* By lemma 3.12 we get that $e_0 < a_1 = (g + 1)/(2g + 1) \leq 3/5$, and so either $x_0 = (2, 3, 2, 2)$ and $g = 2$, or $x_0 = (3, A_3)$.

Suppose that $x_0 = (3, A_3)$ and hence $e_0 < 1/2$. The local configuration tells us that
$$K_{S_0}^2 = \frac{(K_{S_0} \cdot \Sigma_1)^2}{\Sigma_1^2} = \frac{(-1 + \frac{2g+1}{2g+3})^2}{\frac{6g+1}{2g+3} - 1}$$

By running a computer program we see that the only solutions to lemma 3.20 are $g = 2, u = 3, n_0 = 5, n_2 = 1$ and $g = 2, u = 5, n_0 = 2, n_2 = 1$. In any case, we have $g = 2$. Again, since $a_1 = 3/5 > 1/2$, let $F$ and $M$ be as in [KM99, Chapter 12]. $M = 0$ because $A_1$ is in the smooth locus of $S_1$. Then $F \cdot A_1 = (K_{S_1} + A_1)^2 \cdot A_1 = 2$. By [KM99, Lemma 12.1] $[F]$ is basepoint free, hence we can
choose $F$ such that does not pass through the cusp of $A_1$. Hence the pullback of $F$ through $\pi_1$ is the same as its strict transform. Again by $F \equiv K_{S_1} + A_1$ we get that $\pi_1^*(F) \equiv 1/3E_1 + 5/3\Sigma_1$. Pushing forward on $S_0$ we see that there
must exist $C$ such that $\Sigma_0 \equiv 3C$. Hence $(K_{S_0} + C) \cdot C = -2/21 + 2/21 = 0$, contradiction.

Suppose now that $x_0 = (2, 2, 3, 2)$, which gives $e_0 = 6/11$. For this case, the argument given in [KM99, Chapter 15] goes through without any changes, but we’ll show it here too for sake of clarity. Since $K_{S_0}^2 = 1/(11 \cdot 13)$, by the classification of
singularities of low coefficient, there must be a $(3, A_3)$ point to compensate for the thirteen factor in the denominator in lemma 3.20. The equality in lemma 3.20 is already satisfied with these two singularities, hence there can only be (4) or (3, 3) points by lemma 3.22. By [KM99, Lemma 12.3] all these additional points need to be on $M$, and since $K_{S_1} + M$ is negative, there can be at most one such point. Then $\rho(\bar{S})$ is either 12 or 11, and by [KM99, Lemma 10.8] we have $\rho(\bar{S}) = 11$. This means that $(2, 3, 2, 2)$ and $(3, A_3)$ are the only singularities, so now proceed as in [KM99, Definition-Lemma 15.2]. \hfill \Box

**Lemma 4.8.** Suppose $A_1$ has a double point and $g(A_1) \geq 2$. Then configuration III does not arise.

*Proof.* By lemma 3.12 $a_1 = (g + 1)/(2g + 1)$. We use the expression
$$K_{S_0}^2 = \frac{(K_{S_0} \cdot \Sigma_1)^2}{\Sigma_1^2}$$
to compute $K_{S_0}^2$ for all possible singularities, and see that none of them can occur.

Running a computer program, it’s easy to see that the only cases that give solutions to lemma 3.20 are $x_0 = (4)$ and $x_0 = (4, 2)$, both for $g = 2$. By [KM99, Lemma 12.0] there is an effective integral divisor $L$ in the class of $K_{S_0} + A_1$.

Suppose that we have $x_0 = (4)$. Then a computation shows that $f_*\pi^*(L) \equiv 10/3\Sigma_1$, which implies that there is a curve $C$ such that $\Sigma_1 \equiv 3C$. However, $K_{S_0} \cdot C = -1/30$ and $C^2 = 1/60$, hence $C$ is a tiger. The same reasoning gives us a tiger in the case where $x_0 = (4, 2)$ too.

We show below, just as an illustration, how one can exclude the case in which $e_0 < 1/2$. Let $x_0$ be of type $(3, A_r)$. Then

$$K_{S_0}^2 = \frac{2(g + r + 2)^2}{(2g + 1)(2r + 3)(4gr + 4g - 1)}$$

Since there is (2) point and since $g - 1 \leq r$, in the sum of the singularities of lemma 3.20 we already have a term which is at least $1 + 2g(1 - \frac{1}{2g+1})$. By lemma 3.22 we get that $8 > 2g(1 - \frac{1}{2g+1})$ and hence $g \leq 4$. For the case $g = 4$ note that by lemma 3.20 we have $r = 3$, otherwise the right hand side would be negative. By the above we get $K_{S_0}^2 = 2/(7 \cdot 9)$, which means that there would be an $(3, A_2)$ point. This again contradicts lemma 3.20. Suppose $g = 3$. If $3 \leq r \leq 5$, the $12r + 11$ part of the denominator has big primes in its factorization, contradicting lemma 3.20. If $r = 2$, then $K_{S_0}^2 = 2/35$, hence the only non Du Val singularities are of type $(3, 2)$ and $(3, 2, 2)$. However it’s easy to see that $2/35 = 9 - u - n_1 8/5 - n_2 18/7$ has no solutions in integers.

Finally, for the case $g = 2$, running a computer program shows there are no solutions either.

\[\square\]

**Lemma 4.9.** If $\text{char}(k) \neq 2$ and $\pi_1$ has type $U$ or $V$, then $S_0$ has a tiger.

**Proof.** The proof goes along almost exactly the lines of [KM99, Lemma 15.4], with only minor modifications. In particular, we have that $F$ passes through at least two singular points, $M$ is empty, and $A_1$ is in the Du Val locus of $S_1$, with at most one singularity on it.

Suppose $A_1$ is in the in the smooth locus of $S_1$. By [KM99, Lemma 12.3] all the non Du Val points on $S_1$ are almost Du Val and $F$ does not pass through Du Val points. By [KM99, 15.4.1], there can’t be two non Du Val points on $F$; contradiction.

So there is exactly one Du Val singularity on $A_1$ (necessarily in $A \cap F$), and it must be a (2) point by [KM99, 15.4.1]. We can’t be in configuration $U$, since otherwise $x_0 = (2, 4, 2, 2)$ which has coefficient $e(x_0) > 2/3$. So we are in configuration $V$ and $x_0 = (4, 2)$. Now one can compute $K_{S_0}^2 = -5/77$ and $\Sigma_1 = 15/77$. However, $\pi^*(F) \equiv 11/3\Sigma_1 + 1/3E_1$, hence there is a curve $C \subset S_0$ such that $3C \equiv \Sigma_1$, and $C$ is a tiger. \[\square\]
4.3 $A_1$ has a simple cusp

Here we prove

**Proposition 4.10.** If char$(k) \neq 2, 3, 5$ then $A_1$ can’t have a cusp of genus one.

In the following discussion we always assume that char$(k) \neq 2, 3, 5$. $K_{T_1} + a_1\Sigma_1$ and $K_{S_1} + a_1A_1$ are flush by proposition 3.11. The possible configurations are described in lemma 3.13. As case I has been ruled out in lemma 4.6, we may assume that $a_1 \geq 2/3$. Clearly $A_1$ is not in the smooth locus of $S_1$, for otherwise it would be a tiger by adjunction. By lemma 3.13, $2/3 \leq a_1 < 4/5$. Note that $A_1 \neq \Sigma_2$ because $\Sigma_2$ is smooth. By lemma 3.23, $e_1 \geq 1/3$, so $a_1 + e_1 \geq 1$ and the second hunt step is classified in proposition 3.11. Clearly $A_2$ is still a rational curve with a single cusp, of genus one, by proposition 3.11. Also, at least one of $-(K_{S_2} + A_2)$ and $-(K_{S_2} + B_2)$ is ample, hence $-(K_{S_2} + B_2)$ is ample and $B_2$ is smooth.

**Lemma 4.11.** $(S_2, A_2 + B_2)$ is not a fence.

**Proof.** Assume $S_2$ is a fence. Suppose first that $A_2$ is not in the smooth locus of $S_2$. Then, since $K_{S_2} + 2/3A_2$ is negative, we have a birational map to a Gorenstein log Del Pezzo surface $W$ by [KM99, Lemma-Definition 12.4]. As usual, we let $M$ be as in [KM99, Chapter 12] (and $F = \emptyset$). The rank of $W$ is one more than the difference between the number of singular points on $A_2$ and the number of irreducible components of $M$. By [KM99, Lemma 12.5], no irreducible component of $M$ passes through more than two singular points. Furthermore $0 < (K_{S_2} + A_2) \cdot B_2 < 1$, hence $M$ meets $B_2$ at least once, and always at singular points of $S_2$. If the rank of $W$ were more than one, then there would be a component of $M$ which passes through two singular points of $A_2$, which is absurd since it would also meet $B_2$ at a singular point. So the rank of $W$ is one. Let $G$ be an exceptional divisor adjacent to $A_2$. Then we have that $A_2$ is in the smooth locus of $W$, has a cusp, $A_2 \in | -K_W|$ and $K_W \cdot G = K_W \cdot B_2 = -1$. By the description of section 2.2, we get that the corresponding extremal rational elliptic surface has a fiber of type $II$, and hence it’s Mordell-Weil group is trivial by theorem 2.5. But, since $G$ and $B_2$ don’t intersect on $A_2$, there must be at least two sections, contradiction.

Hence $A_2$ is in the smooth locus of $S_2$, and $S_2$ is Gorenstein. Since $K_{S_2} + B_2$ is log terminal, by adjunction we get $0 < B_2^2 = -2 + \deg(\text{Diff}_{B_2}(0)) - K_{S_2} \cdot B_2$. However $K_{S_2} \equiv -A_2$, hence $\deg(\text{Diff}_{B_2}(0)) > 1$, and there are at least two singularities on $B_2$. Again as above, the corresponding extremal rational elliptic surface has a fiber of type $II$, and hence there is another singular fiber, of type $II^*$. The section meets the the farthest end of the $E_8$ Dynkin diagram, and we see that the only way to obtain two singular points is to blowdown until we reach the center of the diagram. That means that $S_2$ is $S(A_1 + A_2)$, and is obtained by taking a flex cubic in $P^2$ and the tangent line to its flex, blowing up three times to separate them and the blowing down the $-2$ curves. However then we have tigers: if $a \geq 5/6$ there is a tiger over the singular point of $A_2$, the $(-1)$ curve of the configuration, and otherwise $B_2$ is a tiger as $K_{S_2} + (5/6)A_2 + B_2$ is numerically trivial. □
Lemma 4.12. The Gorenstein log Del Pezzo surface \( W \) associated to \( S_1 \) as in [KM99, Lemma-Definition 12.4] has rank at least two.

Proof. Suppose \( W \) has rank one. Suppose also that \( A_1 \) has at least two singularities. Then the image of \( A_2 \) in \( W \) has a cusp and meets two \(-1\) curves. But then we obtain a contradiction as above, because the Mordell-Weil group of the associated extremal rational elliptic surface is trivial.

Hence \( A_1 \) has just one singularity and \( M \) of [KM99, Definition-Lemma 12.0] is irreducible. Consider the morphism \( f : Y \to S_1 \) extracting the exceptional divisor \( G \) adjacent to \( A_1 \) and the morphism \( \pi : Y \to W \) contracting \( M \). Define \( \Gamma \) by \( K_Y + \Gamma = f^*(K_{S_1} + a_1A_1) \) and \( \Gamma' = \lambda(\Gamma + \epsilon G) \) such that \( K_Y + \Gamma' \) is \( \pi \)-trivial. Let \( \Delta' = \pi(\Gamma') \). Clearly \( K_W + \Delta' \) is negative by lemma 3.8. Now \( A_1 \) is in the smooth locus of \( W \) and \( K_W \cdot G = -A_1 \cdot G = 1 \). If \( K_W^2 \geq 2 \), then \( K_W + G \) is anti ample because \( (K_W + G) \cdot A_1 < 0 \), and hence \( G \) is a smooth \((-1)\) curve. If \( K_W^2 = 1 \), then by the same process of reduction to elliptic surfaces as above, we have that \( W = S(E_k), G \in |{-K_W}| \) and by the description of the fibers, it’s smooth. So we see that in any case \( G \) is a smooth \((-1)\) curve.

Now, if \( K_W + G \) is log terminal, \( A_1 + G \) is a fence and we can proceed as in lemma 4.11. Otherwise \( G \) meets a unique curve \( V \) of the minimal resolution since it’s a fiber of the associated elliptic surface. Let \( h : Q \to W \) extract \( V \). \( G \) is a \(-1\) curve in the smooth locus of \( Q \), so we can contract it with \( r : Q \to W_1 \). Notice that \( K_{W_1}^2 = K_W^2 - 1 \). Scaling again as in lemma 3.8 and repeating the process with \( A_1 \) and \( r(V) \), we can induct on \( K_{W_1}^2 \), by lemma 3.8.

Lemma 4.13. \( S_1 \) is singular along \( A_1 \) in at least two points and is Du Val outside \( A_1 \).

Proof. The first part is obvious from the above and second part follows from [KM99, Lemma 12.5]. 

Lemma 4.14. \( S_1 \) is not Gorenstein.

Proof. From now on we suppose that \( S_1 \) is Gorenstein and derive a contradiction. Let’s start by noting that \((K_{S_1} + A_1) \cdot A_1 \geq 1 \) by adjunction. Since \((K_{S_1} + 2/3A_1) \cdot A_1 < 0 \), we have \( A_1^2 > 3 \). Note also that \( K_{S_1}^2 > 4/9A_1^2 \), hence \( K_{S_1}^2 \geq 1 \).

First of all let’s show that \( A_1 \) can’t have three singularities. By the classification of log terminal singularities, these would be either \( 2A_1 + A_0 \) or \( A_1 + A_2 + A_k \), with \( k \leq 4 \). By theorem 2.7 the only possibilities are \( 2A_1 + A_3 \) or \( 3A_1 + D_4 \). In the first case \( 4/5 \leq e_0 \leq a_1 \), contradiction. In the second case \( a_1 > e_0 \geq 2/3 \), and by \((K_{S_1} + A_1) \cdot A_1 = 3/2 \) we get that \( K_{S_1}^2 \geq 3 \) as above, contradiction.

So \( A_1 \) has just two singularities. Note they can’t be both \( A_1 \) points, for otherwise \((K_{S_1} + A_1) \cdot A_1 = 1 \), hence \( K_{S_1} + M \) is log terminal. \( M \) does not contain other singular points by [KM99, Lemma 12.5]. Extracting the \((-2)\) curves of the \( A_1 \) points, contacting \( M \) and one of the extracted curves, we obtain a Gorenstein log Del Pezzo surface with a cuspidal rational curve in the smooth locus. However there is no Gorenstein log Del Pezzo surface in the list of theorem 2.7 such that adding two \( A_1 \) points remains Gorenstein, contradiction.
Also, they can’t be an $A_1$ an one $A_2$ point. In fact, by the classification of theorem 2.7 and the fact that $K_{S_1}^2 \geq 2$, we see that $S_1 = S(A_1 + A_2)$ and $K_{S_1}^2 = 6$. But then $\hat{A}_1^2 = K_{W}^2 > K_{Y}^2 = K_{S_1}^2 = 6$, and we obtain a contradiction thanks to [KM99, Lemma 16.5]. Thus we also have shown that if $A_1$ has a $(2)$ point on it, then it also has either an $A_3$ or an $A_5$, again by the classification of theorem 2.7.

After these preliminaries, we rule out each configuration in lemma 3.13.

Configurations $II, u, w, (u; n), (v; f), (v; f^2)$: $x_0$ is a non chain singularity and $e_0 \geq 4/5$, contradiction.

Now we are left with configurations $v, (v; n); (v; n^2)$. If $K_{S_1}^2 \geq 6$, we are done as above. Also, there are no Gorenstein log Del Pezzo surfaces with $K_{S}^2 = 5$ and at least two singularities, by the list in 2.7. On the other hand, combining $K_{S_1} + 5/7A_1 < 0$ and $(K_{S_1} + A_1) \cdot A_1 \geq 5/4$, we get that $K_{S_1}^2 \geq 3$. Hence $K_{S_1}^2$ is either three or four, and the only possibilities for $S_1$ are $S(2A_1 + A_3)$, $S(A_1 + A_5)$ and $S(3A_2)$.

Configuration $v$: if we have $S(2A_1 + A_3)$, $A_1$ contains the $(2)$ and $(2, 2, 2)$ points and $E_2^2 = -3$. It follows that $K_{S_0}^2 = 1/21$, but there is no singularity to make up for the factor three in the denominator since $S_0$ is Du Val outside $x_0$ and the $(3, 2, 2)$ point. For $S(A_1 + A_5)$ and $S(3A_2)$ one can instead compute $K_{S_0} \cdot \Sigma_1 = -4/35$, $\Sigma_1^2 = 16/35$ and $f_x, \pi^* M = 7/4\Sigma_1$, hence there is a tiger.

Configuration $(v; n)$: By looking at the coefficient we get $E_2^2 = -3$. If we have $S(2A_1 + A_3)$ then the strict transform $\overline{M}$ of $M$ on $S_0$ is such that $\overline{M} \equiv 8/5\Sigma_1$. Hence there is a curve $C$ such that $5C \equiv \Sigma_1$. Since $K_{S_0} \cdot \Sigma_1 = -5/28$ and $\Sigma_1^2 = 25/56$, we get that $C$ is a tiger for $S_0$. If we have $S(A_1 + A_5)$ or $S(3A_2)$ then $K_{S_0}^2 = 9/190$, which has a factor nineteen, contradiction.

Configuration $(v; n^2)$: since there is a point of coefficient $2/3$ we necessarily have $E_2^2 = -4$. In the case $S(2A_1 + A_3)$ one gets that on $S_0$ we have $M \equiv 9/5\Sigma_1$, $K_{S_0} \cdot \Sigma_1 = -5/99$ and $\Sigma_1^2 = 25/99$. Hence again we have tigers. Finally, in the cases $S(A_1 + A_5)$ and $S(3A_2)$, $K_{S_0}^2 = 1/(18 \cdot 19)$, and there is a factor nineteen to contradict lemma 3.20.

\begin{lemma}
$S_2$ is not a banana or a net.
\end{lemma}

\begin{proof}
The proofs in [KM99, Lemma 16.5, Lemma 16.6] carry through. We just point out that the argument in [KM99, Lemma 16.5] does not actually use the Bogomolov bound, and that the reduction to $S(A_1)$, $S(A_1 + A_2)$ and $S(A_4)$ still holds without the simply connected hypothesis thanks to lemma 2.7 and theorem 2.5.
\end{proof}

Since these we ruled out all the possibilities given by proposition 3.11, we see that if $A_1$ has a simple cusp, then $S_0$ has a tiger.

### 4.4 $A_1$ has a simple node

From now on we suppose that $\text{char}(k) \neq 2, 3, 5$, $A_1$ has a simple node and $S_0$ doesn’t have tigers. Let’s start with some preliminary lemmas.
Lemma 4.16. Suppose $S$ is a rank one log Del Pezzo surface such that it has a rational nodal curve $A$ in its smooth locus, and there are two rational curves $C$ and $D$ such $K_S \cdot C = K_S \cdot D = -1$, but do not meet the nodal curve at the same point. Suppose also that $K^2_S \geq 4$. Then $S$ is Gorenstein and $S = S(2A_1 + A_3)$. Furthermore $C$ and $D$ each pass through one of the $A_1$ points and opposite ends of the $A_3$ point.

Proof. Clearly $S$ is Gorenstein as $K_S \equiv -A$ by adjunction. Note that $(K_S + C + D) \cdot A \leq -2$, hence $C$ and $D$ are smooth by adjunction. By the description in section 2.2, there are at least two sections in the corresponding elliptic surface. By looking at the classification in theorem 2.5, we see that starting with an extremal rational elliptic surface with non trivial Mordell-Weil group and a fiber of type $I_1$, the only Gorenstein log Del Pezzo surface with $K^2_S \geq 4$ that we obtain is $S(2A_1 + A_3)$, coming from the elliptic surface with singular fibers $III^*, I_2, I_1$. Now one can get the result by either looking at the description of the elliptic surface, or by following the argument in [KM99, Lemma 3.9.2].

Remark 4.17. The proof of [KM99, Proposition 13.5] still works in our setting. We fix notation as in [KM99, Chapter 17].

Notation 4.18. Let $C$ and $D$ be the two branches of $A_1$ at the node, and $c,d$ be the points of $T_1$ where the branches meet $\Sigma_1$. We can assume that the first two blow ups of $h: \tilde T_1 \to S_1$ are along $C$. Let $r + 1$ be the initial number of blow ups along $C$, $r \geq 1$. Note that $d$ is necessarily singular.

Lemma 4.19. Notation as above.

1. $K_{T_1} + \Sigma_1 + E_1$ is log canonical.
2. $\Sigma_1$ has two smooth branches through $x_0$ and meets no other singularities.
3. If $c$ is smooth, then $d$ in an $A_r$ point, $r \geq 1$ and $a_1 = (r + 1)/(r + 2)$.
4. $T_1$ is singular at some point of $E_1 \setminus E_1 \cap \Sigma_1$.
5. $A_1$ contains exactly one singularity.
6. $S_0$ has exactly two non Du Val points and $e_0 > 1/2$.
7. $a_1 \geq 2/3$, and $a_1 \geq 4/5$, unless we have (3) with $r \leq 2$.

Proof. This is [KM99, Lemma 17.2]. Same proof applies, using lemma 4.16.

Let $x_0$ and $y$ be the non Du Val points on $S_0$, and $z$ the singularity on $A_1$, with index $s$.

Proposition 4.20. $(S_2, A_2 + B_2)$ is a fence. $g(A_2) = 1$ and $B_2$ is smooth.

1. If $x_1 \in A_1$ then $\Sigma_2$ meets $E_2$ at a smooth point, and contains a unique singular point $(A_1, 3, A_{j-2})$ for some $t$. $K_T + \Sigma_2$ is log terminal, and $\Sigma_2$ meets the end of the $A_{j-2}$ chain. $(S_2, A_2 + B_2)$ is given by [KM99, 13.5] and $q_2$ is an $A_{t+1}$ point.
2. If \( x_1 \notin A_1 \), and \( A_2 \) is not in the smooth locus of \( S_2 \), then \( S_2 \) is obtained by starting from \( S(2A_1 + A_3) \), blowing up on the \((-1)\) curve \( C \neq B_2 \) at the \( A_3 \) point once and then contracting \( C \). To obtain \( S_1 \) blow up on the intersection of \( A_2 \) and \( B_2 \) twice along \( A_2 \), then contract \( B_2 \). To obtain \( S_0 \) blow up twice along one of the branches of \( A_1 \) and then contract \( A_1 \). In particular \( x_0 \) is a chain singularity.

3. If \( x_1 \notin A_1 \) and \( A_2 \) is in the smooth locus of \( S_2 \), then \( a_2 < 6/7 \) and \((S_2, A_2 + B_2)\) is given by [KM99, 13.5.1], or we have the following description of \( S_0 \): start by \( S(A_1 + A_2) \) or \( S(3A_2) \), consider a \((-1)\) curve \( B \) passing through two of the singularities, consider a rational nodal curve \( A \) in the smooth locus, blow up twice on \( A \cap B \) along \( B \), three times on the node of \( A \) along the same branch, and then contract all the negative curves with self intersection less than \((-1)\); or start by \( S(2A_1 + A_3) \), blow up on \( A \cap B \) twice along \( B \), blow up on the node either four or five times along the same branch, the contract as above; or start by \( S(2A_1 + A_3) \), blow up three times on \( A \cap B \) along \( B \), blow up four times on the node along the same branch and then contract as above. In particular, if \( S_2 \neq S(A_1 + A_2) \) then \( x_0 \) is a chain singularity.

Proof. Part (1) follows as in [KM99, Proposition 17.3].

Suppose \( x_1 \notin A_1 \). Suppose also that \( A_2 + B_2 \) has a node of genus \( g \geq 2 \). Then we have that \( z \in \Sigma_2, a_1 = 2/3, g = 2, r = 1, E_1^2 \leq -3 \), again as in [KM99, Proposition 17.3]. The point \( x_0 \) is a chain singularity of type \((2, 2, -E_1^2, 2)\), hence \( E_1^2 = -3 \), for otherwise the coefficient would be too high. Thus \( K_2^2 = A_2^2 = r+4+g-k = 4 \). Now notice that \((K_2 + 2/3A_2 + B_2) \cdot B_2 > 0 \) because there are no tigers, hence \((K_S + B_2) \cdot B_2 > -2/3 \). By adjunction there are at least two singularities on \( B_2 \), and by checking the list 2.7 we see that \( S_2 = S(2A_1 + A_3) \).

But then, applying adjunction again, we see that \( B_2 \) has to contain all of these singularities. That means that \( x_1 \) is a non chain singularity. Furthermore, since it's not Du Val by lemma 4.19, \( E_2^2 \leq -3 \) and \( e_1 > 2/3 \), contradiction.

So we must go to a fence. Suppose furthermore that \( A_2 \) is in the smooth locus and \( S_2 \neq S(A_1 + A_2) \).

Let's start by the case where \( x_1 \) is a non chain singularity. Since \( A_2 + B_2 \) is a fence, and \( S_2 \) has no tigers, then \( B_2 \) contains at least two singularities and they can't be both \((2)\) points. Looking at the possibilities of [KM99, Lemma 13.5] and comparing them with the classification of non chain log terminal singularities, we see that \( \Sigma_2 \) meets \( E_2 \) at a point of index two, or three if \( S_2 = S(2A_1 + A_3) \).

If \( \Sigma_2 \) meets \( E_2 \) at a \((2)\) point, then \( z = (A_k, 3) \) and \( E_2^2 = -3-k \), with \( k \geq 0 \). If \( S_2 \neq S(A_1 + A_2) \) we have that \( A_2^2 = K_2^2 \leq 4 \), which means that \( K_S \cdot A_1 \geq -3 \) by adjunction and the description of the configuration. Hence \( K_{S_1} \cdot A_1 \geq -4 + 1/(2k + 3) \). We also have that \((K_S + A_1) \cdot A_1 = (2k + 2)/(2k + 3) \) by adjunction, and therefore by linearity of the intersection product we get

\[
a_1 < \frac{3 - 1/(2k + 3)}{3 - 1/(2k + 3) + (2k + 2)/(2k + 3)} = \frac{3k + 4}{4k + 5}
\]
One checks that the singularity at $x_1$ has always greater coefficient.

Let’s consider now the case where $\Sigma_2$ meets $E_2$ at a (3) point. If it meets $A_1$ at a (2) point, then $E_1$ is a $-2$ curve, $A_1^2 = 2$, $K_{S_1} \cdot A_1 = -2$ and $(K_{S_1} + A_1) \cdot A_1 = 1/2$. That gives us that $a_1 < 4/5 = e_1$, contradiction. Otherwise $\Sigma_2$ meets $A_1$ at an $(A_k, -3, -2)$ point. By making the same computations as above, we get that $\tilde{E}_2^2 = -3 - k$ and

$$a_1 < \frac{3 - 2/(3k + 5)}{3 - 2/(3k + 5) + (3k + 4)/(3k + 5)} = \frac{9k + 13}{12k + 17}$$

Again, the singularity at $x_1$ has always greater coefficient.

The only remaining case is that $\Sigma_2$ meets $E_2$ at an $A_2$ point and $A_1$ at a $(A_k, -4)$ point, and can be ruled out as above.

Let’s consider now the case in which $x_1$ is a chain singularity. Then the configuration is given by (3) of lemma 3.12, $\Sigma_2$ meets $A_1$ at an $A_k$ point and $b_2 + a_2/(k + 1) = 1$. Since $(K_{S_2} + a_2A_2 + b_2B_2) \cdot A_2 < 0$, we get that

$$a_2 < \frac{K_{S_2}^2 - 1}{K_{S_2}^2 - 1/(k + 1)}$$

By lemma 4.19 $K_{S_2}^2 \geq 3$, for otherwise $a_1 < 2/3$.

If $K_{S_2}^2 = 3$, then $S_2 = S(A_1 + A_3)$ or $S(3A_2)$ and $a_1 \leq 3/4$, again by the above inequality and lemma 4.19. Since $B_2$ is a $(-1)$ curve, we get that $\tilde{E}_2^2 = -2 - k$, and hence $k = 1$ because $e_1 < a_2$. One computes $e_1 = 3/5$, hence $\tilde{E}_1^2 = -4$ and $r = 2$.

Suppose now that $K_{S_2}^2 = 4$ and hence $S_2 = S(2A_1 + A_3)$. One computes $e_1 = \frac{4k}{3k+1}$. Setting $e_1 < a_2$ we get that $k = 1$ or $k = 2$, and $a_2 < 6/7$ or $a_2 < 9/11$ respectively. Suppose that $x_0$ is a chain singularity. It follows from lemma 4.19 that $a_1 = (r + 1)/(r + 2)$. Let’s start with $k = 1$. Then $\tilde{E}_1^2 = -1 - r$, and $e_0 = \frac{2r^2 - 2}{2r^2 + r + 1}$. The only solutions to $e_1 < e_0 < a_2$ are $r = 3$ or $r = 4$. In the case with $k = 2$, one finds again $r = 3$, exactly as above. If $x_0$ is a non chain singularity instead there are no solutions, for one can check that if there are many blow ups at the node, the coefficient is at least 6/7, and if there are few then the coefficient the coefficient is lower than $e_1$.

Now if $A_2$ is not in the smooth locus of $S_2$ instead, the proof of [KM99, Proposition 17.3] gives the result. \hfill \box

**Lemma 4.21.** Suppose that $x_0$ is a non chain singularity. Then $x_1 \in A_1$ and $S_0$ is obtained by blowing up the end of the $A_5$ point in $S(A_1 + A_3)$ along the $(-1)$ curve $Y$, then blowing up the node twice along one branch and the once along the nearest point of the other branch, and finally contracting down everything.

**Proof.** The proof is the same as in [KM99, Lemma 17.4], by using proposition 4.20 instead of [KM99, Proposition 17.3]. \hfill \box

We conclude this section by noting now that the same classification as in [KM99, 17.7 - 17.14] carries through, by the previous results.
4.5 Smooth fences

In this subsection we collect some useful facts about abstract smooth fences. We will then apply these results during the hunt in the next sections. Throughout the subsection \((S, X + Y)\) will be a smooth fence, by which we mean \(S\) is a rank one log Del Pezzo surface, and \(X + Y\) is a fence, with \(X\) and \(Y\) both smooth rational curves. No assumption about the existence of tigers is made. We will also define \(\alpha = - (K_S + X) \cdot X\) and \(\beta = - (K_S + Y) \cdot Y\). Let’s begin with the following elementary but fundamental lemma.

**Lemma 4.22.** Suppose that \(\alpha, \beta \neq 1\). Then

1. \(X^2 = \frac{1 - \alpha}{1 - \beta}\). In particular \(\alpha > 1\) if and only if \(\beta > 1\).

2. \((K_S + tX) \cdot X = (1 - \alpha)\frac{t - \beta}{1 - \beta} - 1\).

3. If \(K_S + aX + bY\) is negative and \(\alpha, \beta < 1\), then \(a(1 - \alpha) + b(1 - \beta) < 1 - \alpha \beta\).

**Proof.** Since \(S\) is rank one and \(X + Y\) is a fence, \(K_S + X \equiv -\alpha Y\) and \(K_S + Y \equiv -\beta X\). Hence \(K_S + X + \alpha Y \equiv K_S + Y + \beta X\), from which \(X \equiv \frac{1 - \alpha}{1 - \beta} Y\). Now the results follows easily.

**Lemma 4.23.** If \(\alpha > 0\) then \(\hat{Y}^2 \neq -1\).

**Proof.** Suppose \(Y\) is a \((-1)\) curve. Then \(K_S \cdot Y \geq -1\), hence \((K_S + X) \cdot Y \geq 0\). It follows then that \(K_S + X\) is nef, contradicting the fact that \(\alpha > 0\).

**Lemma 4.24.** Suppose \(Y\) has two singular points and \(X\) has at least two singular points. Then \(\hat{Y}^2 \neq 0\).

**Proof.** Suppose \(\hat{Y}^2 = 0\). Let \(f : T \to S\) be the extraction of the two adjacent curves \(L\) and \(M\) to \(Y\). Since \(\hat{Y}^2 = 0\), then we have a contraction \(\pi : T \to \mathbb{P}^1\). Clearly \(Y\) is a fiber and \(X\) and the two extracted curves are sections. Since the relative Picard number of the contraction is two, there is exactly one reducible fiber. However, since \(X\) has at least two singularities, there is a multiple and irreducible fiber \(F\). This fiber can contain at most two singularities, hence it touches either \(L\) or \(M\) at a smooth point, contradiction.

**Lemma 4.25.** \(X\) and \(Y\) can’t both have exactly two singularities.

**Proof.** Suppose they do. Without loss of generality \(Y^2 \leq 1\). However, this contradicts either lemma 4.23 or lemma 4.24.

The following lemma, taken from [KM99], turns out to be very useful when dealing with fences.

**Lemma 4.26.** Suppose \(C\) is a smooth rational curve in the smooth locus of a rank one log Del Pezzo surface \(S\). Then either \(S = \mathbb{F}^2\) or \(S = \mathbb{F}_n\) and \(C \in |\sigma_n|\).

**Proof.** See [KM99, Lemma 13.7].

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Corollary 4.27. Let \( S(X + Y) \) be a fence such that \( X \) contains at least two singularities and \( Y \) exactly two singularities, one Du Val and the other a \((-n)\) curve. Then \( \hat{Y}^2 \neq -1 \).

Proof. Suppose \( \hat{Y}^2 = -1 \). Let \( f : T \to S \) be the extraction of the \((-n)\) curve \( E \) on \( Y \). Then the strict transform of \( Y \) is the curve that gets contracted in lemma 3.8. Clearly we get a birational morphism \( \pi : T \to S_1 \), and the image of \( E \) is a smooth rational curve in the smooth locus of \( S_1 \). However, since there are at least two singularities along \( X \), we get a contradiction by lemma 4.26.

Corollary 4.28. Let \( S(X + Y) \) be a fence such that \( X \) has three singularities and \( Y \) two singularities. Suppose also that one of the two singularities on \( Y \) is a \((-2)\) curve. If \( \text{char}(k) \neq 2 \) then \( \hat{Y}^2 \neq -1 \).

Proof. Suppose \( \hat{Y}^2 = -1 \). Proceeding as above, on \( S_1 \) we have that \( X \) and \( E \) meet once with order two. If \( E \) is in the smooth locus of \( S_1 \), we are done, otherwise there is a singular point. Let \( f_1 : T_2 \to S_1 \) extract the adjacent divisor \( G \) to \( E \). We have again a contraction, thanks to lemma 3.8. This contraction is not a net, for otherwise \( E \) would be either a fiber or a multi section. It can’t be a fiber because otherwise \( X \) would be a double section with at least four ramification points, contradicting Riemann-Hurwitz. It can’t be a multi section either, because \( K_{S_1} + E \) is negative, so we get a birational morphism \( \pi_2 : T_2 \to S_2 \). The curve \( \Sigma_2 \) does not touch \( E \), because \( K_{S_1} + E \) is negative. Also, it can pass through at most two singularities. If the initial singularity on \( Y \) had at least three exceptional components, then we are done by lemma 4.26 since there is a smooth rational curve in the smooth locus of \( S_2 \), and there are at least two singularities. Suppose instead \( G \) is in the smooth locus of \( T_2 \). We have that \( (K_{S_1} + X) \cdot X \geq 0 \) by lemma 4.23, hence \( (K_{T_2} + X) \cdot E > -1 \). It follows that \( K_{S_2} : E > -3 \), and hence \( E^2 < 1 \) on \( S_2 \), contradiction.

Lemma 4.29. Suppose that \( X \) has at least three singularities, \( Y \) exactly two singularities. Then \( \beta \leq 2/3 \).

Proof. Suppose \( \beta > 2/3 \). Then there is a \((-2)\) singularity on \( Y \). Since \( X \) has at least three singularities \( \alpha \leq 1/2 \), so that \( Y^2 = \frac{1}{\beta^2} < 1 \) and \( \hat{Y}^2 \) is either zero or negative one and we can conclude by lemmas 4.28 and 4.24.

Lemma 4.30. Suppose that \( X \) has three singularities, \( Y \) exactly two singularities. Then \( \alpha \leq 1/6 \), and if \( \alpha > 0 \) then \( \beta < \alpha \).

Proof. Let’s start by noticing that, if \( \alpha \leq \beta \), then \( Y^2 \leq 1 \), and \( Y \) must be a \((-1)\) curve by lemma 4.24. In that case we must have \( \alpha \leq 0 \) by lemma 4.23.

Now, if \( \alpha > 1/6 \) instead, we must have \( \beta < \alpha \) and the singularities on \( X \) are two \((2)\) points and a point of index \( k \), with \( 2 \leq k \leq 5 \). Clearly \( X^2 = l/k \), with \( k \) and \( l \) coprime. Since \( \alpha = 1/k \), we also have \( X^2 = \frac{(k-1)/k}{1-\beta} \), which leads to \( \beta = 1 - \frac{k-1}{l} \). By the above \( \beta < \alpha \), therefore \( \frac{k-1}{l} > k - 1 \), hence \( l < k \). But then \( \beta \leq 0 \), contradiction.
**Lemma 4.31.** Suppose that $X$ has three singularities, $Y$ exactly two singularities. Then $K_S + 4/5X + 2/3Y$ is positive.

*Proof.* This follows from lemma 4.30 and lemma 4.22. ∎

**Lemma 4.32.** If $X$ has at most one singularity, so does $Y$.

*Proof.* Suppose $Y$ has at least two singularities. Then $\beta \leq 1 < \alpha$, but this can’t happen by lemma 4.22. ∎

Now we characterize smooth fences with less than one singular points on each branch.

**Lemma 4.33.** Suppose that $X$ and $Y$ have both exactly one singular point, with $Y^2 \leq X^2$. Then $S$ is obtained by starting with the Hirzebruch surface $\mathbb{F}_n$, picking the section with negative self intersection $C$ and another positive disjoint section $D$, picking a fiber $F$, blowing up once at $F \cap C$ or $F \cap D$, then at the intersection of the exceptional divisor and $F$, followed by blow ups at either end of the $(-1)$ curve, and eventually blowing down all the $K$-positive curves. $X$ will be the strict transform of $D$, $Y$ the strict transform of a fiber different from $F$.

The fence $(S, X + Y)$ is determined by the choice of a point on the negative section of the surface $\mathbb{F}_n$, a positive section, and a fiber not passing through the marked point.

*Proof.* $Y$ is clearly a 0-curve, hence, after extracting it’s adjacent divisor $E$, we get a net $T$. $Y$ is a fiber, $X$ and $E$ are sections. $E$ can’t be in the smooth locus since $T$ is singular, hence has a unique singularity which lies on the same fiber as the singularity on $X$. Now, by considering the minimal resolution and running a relative MMP on the minimal resolution of $T$, we get the result. ∎

### 4.6 $A_1$ is smooth

In this subsection we classify all cases in which $A_1$ is smooth. The first part is dedicated to prove that in the second hunt step we must go to a smooth banana, whereas the second part will classify such configurations. Throughout the discussion $\text{char}(k) \neq 2, 3, 5$. Let’s start by proving the following proposition.

**Proposition 4.34.** If $A_1$ is smooth, then $A_1$ has exactly three singularities and $(S_2, A_2 + B_2)$ is a smooth banana.

Clearly $A_1$ has either three or four singularities. Let’s start with a preliminary lemma.

**Lemma 4.35.** $\tilde{A}_1^2 \geq -1, a_1 > 2/3, e_1 > 1/2$.

*Proof.* $\tilde{A}_1^2 \geq -1$ is clear, since $K_S \cdot A_1 < 0$.

Also, the lemma follows from [KM99, Lemma 18.2.4] in the case in which $A_1$ has three singularities. Suppose then that $A_1$ has four singularities, hence $x_0$ is
a non chain singularity and $\Sigma_1$ meets $E_1$ at a smooth point. If $e_0 = 1/2$, and there is a chain singularity with maximal coefficient, restart the hunt with that singularity. Otherwise we can assume that all chain singularities are either Du Val or almost Du Val. Let the branches of $x_0$ be $(2)$, $(2)$ and $(A_t, 3)$. In order to create the fourth singular point, there is a $(3, A_k)$ point (with possibly $k = 0$), which $\Sigma_1$ can meet at either end. By lemma 3.20 we get that $r + k \leq 4$. Suppose that $\Sigma_1$ meets the end of the $A_k$ chain. Then $K_{S_0} \cdot \Sigma_1 = -1/2 + 1/(2k + 3)$ and $\Sigma_1 = -2/(2k + 3) + r + 3/2$. If $k = 0$ and $r = 0$ then $A_1$ is a tiger by adjunction, and by testing all other possibilities in $k$ and $r$, we see that the denominator of $K_{S_0}^2$ would imply the existence of other singularities, contradicting lemma 3.20.

Suppose then that $\Sigma_1$ meets the $(-3)$-curve. Then $K_{S_0} \cdot \Sigma_1 = -1/2 + (k + 1)/(2k + 3)$ and $\Sigma_1 = -(k + 2)/(2k + 3) + r + 3/2$, and again the same argument shows that there are no solutions.

Let’s now show that $e_1 > 1/2$. If there is either a non Du Val point or a point of index at least four on $A_1$, we conclude by lemma 3.23. If not, then $E_1$ is not a $(-2)$-curve. If there is a branch of index three in $x_0$, we get that $a_1 > e_0 \geq 3/4$ and we conclude again by lemma 3.23. If not, $e_0 = 2/3, E_1^2 = -3$ and the singularity has three $(2)$ branches. The fourth singular point on $A_1$ can’t be $(2)$, for otherwise there would be a tiger, hence it’s $(2, 2)$. That means that $\Sigma_1$ meets the first curve of an $(A_k, 3, 2)$ singularity, with $k$ possibly zero. Now we can compute $K_{S_1} \cdot A_1 = -k$ and $A_1^2 = k + 1/6$, hence $K_{S_1}^2 = 6k^2/(6k + 1)$. If we had $e_1 \leq 1/2$, using lemma 3.20 one obtains immediately a contradiction. 

**Lemma 4.36.** $(S_2, A_2 + B_2)$ is not a tacnode.

**Proof.** Suppose that $S_2$ is a tacnode. As in [KM99, Lemma 18.5], one shows that $S_1$ is Du Val along $A_1$. By lemma 3.12 we have that $c + dg/(g + 1) = 1$, where $d$ is the coefficient of the branch containing the $A_g$ point. Since $a_2 > 2/3$ and $b_2 > 1/2$, the only possibility is that $g = 2$ and the singularity lies on the curve $A_1$. Let $A_1$ be a $(l - 2)$-curve. Then $(K_{S_1} + A_1) \cdot A_1 \geq l/3$ by the argument in [KM99, Lemma 18.5].

Suppose that that there are three singularities on $A_1$ and let $A_t$ and $A_m$ be the other two singular points. By adjunction $(K_{S_1} + A_1) \cdot A_1 < 2/3$, hence $l = 1$. Since $S_1$ is Du Val along $A_1$, we have that $K_{S_1} \cdot A_1 = -1$ and $A_1^2 = -1/3 + t/(t + 1) + m/(m + 1)$. However $(K_{S_1} + 3/4A_1) \cdot A_1 > 0$ by [KM99, Lemma 18.5], hence $-1 - 1/4 + 3/4(t/(t + 1) + m/(m + 1)) > 0$. So the minimum between $t$ and $m$ is at least three, and if one of the two is three, then the other is at least twelve. Hence $x_0$ is a chain singularity.

Suppose for now the singularities coming from the extraction of $E_1$ are $A_t$ and $A_2$. Then $A_m$ was created by contracting $\Sigma_1$, which touches $E_1$ normally at a smooth point, and passes through the end of a $(A_k, 3, A_{m-1})$ singularity. Say $E_1$ is a $-n$ curve, so that $x_0 = (A_2, n, A_t)$. Then $e_0 = \frac{t(3n - 6) + 3n - 6}{t(3n - 5) + 3n - 2} < 3/4$, which gives $n = 3$. This means that $k = 1$ and $e_0 = \frac{3t + 3}{4t + 1}$. If $e_1 \geq 3/5$ then $a_2g/(g + 1) + b_2 > 1$, contradiction. Hence every other singularity on $S_0$ has coefficient less than $3/5$. Applying lemma 3.20 to $S_0$ we get a contradiction, since we already have singularities $(A_2, 3, A_t)$ and $(2, 3, A_{m-1})$ with $t + m \geq 8$.  

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Hence \( x_0 \) is the form \((A_m, n, A_1)\). The \( A_2 \) point was created by contracting \( \Sigma_1 \), which passes through an \((A_k, 3, 2)\) point. Since \( c_0 < 3/4 \), we have \( n = 3 \). Again by applying lemma 3.20 to \( S_0 \) we see that \( t + m \leq 6 \). However, then \((K_{S_1} + 3/4A_1) \cdot A_1 < 0\), contradiction.

Now let’s consider the case in which \( A_1 \) has four singularities. Then \( x_0 \) is necessarily a non chain singularity, and, since \( a_1 < 3/4 \) by [KM99, Lemma 18.5], we get that \( e_0 = 2/3 \), \( E_1^2 = -2 \) and \( l = 1 \). Then \( \Sigma_1 \) necessarily passes through the \((-3)\) curve of a \((3, A_k)\) point. But then \( K_{S_0} \cdot \Sigma_1 > 0 \), contradiction. \( \square \)

Using the lemmas from section 4.5, we prove that \( S_2(A_2 + B_2) \) can't be a fence. Let’s start with the following.

**Lemma 4.37.** Suppose \( S_2(A_2 + B_2) \) is a fence and \( x_1 \notin A_1 \). Then \( \Sigma_2 \) can't meet \( E_2 \) at an \( A_2 \) point and \( A_1 \) at a smooth point.

*Proof.* Suppose it does. Then \( a_2 + b_2/(r + 1) = 1 \), hence \( b_2 < (r + 1)/(r + 2) \).

Also, since \( S_2 \) has no tigers, \( B_2 \) contains exactly two singular points and \( x_1 \) is a non chain singularity. From lemma 4.35 we get that \( b_2 > e_1 > 2/3 \). The two singular points on \( B_2 \) can’t both be \((2)\) because \( S_0 \) would have a tiger, hence \( r \leq 4 \) by classification of log terminal singularities. Note also that \( r \neq 1 \) because \( b_2 > 2/3 \), so that the two singular points on \( B_2 \) are \((2)\) and a point of index \( d \), with \( 3 \leq d \leq 5 \). But then we conclude by lemma 4.29. \( \square \)

**Lemma 4.38.** Suppose \( S_2(A_2 + B_2) \) is a fence and \( x_1 \notin A_1 \). Then \( \Sigma_2 \) can’t meet \( E_2 \) at a smooth point and \( A_1 \) at an \( A_2 \) point.

*Proof.* Suppose it does. Then \( a_2/(r + 1) + b_2 = 1 \). From \( a_2 > b_2 \) we get \( a_2 > (r + 1)/(r + 2) > b_2 \).

On the other hand \((K_{S_1} + A_1) \cdot \Sigma_2 = -1 + e_1 + 1/(r + 1) > 0\), hence \( e_1 > r/(r + 1) \). If \( x_1 \) is a non chain singularity, it’s coefficient is a rational number of the form \( m/(m + 1) \) for some \( m \) by [KM99, Lemma 8.3.9], contradiction. So \( x_1 \) is a chain singularity, and by lemma 4.25 \( A_1 \) must have four singularities. Let’s proceed by cases on \( r \). If \( r = 1 \) then \( 1/2 < e_1 < 2/3 \). Since \( B_2^2 > 0 \), we necessarily have that \( E_2^2 = -3 \), hence \( B_2 \) is a \((-1)\) curve. It follows by lemma 4.23 that \( \alpha \leq 0 \) and the \((2)\) point that gets contracted by \( \Sigma_2 \) was a branch of the non chain singularity \( x_0 \). If \( b_2 < 3/5 \), then \( x_1 \) is of the form \((2, k, A_k)\) for \( 2 \leq k \leq 4 \), but this is impossible by lemma 4.28. So \( b_2 \geq 3/5 \), and we get that \( c_0 < a_2 \leq 4/5 \), hence \( e_0 \) is either \( 2/3 \) or \( 3/4 \). If \( e_0 = 3/4 \), then \( b_2 < 5/8 \) and the branches of \( x_0 \) are \((2), (2)\) and \((2, 2)\), for otherwise one of them would have been chosen in the second hunt step. Since \( \alpha \leq 0 \), the fourth singularity on \( A_1 \) has index at least six. But then it would be chosen by the second hunt step, contradiction. If \( e_0 = 2/3 \), \( x_0 \) can’t have a branch which is either \((A_k, 4)\) or \((A_k, 3, 2)\), because in both cases they would be chosen by the second hunt step. Hence \( E_1^2 = -2 \) and \( x_0 \) has branches \((2), (2, 2)\) and \((3)\). To rule out this case, some extra work is needed. First of all one notices that the restrictions on \( e_1 \) and lemma 4.28 tell us that the singularities on \( B_2 \) are \((2, 2)\) and either \((2, 2)\) or \((2, 2, 2)\), which correspond to \( B_2^2 = 1/3 \) or \( 5/12 \). In each case we must have \( \alpha = 0 \), hence the third singularity on \( A_2 \) is either \((2, 2)\) or \((3)\). If it was obtained by contracting an \((A_k, 4)\) point, the only possible case is \( k = 1 \),
which gives \( A_2^2 = 7/3 \), contradiction. One similarly rules out the case in which it was obtained by contracting an \( (A_k, 3, 2) \) point.

Suppose now that \( r = 2 \). Let’s start with the case in which the \( A_2 \) point was constructed by contracting an \( (A_k, 3, 2) \) along \( \Sigma_1 \). Then \( a_1 = (3k + 3)/(3k + 5) \), with \( k \geq 1 \) since \( a_1 > 2/3 \). If \( k = 1 \) then \( e_0 < 3/4 \), hence \( e_0 = 2/3 \). However, \( e_1 > 2/3 \), contradiction. Hence \( k \geq 2 \), \( a_1 \geq 9/11 > 4/5 \) and we conclude by lemma 4.31. In the case in which the \( A_2 \) point is a branch of \( x_0 \) instead, if \( e_0 \geq 4/5 \) we are done as above. If \( e_0 = 3/4 \), \( \alpha > 0 \) and therefore \( \beta < \alpha \leq 1/6 \) by lemma 4.30 and we can apply lemma 4.22. Finally if \( r \geq 3 \) then we again conclude by lemma 4.31.

Lemma 4.39. Suppose \( S_2(A_2 + B_2) \) is a fence and \( x_1 \notin A_1 \). Then \( \Sigma_2 \) can’t meet both \( A_1 \) and \( E_2 \) at singular points.

Proof. Suppose it does. By lemma 4.32 \( x_1 \) must by a non chain singularity, and by lemma 4.25 \( A_1 \) must have four singular points, so that \( x_0 \) is a non chain singularity as well. The singular points of \( B_2 \) can’t be both (2) points, for otherwise it would be a tiger. If \( \Sigma_2 \) meets \( E_2 \) at a point of index bigger than two, then there is a (2) point on \( B_2 \) we get a contradiction by lemma 4.29.

If \( \Sigma_2 \) meets \( E_2 \) on a (2) point instead, then it meets \( A_1 \) on an (3, \( A_k \), with \( \Sigma_2 \) touching the (–3) curve and \( A_1 \) touching the (–2) curve. By imposing \( K_{S_1} \cdot \Sigma_2 < 0 \) and \( (K_{S_1} + A_1) \cdot \Sigma_2 > 0 \), we get that \( (2k + 2)/(2k + 3) < e_1 < (2k + 4)/(2k + 5) < a_2 \), hence \( e_1 = (2k + 3)/(2k + 4) \). Now we conclude by lemma 4.22.

Lemma 4.40. \( (S_2, A_2 + B_2) \) is not a fence.

Proof. Suppose it is. By the previous lemmas \( B_2 \) must be singular. The argument in [KM99, Lemma 18.6] then proves that \( A_2 \) has at least one non Du Val point, \( A_1 \) has only Du Val or almost Du Val points, that the genus of \( B_2 \) is at least two and that configuration II does not occur. Let’s assume we have configuration III of lemma 3.13. Let \( r \) be the index of a non Du Val point on \( A_2 \). By lemma 3.23 we get \( b_2 > 2/3 - 1/(3r) \). However \( b_2 \leq 3/5 \) by lemma 3.13, hence \( r \leq 4 \). Suppose \( x_1 \) is non Du Val, hence a (3) point. Since \( 1/3 + a_2/3 < b_2 \leq 3/5 \), we get that \( a_2 < 4/5 \). Suppose that \( A_1 \) has four singularities, and hence \( x_0 \) is a non chain singularity. Then \( -E_1^2 \) is either 2 or 3. Now, by looking at the non chain singularities of low coefficient, and remembering that \( A_2^2 \geq -1 \), we get that \( A_2^2 \geq 1/2 \). On the other hand, since \( B_2^2 = 4g - 1 \geq 7 \), we get a contradiction. Hence \( A_2 \) has just two singularities. By lemma 4.22 we see that \( 4g - 1 = (2g - 1)/(1 - \alpha) \), hence \( \alpha = 2g/(4g - 1) \). Since \( b_2 \leq 3/5 \) we can apply lemma 3.20 on \( S_2 \), which gives us a contradiction for the above solutions in \( \alpha \).

For example, if \( g = 2 \), that implies that there is an \( A_{13} \) point. Knowing that \( K_{S_2}^2 = (K_{S_2} \cdot B_2)^2/(B_2)^2 \geq g + 1 \), we conclude the other cases similarly.

Hence \( x_1 \) is Du Val, say an \( A_{j+1} \) point. Since there is a non Du Val point on \( A_1 \), but \( x_1 \) got extracted anyway, we must have \( j \geq 3 \). Now, applying adjunction on \( B_2 \) we get \( (K_{S_2} + B_2) \cdot B_2 = -2 + 2g + j/(j + 1) \) and \( B_2^2 = 4g + j/(j + 1) \). But then \( K_{S_2}^2 \geq 4 \). By lemma 3.20 there must be just two singularities on \( A_2 \), either
two (3) points or a (2) point and a (3) point. In each case, we get $A_2^3 > 2/3$, contradicting the fact that $B_2^2 > 2$.

Suppose now we are in configuration $u, g = 2, b_2 = 9/14$. By [KM99, Lemma 18.6], $x_1 = (2, 2, 2)$. But then $B_2^2 = 8$, and also $B_2^2 = 3/(1 - \alpha)$, hence $\alpha = 5/8$. This means that $A_2$ has two singularities, one of which with index eight. But then that contradicts the choice of the hunt.

Finally, suppose we are in configuration $v$, with $g = 2, b_2 = 7/11$. Let $x_1$ be Du Val, say an $(A_j + 1)$ point. By adjunction, and the description of the configuration, one gets $(K_S + B) \cdot B = 2 + j/(j + 1)$ and $B_2^2 = 9 + j/(j + 1)$. But then $\alpha = (6j + 6)/(10j + 9)$. Hence $A_2$ has only two singularities on it, one (2) point, and the other a Du Val point of index at most ten. However, for each one of these one checks that there is no appropriate $j$, contradiction.

The last case is $x_1$ almost Du Val. This case is exactly analogous to the previous one. \hfill \Box

**Proof of 4.34.** By lemma 4.40 $S_2(A_2 + B_2)$ is not a fence, and lemmas [KM99, Lemma 18.7 - 18.8] show that $A_1$ can't be contracted and $T_2$ is not a net. The only option left is that $(S_2, A_2 + B_2)$ is a smooth banana. \hfill \Box

**Lemma 4.41.** Let $S(X + Y)$ be a smooth banana. Then $X$ and $Y$ can't have exactly two and one singular points respectively.

**Proof.** Suppose that is the case. Let $f : T \to S$ extract the adjacent divisor $E$ to $Y$. We get a contraction $\pi$ from lemma 3.8. This contraction can't be a net: if $Y$ is a fiber, $E$ is a section with at most one singular point, hence there is exactly one singular fiber, but $X$ has two singular points, contradiction; if $Y$ is a multisection, then $Y \cdot F \geq 2$ because $T$ is not smooth, but then $(K_S + Y) \cdot f_* F \geq 0$, contradiction. Hence $\pi : T \to S_1$ is a birational contraction, which contracts a curve $\Sigma$ that doesn't touch $Y$. If $S_1$ is not smooth, then $Y^2 \geq 2$, but on $S$ we have that $X^2 > Y^2$ and hence $Y^2 < 2$, contradiction. On the other hand, if $S_1$ is smooth, then it's $\mathbb{P}^2$. Since $E \cdot Y = 1$, they must both be lines and $X$ is a smooth conic. However $\Sigma$ must pass through both it's singularities, hence $X$ has a node, contradiction. \hfill \Box

Now, noticing that thanks to lemma 4.41 $S_2$ is one of the smooth bananas described in [KM99, Lemma 13.2], we can get the classification of [KM99, Chapter 19].

## 5 Surfaces with tigers

In this section we will classify pairs $(S, C)$ such that $S$ is a rank one log del Pezzo and $K_S + C$ is anti-nef. This will allow to write down a complete classification of log Del Pezzo surfaces in the next section. As usual, we assume $\text{char}(k) \neq 2, 3, 5$.  

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5.1 Log terminal case

Here we furthermore assume that $K_S + C$ is log terminal. In the following we shall slightly change the scaling convention of lemma 3.8. We run the hunt for $(K_S, aC)$ with $1 - \varepsilon < a < 1$ and we don’t rescale by multiplying by $\lambda$. Therefore $K_T + \Gamma = f^*(K_S + aC)$, $\Gamma_\varepsilon = \Gamma + \varepsilon E$ and $\Delta_1 = \pi_\varepsilon(\Gamma_\varepsilon)$. It’s easy to see that the statement of proposition 3.11 remains true.

**Proposition 5.1.** Assume that there are at least three singularities on $C$.

If $-(K_S + C)$ is ample then

1. $S \setminus C$ has exactly one singular point, a non cyclic singularity, $z$. If $Z \to S$ extracts the central divisor $E$ of $z$, then $Z$ is a $\mathbb{P}^1$ fibration and $E$ and $C$ are sections. The pair $(S, C)$ is uniquely determined by $z$, and all non cyclic singularities $z$ occur in this way for some pair $(S, C)$.

If $K_S + C$ is numerically trivial then $S$ is Du Val, and $(S, C)$ is one of five families:

2. $S = S(A_1 + 2A_3)$. $(S, C)$ is given by [KM99, Lemma 19.2], Case 1, $s = 3$ and $r = 1$.

3. $S = S(3A_2)$. $(S, C)$ is given by [KM99, Lemma 19.2], Case 2, $s = 2$ and $r = 1$.

4. $K_S^2 = 1$, $C$ is a $(-1)$ curve and $S$ is one of $S(A_1 + A_2 + A_3)$, $S(2A_1 + A_3)$, $S(4A_1)$. The pairs are obtained from [KM99, Lemma 13.5] by blowing up the node of the nodal curve always along the same branch.

**Proof.** This is the statement of [KM99, Proposition 23.4]. The proof given there goes through, with only minor modifications, which we shall point out.

We run the hunt as described above and we note that $e_1 \geq 1/2$, and $e_1 \geq 2/3$ unless $C$ contains four (2) points. If follows that $2a + b > 2$, hence we can’t have a tacnode or a triple point by [KM99, Lemma 8.3.7].

Suppose $x \notin C$. If we have a birational contraction, we must go to a smooth fence by the above. $\Sigma$ must then pass through a singular point of $C$, for otherwise $(K_T + aC + bE) \cdot \Sigma > 0$. If $C$ passes through four singular points, then its corresponding branch in the fence has three (2) points. The other branch can’t have two singular points by lemma 4.30 or one singular point by lemma 4.32. But then $x$ is a non chain singularity, with coefficient 1/2. The central curve has self intersection $(-2)$, hence the corresponding branch of the fence will be a zero curve, contradicting lemma 4.24. Suppose then that $C$ has exactly three singular points. $\Sigma$ can’t meet both $C$ and $E$ at singular points because of lemma 4.25 and lemma 4.32. From this it follows that $x$ is again a non chain singularity, and that $\Sigma$ must meet $C$ at a singular point. Now, since $K_S + C + bE$ is antinef, using the notation of lemma 4.22 we get that $b \leq -(K_{S_1} + C) \cdot C = \alpha$. Since $\alpha \geq 1/2 \geq \beta > 0$ we get a contradiction by lemma 4.30 (note that in the statement of the lemma $\alpha$ and $\beta$ are interchanged).
If we go to a net instead, we get description (1) by the argument in [KM99, Proposition 23.4].

Suppose now that \( x \in C \). If we get a banana, we get descriptions (2) and (3), using lemma 4.41 and [KM99, Lemma 13.2].

We can’t get a smooth fence because of lemma 4.25 and 4.32. Hence if we get a fence, it will be singular, and we can again use the argument in [KM99, Proposition 23.4].

Finally, if we get a net or \( C \) gets contracted, the proof given [KM99, Proposition 23.4] carries through with no modifications.

**Proposition 5.2.** Assume \( C \) has exactly two singular points. Then \((S,C)\) is one of the following

1. Start from \( \mathbb{F}_n \), pick the negative section \( E \) and a positive section \( C \) disjoint from \( E \). Blow up along \( C \) or \( E \) to create two singular fibers, then blow down the \( K \) – positive curves.

2. Start from a smooth fence as in lemma 4.33, blow up on \( X \cap Y \) and blow down the \( Y \) branch. \( C \) will be the strict transform of \( X \).

3. Start from \( \mathbb{F}_n \), pick the negative section \( E \), a positive section \( C \) meeting \( E \) once, then create a multiple fiber with singularities on \( C \) and \( E \), finally contract \( E \).

4. Start from \( \mathbb{F}_2 \) or \( \mathbb{P}^2 \), pick a banana, blow up one intersection as many times as necessary, then contract one branch. The other branch will be \( C \).

5. Start from either a fence as described in lemma 4.33, or by the fence in \( \mathbb{F}_n \) given by a fiber \( C \) and a positive section \( E \). Then blow up points over \( E \) so that \( E \) gets contractible and log terminal.

6. Start from \((S',C')\) with \( C' \) log terminal with one or two singularities. Blow up one of the singularities (or a smooth point if \( C' \) has only one singularity) always along \( C' \) and finally contract \( C' \). \( C \) is the last \((-1)\) curve.

**Proof.** Let’s run the hunt for \( 1 - \epsilon < a < 1 \). Suppose \( x \notin C \). If we get a net, \( C \) and \( E \) are both multisections because they have non zero self intersection. Since \( 2a + b > 2 \) either they are both sections or \( C \) is a section and \( E \) is a multisection.

In the latter case, \( E \) is a double section, \( e_1 = 1/2 \), and the singularities on \( C \) and \( E \) have index two by lemma 3.17. If \( E \) has a singularity on it, then the corresponding multiple fiber must pass through one of the singularities of \( C \). Since the fiber is a double fiber and the singularities are \( A_1 \) points however, we can never get \( E \cdot F = 2 \) because \( F \) can’t contain three singular points. If \( E \) is smooth on the other hand, then \( E_1 = -4 \) and by running a relative MMP on the minimal resolution, \( E \) would be a double section of self intersection zero in \( \mathbb{F}_n \), contradiction. Suppose therefore that they are both sections. Running a relative MMP as above, one sees that we must go to a Hirzebruch surface, \( C \) is a section of positive self intersection, and \( E \) is a section of negative self
intersection. This process can be reversed to obtain $S$. More precisely, one can start from $\mathbb{F}_n$, pick the $(-n)$ section $E$ and a positive $(n)$ section $C$, and then blow up either on $E$ or $C$ so that to create two multiple fibers, and finally contract everything down to a log Del Pezzo. This is case (1).

If we get a birational contraction instead, we must have a fence and $x_1$ must be a chain singularity. The curve $\Sigma$ meets $C$ at a singular point, and $E$ either at a smooth point (if $E$ has just one singularity on it) or at a singular point (if $E$ has two singularities on it). Smooth fences with one singularity at each branch are classified in lemma 4.33, and then one can obtain $S$ by blowing up at the intersection of the branches of the fence and then contracting everything down to a log Del Pezzo. This is case (2).

Suppose now that $x \in C$. Suppose $T$ is a net. $C$ can’t be a fiber because it’s log terminal and has only one singularity, hence $C$ and $E$ are both sections as above. There is just one multiple fiber, passing through the singularities of $C$ and $E$. This is case (3). Suppose then that we get a birational contraction down to $S_1$. If we go to a banana, $\Sigma$ must meet $C$ because $a > 1 - \epsilon$, hence $C$ is in the smooth locus. Adjunction then tells us that $E$ is also in the smooth locus. Since $C \cdot E = 2$ we must have $S_1 = \mathbb{F}_2$ or $\mathbb{P}^2$. To get back to $S$ it suffices to reverse the process: pick a smooth banana in $\mathbb{F}_2$ or $\mathbb{P}^2$, blow up one of the intersection points, then contract one of the two branches of the banana. This gives case (4).

If we go to a smooth fence, we must have at most one singularity on the branch corresponding to $E$. Then to get $S$ start from either a fence as described in lemma 4.33, or by the fence in $\mathbb{F}_n$ given by a fiber $C$ and a positive section $E$. Then blow up points over $E$ so that $E$ gets contractible and log terminal. This is case (5).

In the case of the singular fence we must have a Gorenstein log Del Pezzo, but then the branch relative to $C$ must have at least two singularities by [KM99, Lemma 13.5], contradiction.

Finally, if $C$ gets contracted, we can induct on $(S_1, E)$, giving description (6). \hfill $\Box$

**Proposition 5.3.** Assume $C$ has exactly one singularity. Then $(S, C)$ is one of the following:

1. Start from $\mathbb{F}^2$, pick two lines $C$ and $E$, blow up at the intersection three times along $E$, then continue blowing up keeping the surface log terminal and with just one $(-1)$ curve and finally contract down all the $K$-positive curves.

2. Start from $\mathbb{F}_n$, pick a positive section $C$ disjoint from the negative section and a fiber $F$. Blow up on $F \cap C$ and then continue blowing up keeping the surface log terminal and with just one $(-1)$ curve. Finally blow down all the $K$-positive curves.

3. Start from $\mathbb{F}_n$, pick the negative section $E$, a positive section $C$ disjoint from $E$ and create one log terminal multiple fiber so that $C$, then contract.
4. \((\mathbb{P}_n, C)\) where \(C\) is any smooth rational curve passing through the singularity.

5. Start from \(\mathbb{P}_n\), pick the negative section \(E\), a point \(p\) on it and a fiber \(C\) not passing through \(p\). Blow up on \(p\) to obtain a multiple fiber, then blow down the \(K\)-positive curves.

6. Start with \(\mathbb{P}^2\) and pick two lines \(C\) and \(E\). Pick a point \(p\in E\setminus C\), blow up \(p\) along \(E\), then contract the \(K\)-positive curves.

7. Start from \(\mathbb{P}_n\). Pick the negative section \(E\) and a positive disjoint section \(C\). Blow up a point in the fiber which does not lie in the negative section, always along the fiber. Then contract all \(K\)-positive curves.

Proof. Let’s run the hunt for \(1 - \epsilon < a < 1\). Suppose \(x \notin C\). As \(2a + b > 2\), if we get a birational contraction, we must go to a fence. Since \(a > 1 - \epsilon\), \(\Sigma\) must pass through the only singularity of \(C\), and \(C\) will be a smooth curve in the smooth locus of \(S_1\). It follows then that \(x_1\) is a chain singularity by lemma 4.22, and \(E\) has either one or two singularities on it. If it has just one singularity, \(S_1\) is \(\mathbb{P}^2\), and the images of \(C\) and \(E\) are lines. But then \(S\) is obtained by successively blowing up on the point of intersection of two lines, and then contracting everything except the last \((-1)\) curve. This is case (1). If \(E\) has two singularities on it, then we get case (2). If we have a net instead, \(C\) and \(E\) are both sections, each with one singular point, contained in the same multiple fiber. These are constructed as above by starting by \(\mathbb{P}_n\), picking the negative section, a positive section, and then producing one multiple fiber. This is case (3).

Suppose now that \(x \in C\). Suppose \(T\) is a net. If \(C\) is a section, then \(T\) is smooth, hence a Hirzebruch surface, and \(S\) is just \(\mathbb{P}_n\). Suppose then that \(C\) is a fiber and hence \(E\) is a section. Running a relative MMP one sees that \(S\) is obtained by blowing up once along the negative section of a Hirzebruch surface, and then along the fibers. \(C\) is the image of a smooth fiber. This is case (5).

Let’s consider the case in which we have a birational contraction. We can’t get a banana because \(C\) has only one singular point of \(S\) and \(a + b > 1\). We can’t get a singular fence otherwise \(C\) would be a \((-1)\) curve, hence contractible. Therefore suppose we get a smooth fence. If \(E\) is in the smooth locus of \(S_1\), then \(S_1\) is \(\mathbb{P}^2\), \(C\) and \(E\) are lines, and we obtain \(S\) by blowing up a point of \(E\) multiple times. This is case (6). If \(E\) is not in the smooth locus of \(S_1\) instead, we get case (7).

Finally note that \(C\) can’t get contracted on \(T\) because it’s in the smooth locus and can’t be a \((-1)\) curve.

5.2 Non log terminal case

In this subsection we classify pairs \((S, C)\) of a log Del Pezzo surface \(S\) and a curve \(C\) such that \(K_S + C\) is anti-nef but not log terminal. As above we run a modified version of the hunt. As we will always have strictly log canonical pairs
in the following, in the first hunt step for $K_S + C$, we pick the coefficient of $E$ to be one. If we go to a second hunt step, the coefficient $a_2$ of $E_2$ will be chosen in such a way that $K_T + C + E_1 + a_2E_2$ is $\pi_2$-trivial.

Clearly $K_S + C + A_1$ is almost log canonical, meaning that the points at which it’s not log canonical are singular, either with just one exceptional curve and with $C$ or $A_1$ tangent to that curve, or with two exceptional curves, with $C$ or $A_1$ passing through their intersection. More in general every result about the hunt still holds, with log terminal replaced by almost log canonical and flush replaced by level.

**Lemma 5.4.** Suppose that $S$ is a log terminal surface and $C$ is a curve such that $(K_S + C) \cdot C \leqslant 0$. Then the dual graph of the singularities on $C$ and the curve $C$ is the dual graph of a log canonical singularity.

**Proof.** If $K_S + C$ is log canonical the statement follows from the classification of log canonical singularities (see, for example [Kol97, Chapter 3]). Suppose then that $K_S + C$ is not log canonical and consider its log canonical threshold $e \leqslant 1$.

Let $\tilde{S} \to S$ be the minimal resolution about $C$, with exceptional divisors $E_i$. We define the numbers $b_i$ by $f^*(K_S + C) = K_T + \tilde{C} + \sum b_iE_i$, the numbers $d_i$ by $f^*(C) = \tilde{C} + \sum d_iE_i$ and $\tilde{C} \cdot E_i = a_i$. Since the statement is about abstract graphs and does not take into consideration $C^2$, we can declare $C$ to be such that $(K_S + eC) \cdot C = 0$. From this and the fact that $(K_S + C) \cdot C \leqslant 0$, we get that $C^2 < 0$.

Define now $D = f^*(C)$ and consider the intersection matrix of the divisors $E_i$ and $D$. It is clearly negative definite on the space generated by the $E_i$, and $D$ is orthogonal to that space. Now, $D^2 = C^2 < 0$, so the total intersection matrix is negative definite. However, note that the intersection matrix of $E_i$ and $D$ is obtained by orthogonalizing $\tilde{C}$ with respect to the subspace generated by the $E_i$, hence the intersection matrix of $E_i$ and $\tilde{C}$ is also negative definite. Now we can conclude by using [Kol97, Lemma 2.19.3] and the argument in [Kol97, Chapter 3].

**Lemma 5.5.** If $K_S + C$ is anti-ample, then $C$ can pass through at most two points, and if it passes through two points, one of them is log terminal. If $K_S + C$ is numerically trivial, $C$ can pass through at most three points, and if it passes through three points, two of them are $A_1$ points and $C$ is log terminal at these points, log canonical at the other point.

**Proof.** Suppose $K_S + C$ is negative, and that $C$ has at least three singular points on it, say $p, q, r$, with $p$ non log terminal. By inversion of adjunction, the coefficient of $q$ is $\text{Diff}_C(0)$ is at least one. Since the coefficient of any singular point is at least $1/2$ however, then we would have that $(K_S + C) \cdot C = \text{deg}(\text{Diff}_C(0)) \geqslant 0$, contradiction.

The same reasoning gives us the rest of the statement.

**Lemma 5.6.** Suppose $(S, C)$ is a pair of a rank one log Del Pezzo $S$ and a curve $C$ such that $K_S + C$ is anti-nef, log canonical but not log terminal. If $C$
passes through three singularities then \((S, C)\) is obtained as follows: start from \(F_n\), pick a positive section \(C\) and the negative section \(E\), create two log terminal multiple fibers of multiplicity two. Now you can either stop or create a third fiber such that the singularity is a chain singularity, but the fiber is not log terminal. Finally contract all the \(K\)-positive curves.

**Proof.** Two of the singular points on \(C\) must be \(A_1\) points, at which \(C\) is log terminal. For the last point there are three cases: either it’s a non chain singularity, with two (2) branches, and \(C\) meets the opposite end of the third branch or \(C\) has a node at a singular point of \(S\), which is a chain singularity and \(\tilde{C}\) meets the opposite ends of the chain, or \(C\) has an \((2, n, 2)\) point, and \(\tilde{C}\) meets the central curve.

Let’s start with the first case. The hunt extracts the central curve of the non chain singularity at \(C\). Suppose we get a net. Clearly \(C\) can’t be a fiber since it contains three singularities. Hence \(C\) and \(E\) are multissections, and they are both sections since \(K_T + C + E\) is numerically trivial. There are two multiple fibers with multiplicity two covering the (2) points, and the last multiple fiber is made in such a way to make \(C\) and \(E\) intersect. This is precisely the situations described in the statement.

Suppose now we get a birational contraction. Clearly \(C\) does not get contracted as it contains three singular points. \(K_S + C + E\) is log canonical at singular points and doesn’t have triple points because it’s level. \(\Sigma\) meets \(C\) and \(E\) only at singular points, hence the only possibility is that it passes through \(C \cap E\), by contractibility considerations. In this case however, we get that \(C^2 = E^2 = 1\) by symmetry, and after extracting the two (2) curves on \(C\), we get a net of relative Picard number two. However this is impossible, since \(C\) would be an irreducible log terminal multiple fiber with only one singularity, contradiction.

Let’s consider now the second case. The hunt extracts one of the two adjacent curves \(E\) of \(C\) at the node. As above, \(C\) and \(E\) can’t be neither sections or fibers of a net and \(C\) doesn’t get contracted by the birational contraction. \(\Sigma\) must pass through the intersection of \(C\) and \(E\) at the singular point. After the birational contraction \(K_S + C + E\) is log canonical and the next hunt step extracts the curve adjacent to \(E\). Again, we can’t have a net and \(\Sigma_2\) passes through the third singularity of \(C\). Now \(E\) is however in the smooth locus, but there are at least two singularities on \(S_2\), contradiction.

Finally suppose we are in the third case. Consider the extraction of the adjacent divisor \(E\) of \(C\) at the log canonical point. Suppose \(T\) is a net. If \(C\) is a fiber, the it has multiplicity two and \(E\) is a double section, which is impossible since \(E^2 < 0\). Therefore \(C\) and \(E\) are both sections, meeting at a smooth point. This case however is already described in the statement.

On the other hand, we can’t get a birational contraction either, since there are only \(A_1\) points and \(\Sigma\) must pass through two of them.

**Lemma 5.7.** Suppose \((S, C)\) is a pair of a rank one log Del Pezzo \(S\) and a curve \(C\) such that \(K_S + C\) is anti-nef, log canonical but not log terminal. If
C passes through two singularities, one of which, p, is log terminal, then the minimal resolution of \((S, C)\) is one of the following:

1. Pick a pair \((S, E)\) such that \(E\) has a \((2, n, 2)\) point where \(E\) touches the \((n)\) component, and possibly a log terminal point. Blow up above the log terminal point (or at a smooth point if \(E\) has just one singularity), always along \(E\), then contract \(E\). \(C\) is the final \((-1)\) curve.

2. Start with a pair \((S, E)\) such that \(E\) is log terminal, with two \(A_1\) points on it, and possibly a third one. If there are just two singular points, blow up above a smooth point of \(E\) always along \(E\), otherwise blow up above the third point, always along \(E\). Finally contract \(E\). \(C\) is the last \((-1)\) curve.

3. Start with a fence as described in lemma 4.33 with a branch, \(E\), containing an \(A_1\) point. \(C\) is the other branch. Now blow up a smooth point on \(E\) to create another \(A_1\) point and make \(E\) contractible. Finally contract \(E\).

Proof. The cases for the non log terminal point are exactly as above.

Let’s start with the first case. This time we will choose the adjacent curve \(E\) of the non chain singularity on \(C\). Suppose \(T\) is a net. \(C\) and \(E\) can’t be sections since they don’t touch on the multiple fiber, but one of the singularities in non log terminal. \(C\) can’t be a fiber either because it’s log terminal and has only one singularity on it. Hence we get a birational contraction. If \(C\) gets contracted, then we are in the same situation as in the hypothesis of the lemma, but with a smaller log canonical point, therefore we can apply induction. This is description (1).

If \(\Sigma\) passes through both the singularities on \(C\) and \(E\), then we would get two smooth rational curves meeting at a smooth point and at a chain singularity at the opposite ends of it. By symmetry, \(C^2 = E^2 = 2\), however adjunction would give then a non integer value for \(C^2\), contradiction. Also, \(\Sigma\) can’t meet \(E\) at a smooth point, for otherwise it would contract an \(A_r\) point and \(C + E\) would be a smooth fence. However, \((K_S + C) \cdot C < -1\) and \((K_S + E) \cdot D = 1\), contradiction by lemma 4.22. The only option left is that \(\Sigma_1\) meets one of the two \((2)\) ends of the non chain singularity in \(E\), and therefore \(S_1\) is a smooth fence. However then \(E\) would not be contractible, contradiction.

Let’s deal now with the second case, \(C\) has a node. Suppose \(T\) is a net. \(C\) can’t be a fiber because \(C \cdot E > 1\). However \(C\) and \(E\) can’t be sections either because the multiple fiber over \(p\) would meet \(E\) at a smooth point. Hence we have a birational contraction. \(C\) doesn’t contract for the same reason as above. The only way to contract \(\Sigma\) is that \(\Sigma\) meets \(E\) at a smooth point and contracts an \(A_r\) point or \(\Sigma\) passes through \(C \cap E\) and leaves it singular. Hence on \(S_1\), \(C\) and \(E\) are smooth rational curves meeting at a smooth point, and log canonically at a chain singularity. Now extract the divisor \(D\) of the log canonical point adjacent to \(E\). \(T_2\) can’t be a net: \(D\) and \(E\) are clearly not fibers, but therefore must be sections, \(C\) can’t be a fiber otherwise it would be multiple, and \(C, D, E\) are all sections, contradicting the fact that \(K_{T_2} + C + D + E\) is anti-nef. \(\Sigma_2\) must either pass through \(C \cap D\), making it smooth and \(S_2 = \overline{F_n}\). However, \(D\) and \(E\) are both \((n)\) curves meeting just once, therefore \(n = 1\), contradiction.
Third case: as above one can show that $T$ is not a net. If $C$ gets contracted, $E$ is log terminal with two $A_1$ points and possibly another log terminal point on it. We can then use lemmas 5.1 and 5.2 to classify such configurations and get back to the original pair $(S, C)$. This is description (2).

If $Σ$ meets $C$ and $E$ at singular points, we go to a log canonical banana. After extracting the $−2$ curve $D$ on $E$ we see that we get another birational contraction to $S_2 = \mathbb{F}_2$ where $C + D$ is a banana. By symmetry $C^2 = D^2 = 2$, but they pass through a singular point, contradiction. If $Σ$ meets $E$ at a smooth point, we get a smooth fence which one can rule out by using lemma 4.22. The only remaining option is for $Σ$ to pass through a $−2$ point of $E$ and another singular point on $T_1$ and contract to a smooth point on $E$. We get then a fence with one singular point on each branch, one of whose is a $A_1$ point. It is possible to get then $S$ by starting with such a fence, classified in lemma 4.33 and then blowing up a point in along $E$ multiple times, and finally blowing up once away, to create $Σ$. Finally one can contract $E$. This is case (3).

**Lemma 5.8.** Suppose $(S, C)$ is a pair of a rank one log Del Pezzo $S$ and a curve $C$ such that $K_S + C$ is anti-nef, log canonical but not log terminal. If $C$ passes through two singularities, both non log terminal, then the minimal resolution of $(S, C)$ is one of the following:

1. Start with $\mathbb{F}_n$, pick a double section $E$ and let the two fibers tangent to $E$ be $F$ and $G$. Blow up on the intersection $F \cap E$ any number of times, always along $E$. $C$ is the last $−1$ curve. Now blow up at $E \cap G$ twice to separate them and call $H$ the $−1$ curve. Finally blow up on $H$ at a point not contained in any of the other components, always along $H$ and contract everything down.

2. Start with $\mathbb{F}_n$, pick a double section $E$, a tangent fiber $F$ and a transverse fiber $G$. Blow up on the intersection $F \cap E$ any number of times, then blow on one of the points in $G \cap E$, and once at the intersection of the two $−1$ curves. Finally contract everything down.

3. Start from $\mathbb{F}_n$, pick a positive section $E$ and a negative section $F$. Blow up to create two log terminal double fibers. Now blow up any point on $E$ always along $E$, and if the point was singular then make a last blow up away from $E$ to create an $A_1$ point on $E$. Now contract everything.

**Proof.** Let $p$ and $q$ be the two singularities. There are six cases.

In the first case, both of them are non chain singularities. Extract the adjacent divisor $E$ of $q$ and suppose $T$ is a net. $C$ and $E$ can’t be both sections because otherwise they would meet on the singular fiber. If $C$ is a fiber instead, it must have multiplicity two because the coefficient of $E$ is one. This is description (1). Suppose we get a birational contraction instead. If $C$ gets contracted, then $p$ must be Du Val because $E$ has coefficient one, however then we get a contradiction, since $C$ can’t contract. Thus $Σ \neq C$. If we go to a smooth banana however, we can’t reverse the process, creating two non chain singularities, contradiction. Hence the only possibility we go to a smooth fence,
with one non chain singularity on each branch. By symmetry \( C^2 = E^2 = 1 \), hence after extracting the adjacent curve \( D \) of \( p \) at \( C \), we get that \( C^2 = 0 \), \( E \) and \( D \) are sections. But then we would have a multiple fiber with two non log terminal sections, impossible.

In the second case \( C \) has a node at \( p \) and has a non chain singularity at \( q \). After extracting the adjacent divisor \( E \) of \( q \), \( T \) clearly is not a net, as \( C \) can’t be neither a section or a fiber since it’s singular. So we get a birational contraction. Again, \( C \) can’t contract because it’s singular. The only possibility is that we go to a fence, with \( C \) nodal at a singular point and \( E \) smooth, passing through a non chain singularity. This case can be ruled out exactly as before, as after extracting the adjacent divisor of \( E \), we must have a net.

In the case in which the singularity at \( q \) is chain but \( C \) has not a node on \( q \), the above argument goes through exactly the same way. In the case in which \( C \) has two nodes does not occur either: after running the hunt we must get a birational contraction to a banana in \( \mathbb{F}_n \) where \( C \) has a node, and the intersection points of the banana are smooth. However, this can’t happen because \( C \) would then be numerically equivalent to a double fiber on \( \mathbb{F}_n \), contradiction.

Let’s now deal with the case in which \( p \) in a non chain singularity, and \( q \) is a \((2, n, 2)\) point. Extract the \((-n)\) curve \( E \). If \( T \) is a net, then \( C \) must be a double fiber and \( E \) a double section. This is description (2) in the lemma. If we get a birational contraction, we must go to a fence, where \( C \) has a non chain point and \( E \) has two \((2)\) points. After extracting the adjacent divisor of the non chain singularity we would have a net, with \( E \) and the other exceptional divisor being a sections, which is impossible.

Finally, suppose \( p \) is a \((2, n, 2)\) point and \( q \) is a \((2, m, 2)\) point. Extract the \((-m)\) curve \( E \). If \( T \) is a net, we get again description (2). Suppose therefore that we have a birational contraction to \( S_1 \). Using the usual arguments we deduce that \( C + E \) is a fence. Now, after extracting the \((-n)\) curve \( D \) from \( C \), we clearly get a net where \( C \) is a fiber, \( D \) and \( E \) are sections. This is description (3).

\[ \square \]

**Lemma 5.9.** Suppose \((S, C)\) is a pair of a rank one log Del Pezzo \( S \) and a curve \( C \) such that \( K_S + C \) is anti-nef, log canonical but not log terminal. If \( C \) passes through only one singularity then \((S, C)\) is obtained as follows.

1. Start with \( \mathbb{F}_n \) and pick the negative section \( E \). Make a multiple fiber so that \( E \) is log canonical but not log terminal, then contract.

2. Start with \( \mathbb{F}_n \), pick the negative section \( E \), and a positive section \( C \) meeting \( E \) just once and transversally. Pick a point in \( (C \cup E) \setminus (C \cap E) \) and blow up any number of times to make a log terminal fiber. Then perform a blow up on an interior point of the last \((-1)\) curve so that it creates a chain singularity connecting \( C \) and \( E \), and finally continue blowing up on the \((-1)\) curve at the nearest point to \( C \) and \( E \).

3. Start with a Gorenstein log Del Pezzo surface of rank one, with a nodal rational curve \( E \) in its smooth locus. Then blow up at the node of \( E \) always along the same branch, finally contract. \( C \) is the last \((-1)\) curve.
4. Start with a smooth banana in either $\mathbb{P}^2$ or $\mathbb{F}_2$. Then blow up on one intersection until one curve is negative enough, continue blowing up to make a chain singularity connecting the two branches. Finally, contract the negative curve. $C$ is the other branch of the banana.

5. Start with $\mathbb{F}_n$, pick the negative section $E$ and a fiber $C$. Then create two log terminal double fibers away from $C$ and contract everything.

6. Start from $\mathbb{F}_2$, pick a fiber $E$ and a positive section $C$. Blow up at a smooth point of $E$ so that $E$ is negative and has a $(-2)$ point on it, and eventually contract everything down.

**Proof.** Suppose the singularity is non chain. Then after extracting the adjacent divisor $E$, $C$ is in the smooth locus of $T$. Suppose $T$ is a net. $C$ can’t be a section as $T$ is singular. If $C$ is a fiber, $E$ is a section. This is description (1).

Suppose we get a birational contraction to $S_1$. $C$ can’t get contracted because $K_S \cdot \tilde{C} \leq -2$. $\Sigma$ can only meet $E$ and we get a smooth fence. $\Sigma$ must make the singularity on $E$ a $(-n)$ singularity, as $S_1$ must be $\mathbb{F}_n$, impossible.

Suppose now that the singularity is on a node of $C$ and extract an adjacent divisor $E$. Suppose $T$ is a net. $C$ can’t be a fiber as then $C \cdot E > 2$. If $C$ and $E$ are sections, we get description (2). Hence let’s assume we get a birational transformation. If $C$ contracts, then $E$ is a nodal rational curve in the smooth locus of $S_1$, therefore we get description (3). If we go to a smooth banana instead, $C \cap E$ must be both smooth points and we get description (4).

The last case is the one in which $C$ has a $(2, n, 2)$ point on it. Extract the $(-n)$ curve. If $T$ is a net, then $C$ must be a fiber, $E$ a section. This is description (5). If we have a birational contraction instead, $C$ can’t contract and we must have a fence. Clearly there must be only one $A_1$ point on $E$ since $C$ is in the smooth locus, therefore $\Sigma$ contracts the other $A_1$ point down to a smooth point. This is description (6).

Our aim for the rest of the section is to give a couple of lemmas that allow us to reduce the general case of a surface with tigers to a pair $(S, C)$ as above, where the tiger $C$ is already in $S$ and it’s log canonical. Our convention for the hunt will be again the one in proposition 3.8. Let’s start with a definition.

**Definition 5.10.** A tiger $E$ for $S$ is called exceptional if $E$ does not lie in $S$.

**Lemma 5.11.** Suppose $S$ has an exceptional tiger. Then $S$ is obtained by starting with a pair $(S_1, C)$ where $S_1$ is a log Del Pezzo, $K_{S_1} + C$ is anti-nef and log canonical, then blowing up a point in $C$ any number of times so that $C$ gets contractible, and then contracting all the $K$-positive curves.

**Proof.** Extract a tiger $E$ of maximal coefficient in the minimal resolution. If in the hunt $a \geq 1$, then $K_{S_1} + C$ is log canonical because $K_{T_1} + E_1$ is log canonical, and $\Sigma$ has non positive coefficient for $\pi_1$. If $a < 1$ and $K_{S_1} + C$ is not log canonical, then consider the log canonical threshold $e$. Clearly $a < e < 1$ because $K_{S_1} + aC$ is log terminal, and there is an exceptional tiger $F$ for $K_{S_1} + eC$. But
then $F$ is a tiger for $S$, with coefficient higher than $E$, contradiction. Hence $K_{S_1} + C$ is log canonical and the statement follows.

**Lemma 5.12.** Suppose $S$ has a non exceptional tiger $C$. Then either $C$ is log canonical, or there is an exceptional tiger.

**Proof.** Let $C$ be such a tiger. If $K_S + C$ is not log canonical, let $e < 1$ the log canonical threshold. Any divisor exceptional divisor $D$ with coefficient one for $K_S + eC$ is a tiger. 

### 6 Classification

In this section summarize the previous results and classify all rank one log Del Pezzo surfaces, over an algebraically closed field of characteristic zero, or higher than five. Let’s start by listing the relevant surfaces.

**Definition 6.1 (LDP 1).** Let $A$ and $B$ be two smooth conics meeting to order four in $\mathbb{P}^2$ at a point $c$. Pick a point $b \neq c$ of $B$. Pick a point $a$ in the intersection of $A$ with the tangent line $M_b$ to $B$ at $b$. Let $b'$ be the other point of the intersection of $B$ with the line passing through $a$ and $c$. Let $a'$ be a point of the intersection of $A$ with the line through $b$ and $b'$.

Let $\tilde{S} \to \mathbb{P}^2$ blow up once at $a,a',b'$, twice along $M_b$ at $b$, and five times along $A$ at $c$.

**Definition 6.2 (LDP 2).** Let $B$ and $C$ be two log terminal ($-1$) curves meeting a nodal curve of $S(2A_1 + A_3)$ at smooth points, each passing through an $A_1$ point, and meeting at opposite ends of the $A_3$ point.

Let $\tilde{S} \to \tilde{S}(2A_1 + A_3)$ blow up once at the intersection of $C$ with the $(-2)$ curve at the $A_3$ singularity, twice at $A \cap B$ along $A$ and twice along on the branches of the node of $A$.

**Definition 6.3 (LDP 3).** Let $B$ a $(-1)$ curve in either $S_2 = S(A_1 + A_3)$ or $S_2 = S(3A_2)$ passing through two singularities, $B$ log terminal. Let $A$ be a nodal rational curve in the smooth locus of $S_2$.

Let $\tilde{S} \to S_2$ blow up twice at $A \cap B$ along $B$ and three times on the node of $A$ along the same branch.

**Definition 6.4 (LDP 4).** Let $A$ be a nodal curve in the smooth locus of $S(2A_1 + A_3)$, $B$ a $(-1)$ curve through $A_1$ and $A_3$, log terminal.

Let $\tilde{S} \to \tilde{S}(2A_1 + A_3)$ blow up on $A \cap B$ twice along $B$, blow up the node of $A$ four or five times along the same branch.

**Definition 6.5 (LDP 5).** Let $A$ be a nodal curve in the smooth locus of $S(2A_1 + A_3)$, $B$ a $(-1)$ curve through $A_1$ and $A_3$, log terminal.

Let $\tilde{S} \to \tilde{S}(2A_1 + A_3)$ blow up three times on $A \cap B$ along $B$, blow up four times on the node along the same branch.
**Definition 6.6** (LDP 6). Let $A$ be a nodal curve in the smooth locus of $S(A_1 + A_5)$ and $B$ a log terminal $(-1)$ curve.

Let $\bar{S} \to \bar{S}(A_1 + A_5)$ blow up at the intersection of $B$ with the $(-2)$ curve in the $A_5$ singularity, blow up the node of $A$ twice along one branch and then once along the nearest point of the other branch.

**Definition 6.7** (LDP 7). Let $A$ be a nodal curve in the smooth locus of the Gorenstein log Del Pezzo $S_2$ and $B$ a log terminal $(-1)$ curve that passes through two singular points. Let $\bar{S} \to \bar{S}_2$ blow up $t$ times on the intersection of $B$ with the $(-2)$ curve relative to the specified point $p$, always along $B$, then blow up $s$ times at the node of $A$, always along the same branch, for $p,t$ and $s$ as follows.

1. $S_2 = S(A_1 + A_2)$, $p = A_1$ and $(t, s) = (2, 6), (1, 6), (1, 7)$.
2. $S_2 = S(A_1 + A_2)$, $p = A_2$ and $(t, s) = (3, 6), (2, 6), (1, 6), (1, 7), (1, 8)$.
3. $S_2 = S(A_1 + A_5)$, $p = A_1$ and $(t, s) = (1, 3)$.
4. $S_2 = S(A_1 + A_5)$, $p = A_5$ and $(t, s) = (2, 3), (1, 3), (1, 4)$.
5. $S_2 = S(3A_1)$, $p = A_2$ and $(t, s) = (1, 3)$.
6. $S_2 = S(A_2 + A_5)$, $p = A_5$ and $(t, s) = (1, 2)$.
7. $S_2 = S(A_2 + A_5)$, $p = A_2$ and $(t, s) = (1, 2)$.
8. $S_2 = S(A_1 + 2A_3)$, $p = A_3$ and $(t, s) = (1, 2)$.
9. $S_2 = S(2A_1 + A_3)$, $p = A_3$ and $(t, s) = (1, 5), (1, 4), (2, 4)$.
10. $S_2 = S(2A_1 + A_3)$, $p = A_1$ and $(t, s) = (1, 4)$.

**Definition 6.8** (LDP 8). Let $A$ be a nodal rational curve in the smooth locus of $S(A_1 + A_2)$, and $B$ a log terminal $(-1)$ curve passing through the two singularities.

Let $\bar{S} \to \bar{S}(A_1 + A_2)$ blow up $t$ times $A \cap B$ along $B$ and $s$ times at the node of $A$, always along the same branch for $(t, s) = (2, 5), (2, 6), (3, 6), (4, 6), (5, 6), (2, 7), (3, 7), (2, 8), (2, 9)$.

**Definition 6.9** (LDP 9). Let $A$ be a nodal rational curve in the smooth locus of $S(A_1 + A_2)$, and $B$ a log terminal $(-1)$ curve passing through the two singularities.

Let $\bar{S} \to \bar{S}(A_1 + A_2)$ blow up $A \cap B$ twice along $A$, and once near $B$, then blow up the node at $A$ five, six or seven times along the same branch.

**Definition 6.10** (LDP 10). Let $A$ be the nodal rational curve in the smooth locus of $S(A_1 + A_2)$, and $B$ the log terminal $(-1)$ curve passing through the two singularities.

Let $\bar{S} \to \bar{S}(A_1 + A_2)$ blow up $A \cap B$ twice along $B$, and once near $A$, then blow up the node at $A$ six times.
Definition 6.11 (LDP 11). Let $A$ and $B$ be two positive sections disjoint from the negative section in $\mathbb{F}_2$ intersecting at $p$ and $q$. Let $F$ be a fiber, not passing through $p$ and $q$. Blow up $r + 1$ times at $u = A \cap F$ along $F$ and $s + 1$ times at $p$ along $B$. Now continue in one of the following manners.

If $(s, r) = (3, 2)$ then
1. Blow up above $q$ along $A$ four or five times, or
2. Blow up above $p$ along $A$ three or four times, or
3. Blow up on $u$ along $A$ three times.

If $(s, r) = (4, 1)$ then
1. Blow up above $q$ along $A$ five, six, seven or eight times, or
2. Blow up above $p$ along $A$ four times, or
3. Blow up on $u$ along $A$ three or four times.

Definition 6.12 (LDP 12). Let $A$ and $B$ be two positive sections not intersecting the negative section in $\mathbb{F}_2$ intersecting at $p$ and $q$. Let $F$ be a fiber, not passing through $p$ and $q$. Blow up $r + 1$ times at $u = B \cap F$ along $F$ and $s + 1$ times at $p$ along $B$. Now continue in one of the following manners.

If $(s, r) = (3, 2)$ then blow up $A \cap F$ along $A$ four times.

If $(s, r) = (4, 1)$ then
1. Blow up $q$ along $A$ six times, or
2. Blow up above $A \cap F$ along $A$ four times, or
3. Blow up on $p$ along $A$ four times.

If $(s, r) = (3, 1)$ then
1. Blow up $q$ along $A$ $k$ times, with $5 \leq k \leq 9$ times, or
2. Blow up $A \cap F$ $k$ times, with $4 \leq k \leq 7$, or
3. Blow up $p$ along $A$ $k$ times, with $4 \leq k \leq 8$.

Definition 6.13 (LDP 13). Let $A$ and $B$ be two positive sections disjoint from the negative section in $\mathbb{F}_2$ intersecting at $p$ and $q$. Let $F$ be a fiber, not passing through $p$ and $q$. Blow up twice at $u = A \cap F$ along $A$, and five times at $p$ along $B$, twice along $A$ and once away from $A$.

Definition 6.14 (LDP 14). Let $A$ be a positive section on $\mathbb{F}_n$ and $E$ the negative section. Pick three points on $E$ or $A$ all lying in distinct fibers, blow them up once, then blow up at the intersection of the $(-1)$ curves and continue blowing up to get log terminal fibers such that $E$ and the singularities on it form a log terminal non chain singularity. Now to define $\tilde{S}$ pick any point on $A$ intersecting a $K$-positive curve, and blow up that point in any fashion such that the final result is log terminal and makes $A$ an exceptional divisor.
**Definition 6.15** (LDP 15). Let $S_1$ be one of $S(A_1 + 2A_3), S(3A_2), S(2A_1 + A_3), S(A_1 + A_2 + A_5), S(4A_1)$ and $C$ a smooth curve through all the singularities such that $K_S + C$ is log terminal. Let $S$ pick any point singular on $A$, and blow up that point in any fashion such that the final result is log terminal and makes $A$ an exceptional divisor.

**Definition 6.16** (LDP 16). Let $(S_1, C)$ be one of the following surfaces in lemma 5.2, lemma 5.3, lemma 5.6, lemma 5.7, lemma 5.8 or lemma 5.9. Then define $\tilde{S}$ by a sequence of blow ups on $C$ such that the resulting surface is log terminal and $C$ is a log terminal contractible divisor.

**Definition 6.17** (LDP 17). Let $(S_1, C)$ be one of the following surfaces in lemma 5.2, lemma 5.3, lemma 5.6, lemma 5.7, lemma 5.8 or lemma 5.9. Define $\tilde{S} = S_1$.

We are now ready to state the classification theorem.

**Theorem 6.18** (Classification of rank one log Del Pezzo surfaces). Let $S$ be a rank one log Del Pezzo surface over an algebraically closed field of characteristic different from two, three and five. If $S$ is smooth then $S = \mathbb{P}^2$; if $S$ is Gorenstein, then $S$ is one of the surfaces in theorem 2.7, otherwise it is one of the log Del Pezzo surfaces in the families LDP1 to LDP17.

*Proof.* If $S$ is Gorenstein, this was done in theorem 2.7. Suppose then that $S$ is not Gorenstein. If $S$ has no tigers the result follows from sections 4.1 to 4.6. If $S$ has a tiger $C \subset S$ such that $K_S + C$ is log canonical, the result follows from lemma 5.6, lemma 5.7, lemma 5.8 and lemma 5.9. If $K_S + C$ is not log canonical, or all the tigers for $S$ are exceptional, the theorem follows from lemma 5.12 and 5.11. \qed

7 Liftability to characteristic zero

As an application of theorem 6.18 we show that every rank one log Del Pezzo surface over an algebraically closed field of characteristic higher than five, lifts with smooth base over characteristic zero.

This answers the question after [CTW17, Theorem 1.1]. Let’s start first with the definition of liftability as in [CTW17].

**Definition 7.1.** Let $X$ be a smooth variety over a perfect field $k$ of characteristic $p > 0$, and let $D$ be a simple normal crossing divisor on $X$. Write $D = \sum_i D_i$, where $D_i$ are the irreducible components of $D$. We say that a pair $(X, D)$ is liftable to characteristic zero with a smooth base if there exists

1. a scheme $T$ smooth and separated over Spec $\mathbb{Z}$
2. a smooth and separated morphism $\mathcal{X} \to T$
3. effective Cartier divisors $\mathcal{D}_1, ..., \mathcal{D}_r$ on $\mathcal{X}$ such that the scheme-theoretic intersection $\bigcap J \mathcal{D}_i$ for any subset $J \subseteq \{1, ..., r\}$ is smooth over $T$.

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4. a morphism $\alpha : \text{Spec } k \to T$ such that the base changes of the schemes $X, D_i$ over $T$ by $\alpha$ are isomorphic to $X, D_i$ respectively.

**Theorem 7.2.** Let $S$ be a rank one log Del Pezzo surface over an algebraically closed field of characteristic $p > 5$. Then there is a log resolution $\nu : V \to S$ such that $(V, \text{Exc(}\nu))$ is liftable to characteristic zero with smooth base.

**Proof.** If $S$ is Gorenstein, the result follows by writing the integral Weierstrass model of the corresponding elliptic surface (see [JLR12]). Suppose therefore that $S$ is not Gorenstein. Notice that in the classification of theorem 6.18 $S$ is determined by the geometry of the first surface from which we start the construction of $S$. Since these clearly lift to characteristic zero, and the smooth blow ups also do, we deduce that $S$ has a log resolution that lifts as well. 

### 7.1 Non liftable examples in low characteristic

In this subsection we will see that in characteristic two, three and five, there are log Del Pezzo surfaces of rank one that do not lift to characteristic zero with smooth base, and therefore the conclusion of theorem 7.2 is sharp. The first such example was shown in [KM99, Chapter 9] for characteristic two. The following examples are not liftable to characteristic zero because they do not satisfy the Bogomolov bound of [KM99, Chapter 9] (see the proof of [CTW17, Theorem 1.3]).

**Example 7.3 (characteristic 2).** Start with $D$ given by $y = x^3$ in $\mathbb{P}^2$, and a conic $C$ such that its strange point is in the flex of $D$, and such that there are three distinct point of intersection with $D$. Now blow up once at the flex, twice two of the intersection points and three times the third one, always along $D$.

This gives a Gorenstein log Del Pezzo surface with singularities $D_4 + 4A_1$.

**Example 7.4 (characteristic 2).** Take the previous example, blow up the cusp of $D$, and the down blow down $D$. This gives a log Del Pezzo surface with singularities $D_4 + 4A_1 + (3)$.

One can of course continue producing non liftable examples in this fashion, and all of these examples eventually boil down to the fact that covers of degree two can have any number of ramification points in characteristic two.

Let’s now give two examples in characteristic three.

**Example 7.5 (characteristic 3).** Pick $C$ to be $y = x^3$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Choose three points on $C$ and blow up three times each along $C$. This gives a log Del Pezzo surface with singularities $4(3) + 3A_2$.

**Example 7.6 (characteristic 3).** Pick $C$ to be $y = x^3$ in $\mathbb{P}^2$. Choose three flex lines $L_1, L_2, L_3$ such that their intersection with $C$ lie on a line $L_4$. Blow up three times two flex points, blow up twice the third one, always along $C$ and then blow up $L_1 \cap L_2 \cap L_3$. Finally blow up the cusp of $C$. This gives a log Del Pezzo surface with singularities $2(3) + E_6 + (2,3)$.
Finally let’s conclude with an example in characteristic five.

**Example 7.7** (characteristic 5). Thanks to [Lan94, Theorem 4.1], in characteristic five there is a Gorenstein log Del Pezzo $S(2A_4)$ such that it has a cuspidal rational curve in its smooth locus. It’s constructed as follows. Choose four points $a,b,c,d$ in $\mathbb{P}^2$ and consider the lines $l_{ab}, l_{bc}, l_{cd}, l_{ac}, l_{ad}, l_{bd}$ between them. Resolve the base locus of the pencil given by $l_{ab} \cdot l_{bc} \cdot l_{cd}$ and $l_{ac} \cdot l_{ad} \cdot l_{bd}$ at $a,b,c,d$. This gives the desired Gorenstein log Del Pezzo $S(2A_4)$. Now the anti-canonical class has a cuspidal rational curve with self intersection one. Blow up its cusp three times, then contract the resulting rational curve and two of the exceptional divisors. Therefore we obtain a log Del Pezzo surface with singularities $2A_4 + (5) + (3) + (2)$.

We write now down the explicit details of the above example. Let $k$ be an algebraically closed fields of characteristic five and consider the following pencil in $\mathbb{P}^2_k$:

$$(y^2 - z^2)(x + y) + t(x^2 - z^2)(y - x)$$

Over any algebraically closed field of characteristic higher than five, this pencil contains exactly two other singular curves, both nodal. We claim that in characteristic five, however, there is a cuspidal curve $C$ at $t = 2$, with cusp at $[1, 3, 0]$. First of all, one easily checks that $C$ passes through $[1, 3, 0]$.

By looking at the equation in the chart $x \neq 0$ and setting $x = 1$, we get the following non-homogeneous equation for $C$:

$$(y^2 - z^2)(y + 1) + 2(1 - z^2)(y - 1) = 0$$

The partial derivatives are

$$\frac{\partial}{\partial y} : 2y(y + 1) + y^2 - z^2 + 2(1 - z^2)$$

and

$$\frac{\partial}{\partial z} : -2z(y + 1) - 4z(y - 1)$$

It’s easy to see that they both vanish at $(3, 0)$. Let’s now show that $C$ has a cusp at $[1, 3, 0]$.

$$\frac{\partial^2}{\partial y^2} : y + 2$$

$$\frac{\partial^2}{\partial y \partial z} : -6z$$

$$\frac{\partial^2}{\partial z^2} : -y + 2$$

By evaluating at $(3, 0)$ one sees that the Hessian is singular.
Now that we have fully determined the geometric situazion, consider the following blowups. First blow up twice each of the four vertices \([1, 1, 1], [1, -1, 1], [-1, 1, 1], [-1, -1, 1]\) along the cubic equation that has only one line passing through the vertex. Now contract the eight \((-2)\) curves to get a Gorenstein log Del Pezzo surface \(S = S(2A_4)\) of Picard number one. The strict image of \(C\) is a cuspidal rational curve of self intersection \(C^2 = 1\) and contained in the smooth locus of \(S\). Now resolve the cusp of \(C\), and blow down all the exceptional divisors with self intersection at most \(-2\). This is a log Del Pezzo surface \(S'\) of Picard number one such that

1. \(C\) contains three singularities (a (2), a (3) and a (5)).
2. \((S', C)\) is dlt.
3. \(K_{S'} + C\) is anti-ample.
4. There are two \(A_4\) singularities outside \(C\).

Therefore, \(S'\) does not lift to characteristic zero with smooth base.

References


