QUAL PREP SESSION 4

Problem 1

Define $f * g$ for $f, g \in L^1(\mathbb{R})$ as

$$f * g(x) = \int f(x-y)g(y)dy.$$

Show that $f * g$ is in L^1 and in fact

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1 .$$

Problem 2

Suppose there is a sequence of continuous functions $f_n\colon [0,1] \to \mathbb{R}$ such that

$$\int_0^1 (f_n(x) - f_m(x))^2 dx \to 0 \text{ as } m, n \to 0.$$

Let $k\colon [0,1] \times [0,1] \to \mathbb{R}$ be a continuous function and define

$$\hat{f}_n(x) := \int_0^1 k(x,y) f_n(y) dy$$

Show that $\hat{f}_n$ is uniformly convergent.

Problem 3

Let c_0 be the space of sequences $x = \{x_n\}$ of real numbers such that $\lim x_n = 0$. With the norm $\|x\| = \sup |x_n|$, c_0 is a Banach space.

Show that the set

$$S := \{x \in c_0 \mid \sup n\, |x_n| < \infty\}$$

is of first category (countable union of nowhere dense sets)
