ON THE BIRATIONAL AUTOMORPHISMS OF VARIETIES OF GENERAL TYPE

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In memory of Eckart Viehweg

ABSTRACT. We show that the number of birational automorphisms of a variety of general type X is bounded by $c \cdot \operatorname{vol}(X, K_X)$, where c is a constant which only depends on the dimension of X.

CONTENTS

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1. Introduction

Throughout this paper, unless otherwise mentioned, the ground field k will be an algebraically closed field of characteristic zero.

Theorem 1.1. *If n is a positive integer, then there is a constant c such that the birational automorphism group of any projective variety X of general type of dimension n has at most $c \cdot \operatorname{vol}(X, K_X)$ elements.*

For curves, this is a weak form of the classical Hurwitz Theorem which says that if C is a curve of genus $g \geq 2$ with automorphism group G, then $|G| \leq 84(g-1)$. Note that $\operatorname{vol}(C, K_C) = 2g - 2$ and so this bound may be rephrased as $|G| \leq 42 \cdot \operatorname{vol}(C, K_C)$.

This problem has been extensively studied in higher dimensions, see for example, [1], [3], [8], [10], [16], [30], and [31], for surfaces, [9], [26], [28], [32], and [33], in higher dimensions, and [5], for surfaces in characteristic p.

Xiao, [31], proved that if S is a smooth projective surface of general type, with automorphism group G, then $|G| \leq (42)^2 \operatorname{vol}(S, K_S)$ (if S is minimal, then $\operatorname{vol}(S, K_S) = K_S^2$; for the general definition of the volume, see [21, 2.2.31] or (2.3.1)). Xiao shows that we have equality if and only if S is a quotient of $C \times C$, where C is a curve whose automorphism group has cardinality $42(2g-2)$, by the action of a very special subgroup of the automorphism group of $C \times C$.

Question 1.2. *Find an explicit bound for the constant c appearing in* (1.1).

If C is a curve with automorphism group of maximal size, that is, $|\operatorname{Aut}(C)| = 84(g-1)$ and

$$X = C \times C \times \cdots \times C,$$

then $\operatorname{Aut}(X) = n!(42)^n(2g-2)^n$ and $\operatorname{vol}(X, K_X) = n!(2g-2)^n$, so that $c \geq 42^n$. If we consider the example of the Fermat hypersurface

$$X = (X_0^m + X_1^m + \cdots + X_{n+1}^m = 0) \subset \mathbb{P}^{n+1},$$

then $\operatorname{Aut}(X) \geq (n+2)!m^{n+1}$ and $\operatorname{vol}(X, K_X) = m(m-n-2)^n$. If we take $m = n+3$ then the ratio

$$\frac{\operatorname{Aut}(X)}{\operatorname{vol}(X, K_X)} \geq (n+2)!(n+3)^n,$$

exceeds 42^n for n sufficiently large (indeed, $n \geq 5$ suffices), so that c is eventually greater than 42^n. In fact, c grows faster than n^n, so that c grows faster than any exponential function.

It is all too easy to give examples which show that (1.1) fails spectacularly in characteristic p. Consider the finite field $\mathbb{F}_{q^2}$ with q^2 elements, where $q = p^k$ is a power of a prime p. Note that the function

$$\mathbb{F}_{q^2} \longrightarrow \mathbb{F}_{q^2} \qquad \text{given by} \qquad b \longrightarrow \bar{b} = b^q,$$

is an involution which plays the role of complex conjugation in characteristic p. Suppose that $V = \mathbb{F}_{q^2}^m$ is the standard vector space of dimension m over the field $\mathbb{F}_{q^2}$. Then there is a sesquilinear pairing

$$V \times V \longrightarrow \mathbb{F}_{q^2} \qquad \text{given by} \qquad (a, b) \longrightarrow \sum_i a_i \bar{b}_i.$$

Let $U_m(q)$ denote the group of $m \times m$ unitary matrices over the field $\mathbb{F}_{q^2}$, so that $U_m(q)$ is the group of linear maps of V preserving the pairing. Recall that $U_m(q)$ is a finite simple group of Lie type, see for example [13], whose notation we follow. Note that the Fermat hypersurface

$$X = (X_0^{q+1} + X_1^{q+1} + \cdots + X_{n+1}^{q+1} = 0) \subset \mathbb{P}^{n+1},$$

is the projectivisation of the null cone of the pairing, so that $\operatorname{Aut}(X) \supset U_{n+2}(q)$. We have

$$|U_{n+2}(q)| = \frac{1}{(n+2, q+1)} q^{\binom{n+2}{2}} \prod_{i=2}^{n+2} (q^i - (-1)^i),$$

see, for example, the table on page 8 of [13]. Note that both the order of the automorphism group and the volume of the Fermat hypersurface are polynomials f and g in q. f has degree

$$\binom{n+2}{2} + \binom{n+3}{2} - 1,$$

and g has degree

$$(n+1).$$

If $n = 1$, the genus is a quadratic polynomial in q and the order of the automorphism group is bounded by a polynomial of degree 4 in g.

Question 1.3. *Fix a positive integer n. Can we find positive integers c and d such that if X is any n-dimensional smooth projective variety of general type over an algebraically closed field of arbitrary characteristic, then*

$$|\operatorname{Aut}(X)| \leq c \cdot \operatorname{vol}(X, K_X)^d?$$

It is known that if $n = 1$ then we may take $c = 216$ and $d = 4$ (cf. [25]).

We now explain how to derive (1.1) from a result about the quotient. If Y is a variety of general type, then the automorphism group $G =$

$\mathrm{Aut}(Y)$ is known to be finite, see [22]. If $f\colon Y \longrightarrow X = Y/G$ is the quotient map, then there is a $\mathbb{Q}$-divisor Δ on X such that $K_Y = f^*(K_X + \Delta)$. We call any such log pair (X, Δ) a *global quotient*, cf. (2.2.1). As

$$\mathrm{vol}(Y, K_Y) = |G| \cdot \mathrm{vol}(X, K_X + \Delta),$$

the main issue is to bound $\mathrm{vol}(X, K_X + \Delta)$ from below:

Theorem 1.4. *Fix a positive integer n. Let $\mathfrak{D}$ be the set of log pairs (X, Δ), which are global quotients, where X is projective of dimension n.*

(1) the set

$$\{\, \mathrm{vol}(X, K_X + \Delta) \,|\, (X, \Delta) \in \mathfrak{D} \,\},$$

satisfies the DCC.

Further, there are two constants $\delta > 0$ and M such that if $(X, \Delta) \in \mathfrak{D}$ and $K_X + \Delta$ is big, then

(2) $\mathrm{vol}(X, K_X + \Delta) \geq \delta$, and
(3) $\phi_{M(K_X+\Delta)}$ is birational.

DCC is an abbreviation for the descending chain condition. Note that, by convention, $\phi_{M(K_X+\Delta)} = \phi_{\lfloor M(K_X+\Delta) \rfloor}$ (cf. (2.1)). Note also that the set of volumes of smooth projective varieties of fixed dimension is a discrete set (cf. [29], [15] and [27]). The situation for log pairs is considerably more subtle.

Remark 1.5. [29], [15] *and* [27] *show that if we fix a positive integer n, then the set*

$$\{\, \mathrm{vol}(X, K_X) \,|\, X \text{ is a smooth projective variety of dimension } n \,\},$$

is discrete. However, the corresponding statement fails for kawamata log terminal surfaces, whence also for surface with reduced boundary with simple normal crossings. See [19] *for an example.*

However we do have:

Conjecture 1.6 (Kollár, cf. [17], [1])**.** *Fix $n \in \mathbb{N}$ and a set $I \subset [0, 1]$ which satisfies the DCC.*

If $\mathfrak{D}$ is the set of simple normal crossings pairs (X, Δ), where X is projective of dimension n, and the coefficients of Δ belong to I, then the set

$$\{\, \mathrm{vol}(X, K_X + \Delta) \,|\, (X, \Delta) \in \mathfrak{D} \,\},$$

satisfies the DCC.

Alexeev, cf. [1] and [2], proved (1.6) for surfaces. Note that if (X, Δ) is a global quotient, then the coefficients of Δ belong to the set $I = \{ \frac{r-1}{r} \,|\, r \in \mathbb{N} \}$, so that (1.4) is a special case of (1.6).

We hope to give an affirmative answer to (1.6), using some of the techniques developed in this paper. Let

$$I = \{ \frac{r-1}{r} \,|\, r \in \mathbb{N} \}.$$

Assuming an affirmative answer to (1.6) for this particular set, it is interesting to wonder what is the smallest possible volume. Let

$$(X, \Delta) = (\mathbb{P}^n, \frac{1}{2}H_0 + \frac{2}{3}H_1 + \frac{6}{7}H_2 + \cdots + \frac{r_{n+1}}{r_{n+1}+1}H_{n+1}),$$

where $H_0, H_1, \dots, H_{n+1}$ are $n+2$ general hyperplanes and $r_1, r_2, \dots$ are defined recursively by:

$$r_0 = 1 \qquad \text{and} \qquad r_{n+1} = r_n(r_n + 1).$$

Note that $(X, \Delta) \in \mathfrak{D}$. It is easy to see that the volume of $K_X + \Delta$ is

$$\frac{1}{r_{n+2}^n}.$$

Question 1.7. *Find an explicit bound for*

$$\delta = \min\{ \operatorname{vol}(X, K_X + \Delta) \,|\, (X, \Delta) \in \mathfrak{D} \}.$$

The most optimistic answer to (1.7) would be

$$\delta = \frac{1}{r_{n+2}^n}.$$

Note that $c \geq \frac{1}{\delta}$. When $n = 1$, we have

$$\delta = \frac{1}{r_3} = \frac{1}{42},$$

and the reciprocal is precisely the constant $c = 42$. On the other hand, one can check that r_n grows roughly like

$$a^{2^n},$$

for some constant $a > 1$, so that in general there is a huge difference between r_n and c_n. In fact it is easy to check that

$$r_{n+1} = \prod_{i=0}^{n} (r_i + 1),$$

see §8 of [18] for more details.

Theorem 1.8 (Deformation invariance of log plurigenera)**.** *Let $\pi\colon X \longrightarrow T$ be a projective morphism of smooth varieties. Suppose that (X, Δ) is log canonical and has simple normal crossings over T.*

(1) If $K_X + \Delta$ is kawamata log terminal, and either $K_X + \Delta$ or Δ is big over T, and m is any positive integer such that $m\Delta$ is integral, then $h^0(X_t, \mathcal{O}_{X_t}(m(K_{X_t} + \Delta_t)))$ is independent of $t \in T$.
(2) $\kappa_\sigma(X_t, K_{X_t} + \Delta_t)$ is independent of $t \in T$.
(3) $\operatorname{vol}(X_t, K_{X_t} + \Delta_t)$ is independent of $t \in T$.

For the definitions of κ_σ and simple normal crossings over T, see (2.1). We will prove a similar but stronger statement (4.2) which implies (1.8). Obviously, (1.8) is a generalisation of Siu's theorem on invariance of plurigenera, cf. [24]. We recently learnt that (1) of (1.8) holds even without the assumption that $K_X + \Delta$ is big, see Theorem 0.2 of [6]. We use (1.8) to prove:

Theorem 1.9. *Fix a set $I \subset [0, 1]$ which satisfies the DCC. Let $\mathfrak{D}$ be a set of simple normal crossings pairs (X, Δ), which is log birationally bounded (cf. (2.4.1)), such that if $(X, \Delta) \in \mathfrak{D}$, then the coefficients of Δ belong to I.*

Then the set

$$\{ \operatorname{vol}(X, K_X + \Delta) \,|\, (X, \Delta) \in \mathfrak{D} \},$$

satisfies the DCC.

1.1. **Sketch of the proof of (1.4).** The proof of (1.4) is by induction on the dimension n and the proof is divided into two steps. The first step uses some ideas of Tsuji which are used to prove that some fixed multiple of K_X defines a birational map, for a variety X of general type, see [29], [15] and [27]. In this step we establish that modified versions of (2) and (3) of (1.4) are equivalent, given that $(1.4)_{n-1}$ holds (that is, (1.4) holds in dimension $n-1$). Namely, consider

(2) $\operatorname{vol}(X, r(K_X + \Delta)) > \delta$, and
(3) $\phi_{Mr(K_X+\Delta)}$ is birational.

We show that if (X, Δ) is a global quotient of dimension n, then there are constants $\delta > 0$ and M such that for every positive integer r, (2) implies (3), see (6.1).

It is clear that if some fixed multiple of $r(K_X+\Delta)$ defines a birational map, then the volume of $r(K_X + \Delta)$ is bounded from below, (2.3.2), so that there are constants δ and M such that (3) implies (2). To go the other way, we need to construct a divisor $0 \leq D \sim_{\mathbb{Q}} mr(K_X + \Delta)$, where m is fixed, which has an isolated non kawamata log terminal centre at a very general point and is not kawamata log terminal at another very general point, (2.3.4). As we know that log canonical models exist by [7], we may assume that $K_X + \Delta$ is ample, so that lifting divisors from any subvariety is simply a matter of applying Serre vanishing. In this case, it is well known, since the work of Anghern and Siu, [4], that to construct D we need to bound the volume of $K_X + \Delta$ from below on special subvarieties V of X (specifically, any V which is a non kawamata log terminal centre of $(X, \Delta + \Delta_0)$, where Δ_0 is proportional to $K_X + \Delta$), see (2.3.5) and (2.3.6).

If $V = X$, there is nothing to prove, as we are assuming that $\mathrm{vol}(X, r(K_X + \Delta)) > \delta$. Otherwise, the dimension of V is less than the dimension of X, and we may proceed by induction on the dimension; as V passes through a very general point of X, V is birational to a global quotient. In fact, we even know that $\mathrm{vol}(V, (K_X + \Delta)|_V)$ is bounded from below.

So from now on we assume that (2) implies (3). We may suppose that the constant δ appearing in (2) is at most one. The next step is to prove that (3) holds when $r = 1$ (that is, (3) of (1.4) holds). There are two cases. If the volume of $K_X + \Delta$ is at least one, then the volume of $K_X + \Delta$ is certainly bounded from below, and there is nothing to prove. Otherwise, we may find $r > 0$ such that $\delta < 1 \leq \mathrm{vol}(X, r(K_X + \Delta)) < 2^n$. But then $\phi_{m(K_X+\Delta)}$ is birational, where $m = Mr$ and, at the same time, the volume of $m(K_X + \Delta)$ is bounded from above. In this case, the degree of the image of $\phi_{m(K_X+\Delta)}$ is bounded from above, and so we know that the image belongs to a bounded family. In fact, one can prove that both the degree of the image and the degree of the image of Δ and the exceptional locus have bounded degree, (3.1), so we only need to concern ourselves with a log birationally bounded family of log pairs, (2.4.1). This finishes the first step.

To finish the argument, we need to argue that the volume is bounded from below if we have a log birationally bounded family of log pairs. This is the most delicate part of the argument and is the second step. We use some ideas which go back to Alexeev. Firstly, it is not much harder to prove that the volume of global quotients satisfies the DCC. The first part of the second step is to argue that we only need to worry about log pairs (X, Δ) which are birational to a single pair (Z, B), rather than a bounded family of log pairs. For this we prove a version of deformation invariance of log plurigenera for log pairs, see (1.8). Deformation invariance fails in general (cf. [11, 4.10]); we need to assume that the family has simple normal crossings over the base (which roughly means that every component of Δ is smooth over the base). To reduce to this case involves some straightforward manipulation of a family of log pairs, see the proof of (1.9) in §5. We prove, cf. (4.2), a version of (1.8) which is better suited to induction; to this end, we first show that if we have a family of log pairs over a curve then we can run the MMP in a family, (4.1).

So we are reduced to the most subtle part of the argument. We are given a simple normal crossings pair (Z, B), a set I which satisfies the DCC, and we want to argue that if (X, Δ) is a simple normal crossings pair such that there is a birational morphism $f\colon X \longrightarrow Z$ with $f_*\Delta \leq B$, then the volume of $K_X + \Delta$ belongs to a set which satisfies the DCC, see (5.1). To fix ideas, let us suppose that Z is a smooth surface (this is the case originally treated by Alexeev). We are given a sequence of simple normal crossings surfaces (X_i, Δ_i) and birational morphisms $f_i\colon X_i \longrightarrow Z$.

We have that $\Phi_i = f_{i*}\Delta_i \leq B$ and the coefficients of Φ_i belong to I. Note that we are free to pass to an arbitrary subsequence, so that we may assume that $\Phi_i \leq \Phi_{i+1}$. In particular, the volume of $K_Z + \Phi_i$ is not decreasing. The problem is that if we write

$$K_{X_i} + \Delta_i = f_i^*(K_Z + \Phi_i) + E_i,$$

then E_i might have negative coefficients, so that the volume of $K_{X_i}+\Delta_i$ is less than the volume of $K_Z + \Phi_i$. Suppose that we write $E_i = E_i^+ - E_i^-$, where $E_i^+ \geq 0$ and $E_i^- \geq 0$ have no common components. Note that E_i^+ has no effect on the volume, as E_i is supported on the exceptional locus. What bothers us is the possibility that the E_i^- involve exceptional divisors that live on higher and higher models.

Clearly, we should consider the limit $\Phi = \lim_i \Phi_i$. However, this is not enough, we need to take the limit of the divisors Δ_i on various models, and to work with linear systems on these models. It was for just this purpose that the language of **b**-divisors was introduced by Shokurov. Recall that a **b**-divisor $\mathbf{D}$ is just the choice of a divisor $\mathbf{D}_X$ on every model X, which is compatible under pushforward.

For us there are three relevant **b**-divisors. Since we want to work on higher models without changing the volume, or the fact that the coefficients lie in the set I, we introduce, (5.5), the **b**-divisor $\mathbf{M}_{\Delta_i}$ associated to a log pair (X_i, Δ_i): given a model $Y \longrightarrow X_i$ we just throw in any exceptional divisor with coefficient one. We next take the limit $\mathbf{B}$ of the sequence of **b**-divisors $\mathbf{M}_{\Delta_i}$; on Z we just recover the divisor Φ. Finally, we define (5.2), a **b**-divisor $\mathbf{L}_{\Delta_i}$ that assigns to a model $\pi\colon Y \longrightarrow X_i$ the positive part of the log pullback. Actually, this is the most complicated of the three **b**-divisors, and it is the subtle behaviour of the **b**-divisors $\mathbf{L}_{\Delta_i}$ which complicates the proof.

If we set $\Delta'_i = \Delta_i \wedge \mathbf{L}_{\Phi_i, X_i}$, then,

$$\mathrm{vol}(X_i, K_{X_i} + \Delta'_i) = \mathrm{vol}(X_i, K_{X_i} + \Delta_i),$$

see (2) of (5.3). If we knew that the coefficients of Δ'_i belong to a set $I \subset J$ that satisfies the DCC, then we would be done. In fact, if $\mathbf{L}_\Phi \leq \mathbf{B}$, then it is relatively straightforward to conclude that the volume satisfies the DCC, see the proof of (5.1). Unfortunately, since $X_i \longrightarrow Z$ might extract arbitrarily many divisors, it is all too easy to write down examples where the smallest set J which contains the coefficients of every Φ_i does not satisfy the DCC (cf. (1.10)).

Therefore, our objective is to find a model $Z' \longrightarrow Z$ and suitable modifications Δ'_i of Δ_i such that $\mathbf{L}_{\Phi'} \leq \mathbf{B}'$, where $\mathbf{B}'$ is the limit of $\mathbf{M}_{\Delta'_i}$. We choose Δ'_i so that the difference $\Delta_i - \Delta'_i$ is supported only on the strict transform of the exceptional divisors of $Z' \longrightarrow Z$. In this case, it is not hard to arrange for the coefficients of Δ'_i to belong to a set $I \subset J$ that satisfies the DCC.

We construct $Z' \longrightarrow Z$ by induction. For this, we can work locally about a point p in Z. Suppose that p is the intersection of two components B_1 and B_2 of Φ. If B_1 and B_2 appear with coefficient b_1 and b_2 in Φ and $\pi\colon Z' \longrightarrow Z$ blows up p, then working locally about p, we may write

$$K_{Z'} + b_1 B'_1 + b_2 B'_2 + eE = \pi^*(K_Z + b_1 B_1 + b_2 B_2),$$

where $e = b_1 + b_2 - 1$. Here primes denote strict transforms and E is the unique exceptional divisor. We suppose that $\mathbf{L}_\Phi \leq \mathbf{B}$ does not hold, so that there is some valuation ν, with centre p, such that $\mathbf{L}_\Phi(\nu) > \mathbf{B}(\nu)$. Since $e = b_1 + b_2 - 1$, the larger b_1 and b_2, the further we expect to be from the inequality $\mathbf{L}_\Phi \leq \mathbf{B}$. We introduce the **weight** w, which counts the number of components of Φ of coefficient one, that is, the number of i such that $b_i = 1$. In the case of a surface, the weight is 0, 1 or 2 and it clearly suffices to construct $Z' \longrightarrow Z$ so that the weight goes down.

In fact, the extreme cases are relatively easy. If the weight is two, then we just take $Z' \longrightarrow Z$ to be the blow up of p. The key point is that then the base locus of the linear system

$$f'_{i*}|m(K_{X_i} + \Delta_i)| \subset |m(K_Z + \Phi_i)|,$$

contains p in its support for all m sufficiently large and divisible and this forces the strict transform of E to be a component of E_i^+ as well. Therefore we are free to decrease the coefficient of E in Φ' away from 1. At the other extreme, if the weight is zero, then (Z, Φ) is kawamata log terminal and we may find $Z' \longrightarrow Z$ which extracts every divisor of coefficient greater than zero. In this case $\mathbf{L}_{\Phi'}(\rho) = 0$ for every valuation ρ whose centre on Z' is not a divisor and the inequality $\mathbf{L}_{\Phi'} \leq \mathbf{B}'$ is trivial.

The hard case is when the weight is one, so that one of b_1 and b_2 is one. Suppose that $b_2 = 1$. If ν is a valuation such that $\mathbf{L}_{\Phi}(\nu) > 0$ then ν corresponds to a weighted blow up. At this point it is convenient to use the language of toric geometry. A weighted blow up corresponds to a pair of natural numbers $(v_1, v_2) \in \mathbb{N}^2$. In fact, if

$$\mathfrak{F} = \{\, v_1 \in \mathbb{N} \,|\, (1 - b_1)v_1 < 1 \,\},$$

then

$$\{\, (v_1, v_2) \in \mathbb{N}^2 \,|\, v_1 \in \mathfrak{F} \,\},$$

is the set of all valuations ν such that $\mathbf{L}_{\Phi}(\nu) > 0$. The crucial point is that $\mathfrak{F}$ is finite. For every element $f_1 \in \mathfrak{F}$, we pick $v_2 \in \mathbb{N}$, which minimises $\mathbf{B}(f_1, v_2)$ (this makes sense as I satisfies the DCC). We then pick any simple normal crossings model $Z' \longrightarrow Z$ on which the centre of every one of these finitely many valuations is a divisor. Using standard toric geometry, one can check that on Z' every valuation ν', whose centre belongs to a component of Φ' of coefficient one, satisfies the property $\mathbf{L}_{\Phi'}(\nu') \leq \mathbf{B}'(\nu')$. It follows that the weight of (Z', Φ') is zero and this completes the induction. The details are contained in the proof of (5.1).

Example 1.10. *Consider the behaviour of $\mathbf{L}_{\Phi}$ in a simple example. We use the notation above. Note that $e = b_1 + b_2 - 1$ is an increasing and affine linear function of b_1 and b_2, whence a continuous function of b_1 and b_2. We are only concerned with the possibility that $e > 0$, and in this case, $\mathbf{L}_{\Phi,Z'} = b_1 B'_1 + b_2 B'_2 + eE$, by definition. There are two interesting points $p_1 = B'_1 \cap E$ and $p_2 = B'_2 \cap E$ lying over p, and the coefficients of the divisors containing them are b_1, e and b_2, e. The problem is that we can blow up along either point and keep going.*

Suppose that

$$I = \{\, \frac{r-1}{r} \,|\, r \in \mathbb{N} \,\}.$$

Note that if $b_1 = \frac{i}{r}$ and $b_2 = \frac{r-1}{r}$ then

$$e = b_1 + b_2 - 1 = \frac{i}{r} + \frac{r-1}{r} - 1 = \frac{i-1}{r}.$$

So, the smallest set J which contains I and which is closed under the operation of picking any two elements b_1 and b_2 and replacing them by $b_1 + b_2 - 1$ (provided this sum is non-negative) is $\mathbb{Q} \cap [0, 1]$. Clearly, this set does not satisfy the DCC.

2. Preliminaries

2.1. **Notation and Conventions.** We will use the notations in [20] and [21].

If $D = \sum d_i D_i$ is a $\mathbb{Q}$-divisor on a normal variety X, then the *round down* of D is $\lfloor D \rfloor = \sum \lfloor d_i \rfloor D_i$, where $\lfloor d \rfloor$ denotes the largest integer which is at most d, the *fractional part* of D is $\{D\} = D - \lfloor D \rfloor$, and the *round up* of D is $\lceil D \rceil = -\lfloor -D \rfloor$. If $D' = \sum d'_i D_i$ is another $\mathbb{Q}$-divisor, then $D \wedge D' := \sum \min\{d_i, d'_i\} D_i$.

The sheaf $\mathcal{O}_X(D)$ is defined by

$$\mathcal{O}_X(D)(U) = \{\, f \in k(X) \mid (f)|_U + D|_U \geq 0 \,\},$$

so that $\mathcal{O}_X(D) = \mathcal{O}_X(\lfloor D \rfloor)$. Similarly we define $|D| = |\lfloor D \rfloor|$. If X is normal, and D is a $\mathbb{Q}$-divisor on X, the *rational map ϕ_D associated to D* is the rational map determined by the restriction of $\lfloor D \rfloor$ to the smooth locus of X.

If X is a normal projective variety and D is a $\mathbb{Q}$-Cartier divisor, $\kappa_\sigma(X, D)$ denotes the *numerical Kodaira dimension*, which is defined by Nakayama in [23, V.2.5] as follows: let H be a divisor on X, we define

$$\sigma(X, D; H) := \max\{\, k \in \mathbb{N} \mid \limsup_{m \to \infty} \frac{h^0(X, H + mD)}{m^k} > 0 \,\},$$

and

$$\kappa_\sigma(X, D) := \max\{\, \sigma(X, D; H) \mid H \text{ is a divisor} \,\}.$$

If D is pseudo-effective, we define $N_\sigma(X, D)$ as in [23, III.4] or [7, 3.3.1].

A *log pair* (X, Δ) consists of a normal variety X and a $\mathbb{Q}$-Weil divisor $\Delta \geq 0$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. The *support* of $\Delta = \sum_{i \in I} d_i \Delta_i$ is the sum $D = \sum_{i \in I} \Delta_i$. If (X, Δ) has simple normal crossings then a *stratum* of (X, Δ) is an irreducible component of the intersection $\cap_{j \in J} \Delta_j$, where J is a non-empty subset of I (in particular, a stratum is always a proper closed subset of X). If we are given a morphism $X \longrightarrow T$, then we say that (X, Δ) has *simple normal crossings over* T if (X, Δ) has simple normal crossings and both X and every stratum of (X, Δ) is smooth over T. We say that the birational morphism $f\colon Y \longrightarrow X$ only *blows up strata* of (X, Δ), if f is the composition of birational morphisms $f_i\colon X_{i+1} \longrightarrow X_i$, $1 \leq i \leq k$, with $X = X_0$, $Y = X_{k+1}$, and f_i is the blow up of a stratum of (X_i, D_i), where D_i is the sum of the strict transform of D and the exceptional locus.

A *log resolution* of the pair (X, Δ) is a projective birational morphism $\mu\colon Y \longrightarrow X$ such that the exceptional locus is the support of a μ-ample divisor and (Y, G) has simple normal crossings, where G is the support of the strict transform of Δ and the exceptional divisors. Note that the extra assumption that the exceptional locus is the support of a μ-ample divisor is not standard. However it is convenient for our purpose, and it can be always achieved after possibly choosing a higher model. If we write

$$K_Y + \Gamma + \sum a_i E_i = \mu^*(K_X + \Delta)$$

where Γ is the strict transform of Δ, then a_i is called the *coefficient of E_i with respect to* (X, Δ). Note that $-a_i$ is the discrepancy of the pair (X, Δ), with respect to E_i, see [20, 2.25]. A *non kawamata log terminal centre* is the centre of any valuation ν whose coefficient is at least one.

In this paper, we only consider valuations ν of X whose centre on some birational model Y of X is a divisor. We say that a formal sum $\mathbf{B} = \sum a_\nu \nu$, where the sum ranges over all valuations of X, is a **b**-*divisor*, if the set

$$F_X = \{\, \nu \,|\, a_\nu \neq 0 \text{ and the centre of } \nu \text{ on } X \text{ is a divisor} \,\},$$

is finite. The *trace* $\mathbf{B}_Y$ of $\mathbf{B}$ is the sum $\sum a_\nu B_\nu$, where the sum now ranges over the elements of F_Y. In fact, to give a **b**-divisor is the same as to give a collection of divisors on every birational model of X, which are compatible under pushforward.

2.2. **Log pairs.**

Definition 2.2.1. *We say that a log pair (X, Δ) is a **global quotient** if there is a smooth quasi-projective variety Y and a finite subgroup $G \subset \operatorname{Aut}(Y)$ such that $X = Y/G$ and if $\pi \colon Y \longrightarrow X$ is the quotient morphism, then $K_Y = \pi^*(K_X + \Delta)$.*

Note that if (X, Δ) is a global quotient, then X is $\mathbb{Q}$-factorial, (X, Δ) is kawamata log terminal, and the coefficients of Δ belong to the set

$$\{\, \frac{r-1}{r} \,|\, r \in \mathbb{N} \,\},$$

(cf. [20, 5.15, 5.20]).

Lemma 2.2.2. *If (X, Δ) is a log pair and the coefficients of Δ are less than one, then*

$$\lfloor m\Delta \rfloor \leq \lceil (m-1)\Delta \rceil,$$

for every positive integer m, with equality if the coefficients of Δ belong to the set $\{\, \frac{r-1}{r} \,|\, r \in \mathbb{N} \,\}$.

Proof. Easy. □

2.3. **The volume.**

Definition 2.3.1. *Let X be a normal n-dimensional irreducible projective variety and let D be a $\mathbb{Q}$-divisor. The* ***volume*** *of D is*

$$\operatorname{vol}(X, D) = \limsup_{m\to\infty} \frac{n! h^0(X, \mathcal{O}_X(mD))}{m^n}.$$

We say that D is ***big*** *if $\operatorname{vol}(X, D) > 0$.*

For more background, see [21]. We will need the following simple result:

Lemma 2.3.2. *Let X be a projective variety, D a divisor such that the rational map $\phi_D\colon X \dashrightarrow \mathbb{P}^n$ is birational onto its image Z. Then, the volume of D is greater than or equal to the degree of Z. In particular the volume of D is at least 1.*

Proof. This is well known, see for example (2.2) of [15]. □

Definition 2.3.3. *Let X be a normal projective variety and let D be a big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X.*

If x and y are two very general points of X then, possibly switching x and y, we may find $0 \leq \Delta \sim_{\mathbb{Q}} (1-\epsilon)D$, for some $0 < \epsilon < 1$, where (X, Δ) is not kawamata log terminal at y, (X, Δ) is log canonical at x and $\{x\}$ is a non kawamata log terminal centre, then we say that D is ***potentially birational****.*

Lemma 2.3.4. *Let X be a normal projective variety and let D be a big $\mathbb{Q}$-Cartier divisor on X.*

1. *If D is potentially birational, then $\phi_{K_X+\lceil D\rceil}$ is birational.*
2. *If X has dimension n and ϕ_D is birational, then $(2n+1)\lfloor D\rfloor$ is potentially birational. In particular, $\phi_{K_X+(2n+1)D}$ is birational and $K_X + (2n+1)D$ is big.*

Proof. Replacing X by a resolution, we may assume that X is smooth. As D is big, we may write $D \sim_{\mathbb{Q}} A+B$ where A is an ample $\mathbb{Q}$-divisor and $B \geq 0$. Using A to tie break (cf. [18, 6.9]), we may assume that (X, Δ) is kawamata log terminal in a punctured neighbourhood of x. As

$$\lceil D\rceil - (\Delta + \epsilon B + \lceil D\rceil - D) \sim_{\mathbb{Q}} \epsilon A,$$

is ample, Nadel vanishing implies that

$$H^1(X, \mathcal{O}_X(K_X + \lceil D\rceil) \otimes \mathcal{J}(\Delta + \epsilon B + \lceil D\rceil - D)) = 0,$$

where $\mathcal{J}(\Delta + \epsilon B + \lceil D\rceil - D)$ is the multiplier ideal sheaf. But then we may find a section $\sigma \in H^0(X, \mathcal{O}_X(K_X + \lceil D\rceil))$ vanishing at y but not at x. In particular, as y is very general, we may also find τ not vanishing at y. But then some linear combination ρ of τ and σ is a section which vanishes at x and not at y. This is (1).

Replacing D by $\lfloor D \rfloor$, we may assume that D is Cartier. Let X' be the image of ϕ_D. Let $x' = \phi_D(x)$ and $y' = \phi_D(y)$ and let Δ' be the sum of n general hyperplanes through x' and n general hyperplanes through y'. Let Δ be the strict transform of Δ'. As x and y are general, ϕ_D is an isomorphism in a neighbourhood of x and y. It follows that (X, Δ) is not kawamata log terminal at y, (X, Δ) is log canonical at x and if we blow up x then the coefficient of the exceptional divisor is one. It is then easy to see that $(2n+1)D$ is potentially birational and (2) follows from (1). □

We will need the following result from [18]:

Theorem 2.3.5. *Let (X, Δ) be a kawamata log terminal pair, where X is projective. Suppose that x and y are two closed points of X. Let $\Delta_0 \geq 0$ be a $\mathbb{Q}$-Cartier divisor on X such that $(X, \Delta + \Delta_0)$ is log canonical in a neighbourhood of x but not kawamata log terminal at y, and there is a non kawamata log terminal centre V which contains x. Let H be an ample $\mathbb{Q}$-divisor on X such that $\operatorname{vol}(V, H|_V) > 2k^k$, where $k = \dim V$.*

Then, possibly switching x and y, there is a $\mathbb{Q}$-divisor $H \sim_{\mathbb{Q}} \Delta_1 \geq 0$ and rational numbers $0 < a_i \leq 1$ such that $(X, \Delta + a_0\Delta_0 + a_1\Delta_1)$ is log canonical in a neighbourhood of x but not kawamata log terminal at y, and there is a non kawamata log terminal centre V' which contains x, such that $\dim V' < k$.

Proof. By (6.9.1) of [18], we may assume that V is the unique non kawamata log terminal centre which contains x, and we may apply (6.8.1), (6.8.1.3) and (6.5) of [18]. □

Theorem 2.3.6. *Let (X, Δ) be a kawamata log terminal pair, where X is projective of dimension n and let H be an ample $\mathbb{Q}$-divisor. Suppose $\gamma_0 \geq 1$ is a constant such that $\operatorname{vol}(X, \gamma_0 H) > n^n$. Suppose ϵ is a constant with the following property:*

For very general x in X and every $0 \leq \Delta_0 \sim_{\mathbb{Q}} \lambda H$ such that $(X, \Delta + \Delta_0)$ is log canonical at x, if V is the minimal non kawamata log terminal centre containing x, then $\operatorname{vol}(V, \lambda H|_V) > \epsilon^k$, where k is the dimension of V and $\lambda \geq 1$ is a rational number.

Then mH is potentially birational, where

$$m = 2\gamma_0 (1+\gamma)^{n-1} \qquad \text{and} \qquad \gamma = \frac{2n}{\epsilon}.$$

Proof. Let x and y be two very general points of X. Possibly switching x and y, we will prove by descending induction on k that there is a $\mathbb{Q}$-divisor $\Delta_0 \geq 0$ such that:
$(\flat)_k$ $\Delta_0 \sim_{\mathbb{Q}} \lambda H$, for some $1 \leq \lambda < 2\gamma_0(1+\gamma)^{n-1-k}$, where $(X, \Delta + \Delta_0)$ is log canonical at x, not kawamata log terminal at y and there is a non kawamata log terminal centre V of dimension at most k containing x.

As

$$\operatorname{vol}(X, 2\gamma_0 H) > 2n^n,$$

we may find $0 \leq \Phi \sim_{\mathbb{Q}} 2\gamma_0 H$, such that $(X, \Delta+\Phi)$ is not log canonical at either x or y. If

$$\beta = \sup\{\, \alpha \,|\, K_X + \Delta + \alpha\Phi \text{ is log canonical at } x \,\},$$

is the log canonical threshold, then $\beta < 1$. Possibly switching x and y, we may assume that $(X, \Delta + \beta\Phi)$ is not kawamata log terminal at y. Clearly $\Delta_0 = \beta\Phi$ satisfies $(\flat)_{n-1}$, so this is the start of the induction.

Now suppose that we may find a $\mathbb{Q}$-divisor Δ_0 satisfying $(\flat)_k$. We may assume that V is the minimal non kawamata log terminal centre containing x and that V has dimension k. By assumption,

$$\operatorname{vol}(V, \lambda\gamma H|_V) > 2k^k,$$

so that by (2.3.5), possibly switching x and y, we may find $\Delta_1 \sim_{\mathbb{Q}} \mu H$, where $\mu < \lambda\gamma$ and constants $0 < a_i \leq 1$ such that $(X, \Delta + a_0\Delta_0 + a_1\Delta_1)$ is log canonical at x, not kawamata log terminal at y and there is a non kawamata log terminal centre V' containing x, whose dimension is less than k. As

$$a_0\Delta_0 + a_1\Delta_1 \sim_{\mathbb{Q}} (a_0\lambda + a_1\mu)H,$$

and

$$\lambda' = a_0\lambda + a_1\mu \leq (1+\gamma)\lambda < 2\gamma_0(1+\gamma)^{n-1-(k-1)},$$

$a_0\Delta_0 + a_1\Delta_1 + \max(0, 1-\lambda')B$ satisfies $(\flat)_{k-1}$, where the support of $B \sim_{\mathbb{Q}} H$ does not contain either x or y (we only need to add on B in the unlikely event that $\lambda' < 1$). This completes the induction and the proof. $\square$

2.4. **Bounded pairs.**

Definition 2.4.1. *We say that a set $\mathfrak{X}$ of varieties is* ***birationally bounded*** *if there is a projective morphism $Z \longrightarrow T$, where T is of finite type, such that for every $X \in \mathfrak{X}$, there is a closed point $t \in T$ and a birational map $f \colon Z_t \dashrightarrow X$.*

We say that a set $\mathfrak{D}$ of log pairs is ***log birationally bounded*** *if there is a log pair (Z, B), where the coefficients of B are all one, and a projective morphism $Z \longrightarrow T$, where T is of finite type, such that for* *every $(X,\Delta)\in\mathfrak{D}$, there is a closed point $t\in T$ and a birational map $f\colon Z_t \dashrightarrow X$ such that the support of B_t contains the support of the strict transform of Δ and any f-exceptional divisor.*

Lemma 2.4.2. *Fix a positive integer n.*

1. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two sets of varieties, such that if $X\in\mathfrak{X}$, then we may find $Y\in\mathfrak{Y}$ birational to X. If $\mathfrak{Y}$ is birationally bounded, then $\mathfrak{X}$ is birationally bounded.*
2. *Let $\mathfrak{X}$ be a set of varieties of dimension n. If there is a constant V such that for every $X\in\mathfrak{X}$, we may find a Weil divisor D such that ϕ_D is birational and the volume of D is at most V, then $\mathfrak{X}$ is birationally bounded.*
3. *Let $\mathfrak{D}$ and $\mathfrak{G}$ be two sets of log pairs, such that if $(X,\Delta)\in\mathfrak{D}$, then we may find $(Y,\Gamma)\in\mathfrak{G}$, and a birational map $f\colon Y\dashrightarrow X$, where the support of Γ contains the support of the strict transform of Δ and any f-exceptional divisor. If $\mathfrak{G}$ is log birationally bounded, then $\mathfrak{D}$ is log birationally bounded.*
4. *Let $\mathfrak{D}$ be a set of log pairs of dimension n. If there are constants V_1 and V_2 such that for every $(X,\Delta)\in\mathfrak{D}$ we may find a Weil divisor D such that $\phi_D\colon X\dashrightarrow Y$ is birational, the volume of D is at most V_1, and if G denotes the sum over the components of the strict transform of Δ and the ϕ^{-1}-exceptional divisors, then $G\cdot H^{n-1}\leq V_2$, where H is the very ample divisor on Y determined by ϕ_D, then $\mathfrak{D}$ is log birationally bounded.*
5. *If the set $\mathfrak{D}$ of log pairs is log birationally bounded, then*
$$\mathfrak{X}=\{\,X\,|\,(X,\Delta)\in\mathfrak{D}\,\}$$
is birationally bounded.

Proof. (1), (3) and (5) are clear.

We prove (4). Suppose that $Y\subset\mathbb{P}^s$ is a closed subvariety of dimension n and degree at most V_1. Then by the classification of minimal degree subvarieties of projective space, we may assume $s\leq V_1+1-n$. By boundedness of the Chow variety, there are flat morphisms $Z\longrightarrow T$ and $B\longrightarrow T$ such that if $Y\subset\mathbb{P}^s$ has dimension n (respectively $n-1$) and degree at most V_1 (respectively V_2), then Y is isomorphic to the fibre Z_t (respectively B_t) over a closed point $t\in T$. Passing to a stratification of T and a log resolution of the generic fibres of $Z\longrightarrow T$, we may assume that the fibres of $Z\longrightarrow T$ are smooth. In particular, (Z,B) is a log pair.

Now suppose that $(X,\Delta)\in\mathfrak{D}$. By assumption there is a divisor D such that $\phi_D\colon X\dashrightarrow Y$ is birational. The degree of the image is at most the volume of D, that is, at most V_1. So there is a closed point

$t \in T$ such that Y is isomorphic to Z_t. By assumption $G \cdot H^{n-1} \leq V_2$ so that we may assume that G corresponds to B_t. But then $\mathfrak{D}$ is log birationally bounded. This is (4).

The proof of (2) is similar to and easier than the proof of (4). □

3. Birationally bounded pairs

§3 is devoted to a proof of:

Theorem 3.1. *Fix a positive integer n and two constants A and $\delta > 0$. Then the set of log pairs (X, Δ) satisfying*

1. *X is projective of dimension n,*
2. *(X, Δ) is log canonical,*
3. *the coefficients of Δ are at least δ,*
4. *there is a positive integer m such that $\operatorname{vol}(X, m(K_X + \Delta)) \leq A$, and*
5. *$\phi_{K_X + m(K_X+\Delta)}$ is birational,*

is log birationally bounded.

The key result is:

Lemma 3.2. *Let X be a normal projective variety of dimension n and let M be a base point free Cartier divisor such that ϕ_M is birational. Let $H = 2(2n+1)M$.*

If D is a sum of distinct prime divisors, then

$$D \cdot H^{n-1} \leq 2^n \operatorname{vol}(X, K_X + D + H).$$

Proof. Possibly discarding ϕ_M-exceptional components of D, we may assume that no component of D is ϕ_M-exceptional. If $f\colon Y \longrightarrow X$ is a log resolution of the pair (X, D) and G is the strict transform of D, then

$$D \cdot H^{n-1} = G \cdot (f^*H)^{n-1},$$

and

$$\operatorname{vol}(Y, K_Y + G + f^*H) \leq \operatorname{vol}(X, K_X + D + H).$$

Replacing (X, D) by (Y, G) and M by f^*M we may assume that (X, D) has simple normal crossings, and possibly blowing up more, that the components of D do not intersect.

Since no component of D is contracted, we may find an ample $\mathbb{Q}$-divisor A and a $\mathbb{Q}$-divisor $B \geq 0$, such that

$$M \sim_{\mathbb{Q}} A + B,$$

where B and D have no common components. As $K_X + D + \delta B$ is divisorially log terminal for any $\delta > 0$ sufficiently small, it follows that

$$H^i(X, \mathcal{O}_X(K_X + E + pM)) = 0,$$

for all positive integers p, $i > 0$ and any integral Weil divisor $0 \leq E \leq D$. If we let

$$A_m = K_X + D + mH,$$

then

$$H^i(D, \mathcal{O}_D(A_m)) = 0,$$

for all $i > 0$ and $m > 0$ and so there is a polynomial $P(m)$ of degree $n-1$, with

$$P(m) = h^0(D, \mathcal{O}_D(A_m)),$$

for $m > 0$. (2) of (2.3.4) implies that $A_1 = K_X + D + H$ is big and so [7] implies that $K_X + D + H$ has a log canonical model. In particular there is a polynomial of $Q(m)$ of degree n, with

$$Q(m) = h^0(X, \mathcal{O}_X(2mA_1)),$$

for any sufficiently divisible positive integer m. Note that the leading coefficients of $P(m)$ and $Q(m)$ are

$$\frac{D \cdot H^{n-1}}{(n-1)!} \qquad \text{and} \qquad \frac{2^n \operatorname{vol}(X, K_X + D + H)}{n!}.$$

If D_i is a component of D, and $M_i = (D - D_i + (2n+1)M)|_{D_i}$, then

$$H^0(X, \mathcal{O}_X(K_X + D + (2n+1)M)) \longrightarrow H^0(D_i, \mathcal{O}_{D_i}(K_{D_i} + M_i)),$$

is surjective, and so the general section of $H^0(X, \mathcal{O}_X(K_X + D + (2n+1)M))$ does not vanish identically on any component of D. Pick sections

$$s \in H^0(X, \mathcal{O}_X(K_X+D+(2n+1)M)) \quad \text{and} \quad l \in H^0(X, \mathcal{O}_X((2n+1)M)),$$

whose restrictions to each component of D is non-zero. Let

$$t = s^{\otimes 2m-1} \otimes l \in H^0(X, \mathcal{O}_X(2mA_1 - A_m)).$$

Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X(A_m - D) \arrow[r] \arrow[d] & \mathcal{O}_X(A_m) \arrow[r] \arrow[d] & \mathcal{O}_D(A_m) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_X(2mA_1 - D) \arrow[r] & \mathcal{O}_X(2mA_1) \arrow[r] & \mathcal{O}_D(2mA_1) \arrow[r] & 0,
\end{tikzcd}$$

where the vertical morphisms are injections induced by multiplying by t.

Note that

$$H^0(X, \mathcal{O}_X(A_m)) \longrightarrow H^0(D, \mathcal{O}_D(A_m)),$$

is surjective. Hence every element of $H^0(D, \mathcal{O}_D(2mA_1))$ in the image of $H^0(D, \mathcal{O}_D(A_m))$ lifts to $H^0(X, \mathcal{O}_X(2mA_1))$. Therefore

$$P(m) \leq h^0(X, \mathcal{O}_X(2mA_1)) - h^0(X, \mathcal{O}_X(2mA_1 - D)).$$

Note that

$$Q(m-1) = h^0(X, \mathcal{O}_X(2(m-1)A_1)) \leq h^0(X, \mathcal{O}_X(2mA_1 - D)),$$

as $h^0(X, \mathcal{O}_X(2K_X + D + 2H)) \neq 0$. It follows that

$$P(m) \leq Q(m) - Q(m-1).$$

Now compare the leading coefficients of $P(m)$ and $Q(m)-Q(m-1)$. $\square$

Proof of (3.1). Let (X, Δ) be a log pair satisfying the hypotheses of (3.1). If $\pi \colon Y \longrightarrow X$ is a log resolution of (X, Δ) which resolves the indeterminacy of

$$\phi = \phi_{K_X + m(K_X+\Delta)} \colon X \dashrightarrow Z,$$

and Γ is the strict transform of Δ plus the sum of the exceptional divisors, then (X, Δ) is log birationally bounded if and only if (Y, Γ) is log birationally bounded, by (3) of (2.4.2). On the other hand,

$$\operatorname{vol}(Y, m(K_Y + \Gamma)) \leq \operatorname{vol}(X, m(K_X + \Delta)) \leq A,$$

and $\phi_{K_Y+(2n+1)(m+1)(K_Y+\Gamma)}$ is birational, as $(m+1)(K_Y + \Gamma)$ is big.

Replacing (X, Δ) by (Y, Γ) and m by $(2n+1)(m+1)$, we may assume that

$$\phi = \phi_{K_X + m(K_X+\Delta)} \colon X \longrightarrow Z,$$

is a birational morphism. In particular, if we decompose $\lfloor K_X + m(K_X + \Delta) \rfloor$ into its mobile part M and its fixed part E, so that

$$|K_X + m(K_X + \Delta)| = |M| + E,$$

then M is big and base point free. Let H be a divisor on Z such that $M = \phi^* H$, so that H is very ample.

Note that

$$\begin{aligned} \operatorname{vol}(X, K_X + m(K_X + \Delta)) &\leq \operatorname{vol}(X, (m+1)(K_X + \Delta)) \\ &\leq 2^n A. \end{aligned}$$

On the other hand, let G be the sum of the components of the strict transform of Δ on Z. Pick $B \in |\lfloor K_X + m(K_X + \Delta) \rfloor|$. Let

$$\alpha = \max(\frac{1}{\delta}, 2(2n+1)).$$

If D_0 is the sum of the components of Δ and B which are not contracted by ϕ, then

$$D_0 \leq \alpha(B + \Delta).$$

Note that there is a divisor $C \geq 0$ such that

$$\alpha(B+\Delta)+C \sim_{\mathbb{Q}} \alpha(m+1)(K_X+\Delta).$$

As ϕ is a morphism and M is base point free, (3.2) implies that

$$\begin{aligned}
G \cdot H^{n-1} &\leq D_0 \cdot (2(2n+1)M)^{n-1} \\
&\leq 2^n \operatorname{vol}(X, K_X + D_0 + 2(2n+1)M) \\
&\leq 2^n \operatorname{vol}(X, K_X + \alpha(B+\Delta) + 2(2n+1)(M+E+\Delta)) \\
&\leq 2^n \operatorname{vol}(X, K_X + \Delta + \alpha(m+1)(K_X+\Delta) + 2(2n+1)(m+1)(K_X+\Delta)) \\
&\leq 2^n(1+2\alpha(m+1))^n \operatorname{vol}(X, K_X+\Delta) \\
&\leq 2^{3n}\alpha^n \operatorname{vol}(X, (m+1)(K_X+\Delta)) \\
&\leq 2^{4n}\alpha^n A.
\end{aligned}$$

Now apply (4) of (2.4.2). □

4. Deformation invariance of log plurigenera

Proposition 4.1. *Let (X, Δ) be a $\mathbb{Q}$-factorial log pair. Suppose that $X \longrightarrow T$ is a projective morphism to a smooth curve T whose fibres (X_t, Δ_t) are terminal, where every component of Δ dominates T. Let $0 \in T$ be a closed point. Suppose that*

- *either Δ or $K_X + \Delta$ is big over T, and*
- *no component of Δ_0 belongs to the stable base locus of $K_{X_0}+\Delta_0$.*

Then we may find a log terminal model $f \colon X \dashrightarrow Y$ of (X, Δ) over T, such that f is an isomorphism at the generic point of every component of Δ_0 and $f_0 \colon X_0 \dashrightarrow Y_0$ is a weak log canonical model of (X_0, Δ_0).

Proof. We first prove this result under the additional hypothesis that $K_X + \Delta$ is pseudo-effective over T.

Note that X is smooth in codimension two, as the fibres X_t of $X \longrightarrow T$ are Cartier and smooth in codimension two. Hence

$$(K_X+\Delta)|_{X_0} = (K_X + X_0 + \Delta)|_{X_0} = K_{X_0} + \Delta_0.$$

It follows from [7, 1.4.5] that the coefficient of any valuation μ with respect to $(X, X_0 + \Delta)$ is at most zero, if the centre of μ is neither a component of Δ nor a component of Δ_0.

By [7, 1.2], we may run the $(K_X + \Delta)$-MMP over T, that is, we may find a sequence $g^1, g^2, \dots, g^{m-1}$ of divisorial contractions and flips $g^k \colon X^k \dashrightarrow X^{k+1}$ starting at $X = X^1$ and ending with a log terminal model $Y = X^m$ for the pair (X, Δ) over T. Let Δ^k denote the pushforward of Δ under the induced birational map $f^k \colon X \dashrightarrow X^k$. We will prove by induction on k that

(a) g^k is an isomorphism at the generic point of any component of Δ^k_0, and

(b) $g^k_0\colon X^k_0 \dashrightarrow X^{k+1}_0$ is a birational contraction.

Suppose that (a–b)$_{\leq k-1}$ hold. Then $f^k_0\colon X_0 \dashrightarrow X^k_0$ is a birational contraction which does not contract any components of Δ_0 and so (X^k_0, Δ^k_0) is terminal.

(b)$_{\leq k-1}$ implies that no component of Δ^k_0 is a component of the stable base locus of $K_{X^k_0} + \Delta^k_0$. Suppose that g^k is not an isomorphism at the generic point of a divisor D contained in X^k_0. Then D is covered by curves C such that

$$(K_{X^k_0} + \Delta^k_0) \cdot C = (K_{X^k} + \Delta^k) \cdot C < 0.$$

It follows that D is a component of the stable base locus of $K_{X^k_0} + \Delta^k_0$, so that D is not a component of Δ^k_0. Thus (a)$_k$ holds.

Suppose that $G \subset X^{k+1}_0$ is a prime divisor, which is not a component of Δ^{k+1}_0. By the classification of log canonical surface singularities, we may find a valuation ν with centre G on X^{k+1} whose coefficient d with respect to $(X^{k+1}, X^{k+1}_0 + \Delta^{k+1})$ is at least zero. As X^k_0 is the pullback of a divisor from T, g^k is $(K_{X^k} + X^k_0 + \Delta^k)$-negative and so the coefficient c of ν with respect to $(X^k, X^k_0 + \Delta^k)$ is at least d, with equality if and only if g^k is an isomorphism at the generic point of G. By (a)$_k$ the centre of ν on X^k is not a component of Δ^k_0. It follows that $0 \leq d \leq c \leq 0$, so that $c = d = 0$ and g^k is an isomorphism at the generic point of G. Hence g^k_0 is a birational contraction, that is, (b)$_k$ holds. This completes the induction and the proof that (a–b)$_{\leq m-1}$ hold.

As g^k_0 is a birational contraction which is $(K_{X^k_0} + \Delta^k_0)$-negative, for $k \leq m-1$, it follows that f_0 is a $(K_{X_0} + \Delta_0)$-negative birational contraction. But then f_0 is a weak log canonical model.

It remains to prove that $K_X + \Delta$ is pseudo-effective over T. Pick a divisor A which is ample over T and let

$$\lambda = \inf\{\, t \in \mathbb{R} \mid K_X + \Delta + tA \text{ is } \pi\text{-pseudo-effective} \,\},$$

be the π-pseudo-effective threshold. It is proved in [7] that λ is rational. By what we have already proved we may find a log terminal model $f\colon X \dashrightarrow Y$ of $K_X + \Delta + \lambda A$ over T such that $f_0\colon X_0 \dashrightarrow Y_0$ is a weak log canonical model of $K_{X_0} + \Delta_0 + \lambda A_0$. Let $G = f_*(K_X + \Delta + \lambda A)$. If $\lambda > 0$ then $K_{X_0} + \Delta_0 + \lambda A_0$ is big, so that G_0 is big and nef. But then $G^n_0 > 0$ is positive and so $G^n_t = G^n_0 > 0$ for every $t \in T$. As G is nef over T, it is big over T, and so $K_X + \Delta + \lambda A$ is big over T, a contradiction. It follows that $\lambda = 0$ so that $K_X + \Delta$ is pseudo-effective over T. $\square$

Theorem 4.2. *Let $X \longrightarrow T$ be a flat projective morphism of quasi-projective varieties. Let (X, Δ) be a log pair such that the fibres (X_t, Δ_t) are $\mathbb{Q}$-factorial terminal, for every $t \in T$. Assume that every component R of Δ dominates T and that the fibres of the Stein factorisation of $R \longrightarrow T$ are irreducible. Let $m > 1$ be any integer such that $D = m(K_X + \Delta)$ is integral.*

If either $K_X + \Delta$ or Δ is big over T then $h^0(X_t, \mathcal{O}_{X_t}(D_t))$ is independent of $t \in T$.

Proof. Let R be a component of Δ, let $S \longrightarrow T$ be the normalisation of the Stein factorisation of $R \longrightarrow T$ so that $S \longrightarrow T$ is finite and S is normal, and let

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \arrow[d] \\ S \arrow[r] & T, \end{tikzcd}$$

be the fibre square. As $S \longrightarrow T$ is finite, S is irreducible and $Y \longrightarrow S$ is flat, Y is a quasi-projective variety. Y is normal by [14, 5.12.7]. Replacing $X \longrightarrow T$ by $Y \longrightarrow S$ finitely many times, we may assume that the fibres of $R \longrightarrow T$ are irreducible, for every component R of Δ. Fix a closed point $0 \in T$. Replacing T by the intersection of general hyperplane sections containing 0, we may assume that T is a curve. Replacing T by its normalisation and passing to the fibre square, we may assume that T is smooth. As the fibres of $X \longrightarrow T$ are $\mathbb{Q}$-factorial terminal, [11, 3.2] implies that X is $\mathbb{Q}$-factorial.

It suffices to show that $|D_0| = |D|_{X_0}$. In particular we may suppose that $K_{X_0} + \Delta_0$ is pseudo-effective. Further we are free to work locally about 0. In particular we may assume that T is affine. [7, 1.2] implies that the divisor $N_\sigma(X_0, K_{X_0} + \Delta_0)$, defined in (2.1), is a $\mathbb{Q}$-divisor. In particular

$$\Theta_0 = \Delta_0 - \Delta_0 \wedge N_\sigma(X_0, K_{X_0} + \Delta_0),$$

is a $\mathbb{Q}$-divisor. By assumption we may find a $\mathbb{Q}$-divisor $0 \leq \Theta \leq \Delta$ whose restriction to X_0 is Θ_0. If we set

$$\mu = \frac{m}{m-1},$$

then $K_X + \mu\Delta$ is big. Therefore, we may find $\mathbb{Q}$-divisors $A \geq 0$ and $B \geq 0$, where A is ample, the support of A is a prime divisor and X_0 is not a component of B, such that $K_X + \mu\Delta \sim_{\mathbb{Q}} A + B$. Possibly passing to an open subset of T we may assume that the components of B dominate T. Pick $\delta \in (0, 1/2)$ such that $(X_t, \Delta_t + \delta(A_t + B_t))$ is terminal for every $t \in T$. If we let

$$H = \frac{\delta}{m-1-\delta} A,$$

then

$$D - \Xi \sim_{\mathbb{Q}} K_X + \Phi + \delta B + (m-1-\delta)(K_X + \Theta + H),$$

where $\Phi = (1 - \delta\mu + \delta)\Delta$ and $\Xi = (m-1-\delta)(\Delta - \Theta)$.

As H_0 is ample, no component of $\Theta_0 + H_0$ belongs to the stable base locus of $K_{X_0} + \Theta_0 + H_0$. (4.1) implies that we may find a log terminal model $f \colon X \dashrightarrow Y$ for $(X, \Theta + H)$, such that the induced birational map $f_0 \colon X_0 \dashrightarrow Y_0$ is a weak log canonical model of $(X_0, \Theta_0 + H_0)$.

Let $p \colon W \longrightarrow X$ and $q \colon W \longrightarrow Y$ resolve $f \colon X \dashrightarrow Y$, where p is also a log resolution of $(X, \Delta + A + B)$. If we let

$$G = (m-1-\delta) f_*(K_X + \Theta + H),$$

then G is big and nef and

$$(m-1-\delta)p^*(K_X + \Theta + H) = q^*G + F,$$

where $F \geq 0$ is exceptional for q.

Let W_0 be the strict transform of X_0. As $(X_0, \Phi_0 + \delta B_0)$ is kawamata log terminal, inversion of adjunction, [7, 1.4.5], implies that $(X, X_0 + \Phi + \delta B)$ is purely log terminal. Therefore, if we write

$$K_W + W_0 = p^*(K_X + X_0 + \Phi + \delta B) + E,$$

then $\lceil E \rceil \geq 0$ is exceptional for p. Let

$$L = \lceil p^*(D - \Xi) + E - F \rceil.$$

Possibly passing to an open subset of T, we may assume X_0 is $\mathbb{Q}$-linearly equivalent to zero. In particular,

$$L - W_0 \sim_{\mathbb{Q}} K_W + C + q^*G,$$

where C is the fractional part of $-p^*(D - \Xi) - E + F$. Hence (W, C) is kawamata log terminal and Kawamata-Viehweg vanishing implies that

$$H^1(W, \mathcal{O}_W(L - W_0)) = 0.$$

Let $N = p^*(K_X + \Theta) - q^* f_*(K_X + \Theta)$. As H is ample, $p^*H \leq q^* f_* H$, and so

$$mN = (1+\delta)N + (m-1-\delta)N \geq F.$$

As $\Xi \leq m(\Delta - \Theta)$ we have $D - \Xi \geq m(K_X + \Theta)$ and so it follows that

$$\begin{aligned} M &= L - \lfloor mq^* f_*(K_X + \Theta) \rfloor \\ &= \lceil L - mq^* f_*(K_X + \Theta) \rceil \\ &\geq \lceil mN + E - F \rceil \\ &\geq \lceil E \rceil. \end{aligned}$$

Let $q_0 \colon W_0 \longrightarrow X_0$ denote the restriction of q to W_0 and let L_0 and M_0 denote the restriction of L and M to W_0. We have

$$\begin{aligned} |D_0| &= |m(K_{X_0} + \Theta_0)| && \text{by definition of } \Theta_0 \\ &\subset |mf_{0*}(K_{X_0} + \Theta_0)| && \text{since } f_0 \text{ is a birational contraction} \\ &= |mq_0^* f_{0*}(K_{X_0} + \Theta_0)| && \\ &\subset |L_0| && \text{as } M_0 \geq 0 \\ &= |L|_{W_0} && \text{since } H^1(W, \mathcal{O}_W(L - W_0)) = 0 \\ &\subset |D|_{X_0} && \text{since } \lceil E \rceil \text{ is exceptional for } p. \end{aligned}$$

Thus equality holds, as the reverse inequality holds automatically. $\square$

Proof of (1.8). We first prove (1). Let $0 \in T$ be a closed point. Replacing T by an unramified cover, we may assume that the strata of (X, Δ) intersect X_0 in strata of (X_0, Δ_0). Since the only valuations of non-negative coefficient lie over the strata of (X, Δ), replacing (X, Δ) by a blow up, we may assume that (X, Δ) is terminal. Thus (1) follows from (4.2).

Now we prove (2). Pick $m_0 > 0$ such that $m_0\Delta$ is integral and an ample divisor H such that $H + m_0\Delta$ is very ample. Pick a prime divisor $A \sim H + m_0\Delta$, such that $(X, \Delta + A)$ has simple normal crossings over T. If $m \geq m_0$ is any positive integer such that $m\Delta$ is integral, then

$$(X, \Delta' = \frac{m - m_0}{m}\Delta + \frac{1}{m}A)$$

is a simple normal crossings pair and it is kawamata log terminal. Further

$$K_X + \Delta' \sim_{\mathbb{Q}} K_X + \Delta + H/m,$$

and Δ' is big over T. Thus (2) follows from (1).

Note that

$$\operatorname{vol}(X_t, K_{X_t} + \Delta_t) = \lim_{\epsilon \to 0} \operatorname{vol}(X_t, K_{X_t} + (1 - \epsilon)\Delta_t).$$

As $(X_t, (1 - \epsilon)\Delta_t)$ is kawamata log terminal, (3) follows from (1) and (2). $\square$

5. DCC for the volume of bounded pairs

We prove (1.9) in this section. We first deal with the case that T is a closed point.

Proposition 5.1. *Fix a set $I \subset [0,1]$ which satisfies the DCC and a simple normal crossings pair (Z, B), where Z is projective of dimension n. Let $\mathfrak{D}$ be the set of simple normal crossings pairs (X, Δ), where X is projective, the coefficients of Δ belong to I and there is a birational morphism $f\colon X \longrightarrow Z$ with $\Phi = f_*\Delta \leq B$.*

Then the set

$$\{\, \mathrm{vol}(X, K_X + \Delta) \,|\, (X, \Delta) \in \mathfrak{D} \,\},$$

also satisfies the DCC.

Definition 5.2. *Let (X, Δ) be a log pair. If $\pi\colon Y \longrightarrow X$ is a birational morphism, then we may write*

$$K_Y + \Gamma = \pi^*(K_X + \Delta) + E,$$

where $\Gamma \geq 0$ and $E \geq 0$ have no common components, $\pi_*\Gamma = \Delta$ and $\pi_*E = 0$.

*Define a **b**-divisor $\mathbf{L}_\Delta$ by setting $\mathbf{L}_{\Delta,Y} = \Gamma$.*

Lemma 5.3. *Let (X, Δ) be a simple normal crossings pair, where X is a projective variety.*

1. *If $Y \longrightarrow X$ is a birational morphism such that $(Y, \Theta = \mathbf{L}_{\Delta,Y})$ has simple normal crossings, and $\Gamma - \Theta \geq 0$ is exceptional, then*
$$\mathrm{vol}(X, K_X + \Delta) = \mathrm{vol}(Y, K_Y + \Gamma).$$
2. *If $f\colon X \longrightarrow Z$ is a birational morphism such that $(Z, \Phi = \mathbf{L}_{\Delta,Z})$ has simple normal crossings and $\Theta = \Delta \wedge \mathbf{L}_{\Phi,X}$, then*
$$\mathrm{vol}(X, K_X + \Delta) = \mathrm{vol}(X, K_X + \Theta).$$

Proof. (1) is clear, as

$$H^0(X, \mathcal{O}_X(m(K_X + \Delta))) \simeq H^0(Y, \mathcal{O}_Y(m(K_Y + \Gamma))),$$

for all m.

For (2), suppose m is a sufficiently divisible positive integer. We have

$$\begin{aligned} H^0(X, \mathcal{O}_X(m(K_X + \Delta))) &\subset H^0(Z, \mathcal{O}_Z(m(K_Z + \Phi))) \\ &= H^0(X, \mathcal{O}_X(m(K_X + \mathbf{L}_{\Phi,X}))), \end{aligned}$$

and so

$$H^0(X, \mathcal{O}_X(m(K_X + \Delta))) = H^0(X, \mathcal{O}_X(m(K_X + \Theta))). \qquad \square$$

Lemma 5.4. *Let (Z,Φ) be a simple normal crossings pair which is log canonical.*

If ν is a valuation such that $\mathbf{L}_\Phi(\nu) > 0$, then the centre of ν is a stratum W of (Z,Φ) and there is a birational morphism $Y = Y_\nu \longrightarrow Z$ such that $\rho(Y/Z) = 1$, Y is $\mathbb{Q}$-factorial and the centre of ν is a divisor on Y; Y_ν is unique with these properties.

Proof. This is a consequence of the existence of log terminal models, which is proved in [7], and uniqueness of log canonical models. $\square$

Definition 5.5. *Let (X,Δ) be a log pair. Define a* ***b****-divisor $\mathbf{M}_\Delta$ by assigning to any valuation ν,*

$$\mathbf{M}_\Delta(\nu) = \begin{cases} \operatorname{mult}_B(\Delta) & \text{if the centre of } \nu \text{ is a divisor } B \text{ on } X, \\ 1 & \text{otherwise.} \end{cases}$$

Definition 5.6. *Let $\mathbf{B}$ be a* **b***-divisor whose coefficients belong to $[0,1]$ and let $(Z,\Phi = \mathbf{B}_Z)$ be a model with simple normal crossings. Let $Z' \longrightarrow Z$ be a log resolution, and let Σ be a set of valuations σ whose centres are exceptional divisors for $Z' \longrightarrow Z$, such that $\mathbf{L}_\Phi(\sigma) > 0$.*

For every valuation $\sigma \in \Sigma$, let $\Gamma_\sigma = (\mathbf{L}_\Phi \wedge \mathbf{B})_{Y_\sigma}$, where $Y_\sigma \longrightarrow Z$ is defined in (5.4). Let

$$\Theta = \bigwedge_{\sigma\in\Sigma} \mathbf{L}_{\Gamma_\sigma,Z'},$$

the minimum of the divisors $\mathbf{L}_{\Gamma_\sigma,Z'}$.

The ***cut*** *of $(Z,\mathbf{B})$, associated to $Z' \longrightarrow Z$ and Σ, is the pair $(Z',\mathbf{B}')$, where*

$$\mathbf{B}' = \mathbf{B} \wedge \mathbf{M}_\Theta,$$

so that the trace of $\mathbf{B}'$ on Z' is $\Theta \wedge \mathbf{B}_{Z'}$ and otherwise $\mathbf{B}'$ is the same **b***-divisor as $\mathbf{B}$.*

We say that the pair $(Z',\mathbf{B}')$ is a ***reduction*** *of the pair $(Z,\mathbf{B})$, if they are connected by a sequence of cuts, that is, there are pairs, $(Z_i,\mathbf{B}_i)$, $0 \leq i \leq k$, starting at $(Z_0,\mathbf{B}_0) = (Z,\mathbf{B})$ and ending at $(Z_k,\mathbf{B}_k) = (Z',\mathbf{B}')$, such that $(Z_{i+1},\mathbf{B}_{i+1})$ is a cut of $(Z_i,\mathbf{B}_i)$, for each $0 \leq i < k$.*

Lemma 5.7. *Let $\mathbf{B}$ be a* **b***-divisor whose coefficients belong to a set $I \subset [0,1]$ which satisfies the DCC, and let $(Z,\Phi = \mathbf{B}_Z)$ be a model with simple normal crossings.*

Then there is a reduction $(Z',\mathbf{B}')$ of $(Z,\mathbf{B})$ such that

$$\mathbf{L}_{\Phi'} \leq \mathbf{B}',$$

where $\Phi' = \mathbf{B}'_{Z'}$.

Proof. If W is a stratum of (Z, Φ), then define the **weight** w of W as follows:
If there is a valuation ν, with centre W, such that $\mathbf{B}(\nu) < \mathbf{L}_\Phi(\nu)$, then let w be the number of components of Φ with coefficient 1 which contain W. Otherwise, if there is no such ν, then let $w = -1$.

Define the **weight** of $(Z, \mathbf{B})$ to be the maximum weight of the strata of (Z, Φ).

Suppose the weight of $(Z, \mathbf{B})$ is -1. Then $\mathbf{L}_\Phi(\nu) \leq \mathbf{B}(\nu)$ for any valuation ν whose centre is a stratum. If ρ is a valuation, whose centre is not a stratum, we have $0 = \mathbf{L}_\Phi(\rho) \leq \mathbf{B}(\rho)$ (cf. [20, 2.31]). In this case we just take $Z' = Z$.

From now on we suppose that the weight $w \geq 0$. Suppose that $(Z', \mathbf{B}')$ is a cut of $(Z, \mathbf{B})$. Then $\mathbf{B}'$ and $\mathbf{B}$ have the same coefficients, except for finitely many valuations. In particular, the coefficients of $\mathbf{B}'$ belong to a set $I' \supset I$ which still satisfies the DCC. It suffices therefore to prove that we can find a cut $(Z', \mathbf{B}')$ of $(Z, \mathbf{B})$ with smaller weight.

Now if $(Z', \mathbf{B}')$ is a cut of $(Z, \mathbf{B})$, then $\mathbf{B}'_{Z'} \leq \mathbf{L}_{\Phi, Z'}$. On the other hand, if ν is any valuation whose centre is not a divisor on Z', then $\mathbf{B}(\nu) = \mathbf{B}'(\nu)$. It follows that the weight of $(Z', \mathbf{B}')$ is at most the weight of $(Z, \mathbf{B})$. Therefore, as (Z, Φ) has only finitely many strata, we may construct $(Z', \mathbf{B}')$ étale locally about every stratum. Thus, we may assume that $Z = \mathbb{C}^n$ and that Φ is supported on the coordinate hyperplanes.

We will use the language of toric geometry, cf. [12]. $\mathbb{C}^n$ is the toric variety associated to the cone spanned by the standard basis vectors $e_1, e_2, \dots, e_n$ in $\mathbb{R}^n$. If ν is any valuation such that $\mathbf{L}_\Phi(\nu) > 0$, then ν is toric and we will identify ν with an element $(v_1, v_2, \dots, v_n)$ of $\mathbb{N}^n$. Order the components of Φ so that the last w components have coefficient one and let $0 \leq c_1, c_2, \dots, c_s < 1$ be the initial coefficients, so that $n = s + w$. With this ordering, we have

$$\mathbf{L}_\Phi(\nu) = 1 - \sum_i v_i(1 - c_i).$$

(Indeed both sides of this equation are affine linear in $v_1, v_2, \dots, v_n$ and $c_1, c_2, \dots, c_s$ and it is easy to check we have equality when either ν is the zero vector or when $\nu = e_i$, $1 \leq i \leq n$.) Consider the finite set

$$\mathfrak{F} = \{\, (v_1, v_2, \dots, v_s) \in \mathbb{N}^s \mid \sum_i v_i(1 - c_i) < 1 \,\}.$$

Given a valuation $\nu = (v_1, v_2, \dots, v_n)$, note that $\mathbf{L}_\Phi(\nu) > 0$ if and only if $(v_1, v_2, \dots, v_s) \in \mathfrak{F}$.

As I satisfies the DCC, for every $f = (f_1, f_2, \dots, f_s) \in \mathfrak{F}$, we may pick a valuation $\sigma = (f_1, f_2, \dots, f_s, v_{s+1}, v_{s+2}, \dots, v_n)$, such that

$$\mathbf{B}(\sigma) = \inf\{\, \mathbf{B}(\nu) \,|\, \nu = (f_1, f_2, \dots, f_s, u_{s+1}, u_{s+2}, \dots, u_n) \,\}.$$

Let Σ be a set of choices of such valuations σ, so that Σ and $\mathfrak{F}$ have the same cardinality. Let $Z' \longrightarrow Z$ be any log resolution of (Z, Φ) such that the centre of every element of Σ is a divisor on Z'. We may assume that the induced birational map $Z' \longrightarrow Y_\sigma$ is a morphism, for every $\sigma \in \Sigma$. Let $(Z', \mathbf{B}')$ be the cut of $(Z, \mathbf{B})$ associated to $Z' \to Z$ and Σ.

There are two cases. If $w = 0$, then $\Sigma = \mathfrak{F}$ is the set of all valuations of coefficient less than one. It follows that if ν is any valuation whose centre on Z' is not a divisor, then $\mathbf{L}_{\Phi'}(\nu) = 0$, so that the weight of $(Z', \mathbf{B}')$ is -1, which is less than the weight of $(Z, \mathbf{B})$.

Otherwise we may assume that $w \geq 1$. Suppose that ν is a valuation whose centre is not a divisor on Z' such that $\mathbf{B}'(\nu) < 1$ and $\mathbf{L}_{\Phi'}(\nu) > 0$. Then $\mathbf{B}(\nu) = \mathbf{B}'(\nu)$ and $\mathbf{L}_{\Phi}(\nu) > 0$ and so $\nu = (v_1, v_2, \dots, v_n)$ is toric and $(v_1, v_2, \dots, v_s) \in \mathfrak{F}$. By construction, there is an element σ of Σ with the same first s coordinates as ν such that $\mathbf{B}(\sigma) \leq \mathbf{B}(\nu) < 1$. The cone spanned by the standard basis vectors $e_1, e_2, \dots, e_n$ is divided into $m \leq n$ subcones by σ (these are the maximal cones of Y_σ), where m is the number of non-zero entries of σ, and so ν is a non-negative linear combination of σ and $n-1$ vectors taken from $e_1, e_2, \dots, e_n$. It follows that

$$(\sharp) \qquad \nu = \sum_{j \neq l} \lambda_j e_j + \lambda \sigma,$$

for some index $1 \leq l \leq n$ and non-negative real numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ and λ, where the lth entry of σ is non-zero.

If the centre of ν on Z' is contained in w components of Φ' of coefficient one, then the centre of ν on Y_σ is also contained in w components of coefficient one of $\Gamma_\sigma = (\mathbf{L}_\Phi \wedge \mathbf{B})_{Y_\sigma}$. By assumption, the exceptional divisor of $Y_\sigma \longrightarrow Y$ has coefficient strictly less than one, and so the centre of ν on Y_σ must be contained in the strict transform of the last w coordinate hyperplanes. But then, by standard toric geometry, $l \leq s$. Hence the lth entry of ν is non-zero. Comparing the coefficient of e_l in

($\sharp$), we must have $\lambda = 1$, so that $\nu \geq \sigma$. In this case,

$$\begin{aligned}
\mathbf{L}_{\Phi'}(\nu) &\leq \mathbf{L}_{\Gamma_\sigma}(\nu) && \text{by definition of } \mathbf{B}', \\
&\leq \mathbf{L}_{\Gamma_\sigma}(\sigma) && \text{as } \nu \geq \sigma, \\
&\leq \mathbf{B}(\sigma) && \text{since } \Gamma_\sigma \leq \mathbf{B}_{Y_\sigma}, \\
&\leq \mathbf{B}(\nu) && \text{by our choice of } \sigma, \\
&= \mathbf{B}'(\nu) && \text{by definition of } \mathbf{B}'.
\end{aligned}$$

It follows that the weight of $(Z', \mathbf{B}')$ is indeed smaller than the weight of $(Z, \mathbf{B})$ and this completes the induction and the proof. $\square$

Proof of (5.1). Suppose we have a sequence of log pairs $(X_i, \Delta_i) \in \mathfrak{D}$, such that $v_i \geq v_{i+1}$, where $v_i := \operatorname{vol}(X_i, K_{X_i} + \Delta_i)$. We will show that the sequence $v_1, v_2, \dots$ is eventually constant; to this end we are free to pass to a subsequence. Replacing I by $\bar{I} \cup \{1\}$, we may assume that I is closed and $1 \in I$.

By assumption there are projective birational morphisms $f_i \colon X_i \longrightarrow Z$ such that $\Phi_i = f_{i*}\Delta_i \leq B$. Note that if ν is a valuation such that $\mathbf{M}_{\Delta_i}(\nu) \notin \{0, 1\}$, then the centre of ν is a component of Δ_i. On the other hand, if $\mathbf{M}_{\Delta_i}(\nu) = 1$ and $\mathbf{M}_{\Delta_j}(\nu) = 0$ and the centre of ν is not a component of Δ_i, then the centre of ν is not a divisor on X_i and it is a divisor on X_j. Therefore there are only countably many valuations ν such that $\mathbf{M}_{\Delta_i}(\nu) \neq \mathbf{M}_{\Delta_j}(\nu)$ for some i and j. Therefore, as I satisfies the DCC, by a standard diagonalisation argument, after passing to a subsequence, we may assume that $\mathbf{M}_{\Delta_i}(\nu)$ is eventually a non-decreasing sequence, for all valuations ν. In particular, we may define a $\mathbf{b}$-divisor $\mathbf{B}$ by putting

$$\mathbf{B}(\nu) = \lim_{i \to \infty} \mathbf{M}_{\Delta_i}(\nu).$$

Note that the coefficients of $\mathbf{B}$ belong to I. Let $\Phi = \mathbf{B}_Z$.

Suppose that $(Z', \mathbf{B}')$ is a cut of $(Z, \mathbf{B})$ associated to a birational morphism $Z' \longrightarrow Z$ and a set of valuations Σ. Let $f'_i \colon X_i \dashrightarrow Z'$ be the induced birational map. Note that, if $X'_i \longrightarrow X_i$ is a birational morphism and $(X'_i, \Delta'_i = \mathbf{M}_{\Delta_i, X'_i})$ has simple normal crossings, then $\operatorname{vol}(X'_i, K_{X'_i} + \Delta'_i) = v_i$ and the coefficients of Δ'_i belong to I, so that $(X'_i, \Delta'_i) \in \mathfrak{D}$. Therefore, we are free to replace (X_i, Δ_i) by (X'_i, Δ'_i). In particular, we may assume that f'_i is a birational morphism.

Given $\sigma \in \Sigma$, let $\Gamma_{i,\sigma} = (\mathbf{L}_{\Phi_i} \wedge \mathbf{B})_{Y_\sigma}$, where $Y_\sigma \longrightarrow Z$ is defined in (5.4). Suppose we define a sequence of divisors

$$\Theta_i = \bigwedge_{\sigma \in \Sigma} \mathbf{L}_{\Gamma_{i,\sigma}, Z'},$$

as in (5.6). Suppose that B is a prime divisor on Z' which is exceptional over Z. Then the coefficient of B in Θ_i is the minimum of finitely many affine linear functions of the coefficients of Δ_i. It follows that the coefficients of $\Theta_1, \Theta_2, \dots$ belong to a set $I' \supset I$ which satisfies the DCC. Finally, let

$$\Delta'_i = \Delta_i \wedge \mathbf{M}_{\Theta_i, X_i},$$

so that we only change the coefficients of divisors which are exceptional for $Z' \to Z$. In particular, the coefficients of Δ'_i belong to I'. On the other hand,

$$\Delta_i \wedge \mathbf{L}_{\Theta_i, X_i} \leq \Delta'_i = \Delta_i \wedge \mathbf{M}_{\Theta_i, X_i} \leq \Delta_i,$$

so that

$$v_i = \operatorname{vol}(X_i, K_{X_i} + \Delta'_i),$$

by (2) of (5.3). Finally, note that $\mathbf{M}_{\Delta'_i}(\rho)$ is eventually a non-decreasing sequence for any valuation ρ. In particular, we may define a **b**-divisor $\mathbf{B}'$ by putting

$$\mathbf{B}'(\rho) = \lim_{i \to \infty} \mathbf{M}_{\Delta'_i}(\rho),$$

as before.

Hence, (5.7) implies that we may find a reduction $(Z', \mathbf{B}')$, of $(Z, \mathbf{B})$ and pairs (X'_i, Δ'_i), whose coefficients belong to a set I' which satisfies the DCC, such that $v_i = \operatorname{vol}(X'_i, K_{X'_i} + \Delta'_i)$, there is a birational morphism $X'_i \longrightarrow Z'$, and moreover $\mathbf{L}_{\Phi'} \leq \mathbf{B}'$. Replacing (X_i, Δ_i) by (X'_i, Δ'_i), I by I', and Z by Z', we may therefore assume that $\mathbf{L}_\Phi \leq \mathbf{B}$.

Note that

$$\begin{aligned} v_i &= \operatorname{vol}(X_i, K_{X_i} + \Delta_i) \\ &\leq \operatorname{vol}(Z, K_Z + \Phi_i) \\ &\leq \lim \operatorname{vol}(Z, K_Z + \Phi_i) \\ &= \operatorname{vol}(Z, K_Z + \Phi), \end{aligned}$$

as $\lim \Phi_i = \Phi$.

On the other hand, if we fix $\epsilon > 0$, then $(Z, (1-\epsilon)\Phi)$ is kawamata log terminal. In particular, we may pick a birational morphism $f \colon Y \longrightarrow Z$ such that $(Y, \Psi = \mathbf{L}_{(1-\epsilon)\Phi, Y})$ is terminal. If we let $\Theta = \mathbf{L}_{\Phi, Y}$ and $\Gamma = \mathbf{B}_Y$, then

$$\Psi \leq (1-\eta)\Theta \leq \Theta \leq \Gamma$$

for some $\eta > 0$. As Γ is the limit of $\Gamma_i = \mathbf{M}_{\Delta_i, Y}$, it follows that we may find i such that $\Psi \leq \Gamma_i$. As (Y, Ψ) is terminal, we have

$\Psi_i = \mathbf{L}_{\Psi,X_i} \leq \Delta_i$. But then

$$\begin{aligned} \operatorname{vol}(Z, K_Z + (1-\epsilon)\Phi) &= \operatorname{vol}(Y, K_Y + \Psi) \\ &\leq \operatorname{vol}(X_i, K_{X_i} + \Psi_i) \\ &\leq \operatorname{vol}(X_i, K_{X_i} + \Delta_i) = v_i. \end{aligned}$$

Taking the limit as ϵ goes to zero, we get

$$\operatorname{vol}(Z, K_Z + \Phi) \leq v_i \leq \operatorname{vol}(Z, K_Z + \Phi),$$

so that $v_i = \operatorname{vol}(Z, K_Z + \Phi)$ is constant. □

Proof of (1.9). We may assume that $1 \in I$. By assumption there is a log pair (Z, B) and a projective morphism $Z \longrightarrow T$, where T is of finite type, such that if $(X, \Delta) \in \mathfrak{D}$, then there is a closed point $t \in T$ and a birational map $f\colon X \dashrightarrow Z_t$ such that the support of B_t contains the support of the strict transform of Δ_t and any f^{-1}-exceptional divisor.

Suppose that $p\colon Y \longrightarrow X$ is a birational morphism. Then the coefficients of $\Gamma = \mathbf{M}_{\Delta,Y}$ belong to I and

$$\operatorname{vol}(X, K_X + \Delta) = \operatorname{vol}(Y, K_Y + \Gamma),$$

by (1) of (5.3). Replacing (X, Δ) by (Y, Γ), we may assume that f is a morphism and we are free to replace Z and B by higher models.

We may assume that T is reduced. Blowing up and decomposing T into a finite union of locally closed subsets, we may assume that (Z, B) has simple normal crossings; passing to an open subset of T, we may assume that the fibres of $Z \longrightarrow T$ are log pairs, so that (Z, B) has simple normal crossings over T; passing to a finite cover of T, we may assume that every stratum of (Z, B) has irreducible fibres over T; decomposing T into a finite union of locally closed subsets, we may assume that T is smooth; finally passing to a connected component of T, we may assume that T is integral.

Let Z_0 and B_0 be the fibres over a fixed closed point $0 \in T$. Let $\mathfrak{D}_0 \subset \mathfrak{D}$ be the set of simple normal crossings pairs (Y, Γ), where the coefficients of Γ belong to I, Y is a projective variety of dimension n, and there is a birational morphism $g\colon Y \longrightarrow Z_0$ with $g_*\Gamma \leq B_0$.

Pick $(X, \Delta) \in \mathfrak{D}$. Let $\Phi = f_*\Delta$. Let Σ be the set of all valuations ν whose centre on X is a divisor which is exceptional over Z_t such that $\mathbf{L}_\Phi(\nu) > 0$. We may find a birational morphism $f'\colon X' \longrightarrow Z_t$, such that the centre of every element of Σ is a divisor on X', whilst f' only blows up strata of (Z_t, Φ). Suppose that $g\colon W \longrightarrow X$ is a log resolution which resolves the indeterminacy locus of the induced birational map $X \dashrightarrow X'$. If we set $\Delta' = \mathbf{M}_{\Delta,X'}$, then the coefficients of Δ' belong to

I and

$$\operatorname{vol}(X, K_X + \Delta) = \operatorname{vol}(W, K_W + \mathbf{M}_{\Delta,W}) \leq \operatorname{vol}(X', K_{X'} + \Delta'),$$

by (1) of (5.3). If ν is any valuation whose centre is an exceptional divisor for $W \longrightarrow X'$ but not for $W \longrightarrow X$, then the centre of ν is an exceptional divisor for $X \longrightarrow Z_t$ and so $\mathbf{L}_\Phi(\nu) = 0$, by choice of f'. It follows that

$$\mathbf{M}_{\Delta,W} \geq \mathbf{M}_{\Delta',W} \wedge \mathbf{L}_{\Phi,W},$$

and so (5.3) implies that

$$\operatorname{vol}(W, K_W + \mathbf{M}_{\Delta,W}) \geq \operatorname{vol}(W, K_W + \mathbf{M}_{\Delta',W} \wedge \mathbf{L}_{\Phi,W}) = \operatorname{vol}(X', K_{X'} + \Delta').$$

and hence the inequalities above are equalities. In particular

$$\operatorname{vol}(X, K_X + \Delta) = \operatorname{vol}(X', K_{X'} + \Delta').$$

Replacing (X, Δ) by (X', Δ'), we may assume that f only blow ups strata of Φ.

As (Z, B) has simple normal crossings over T and the strata of (Z, B) have irreducible fibres, we may find a sequence of blow ups $g\colon Z' \longrightarrow Z$ of strata of B, which induces the sequence of blow ups determined by f, so that $X = Z'_t$. There is a unique divisor Ψ supported on the strict transform of B and the exceptional locus of g, such that $\Delta = \Psi_t$. If $Y = Z'_0$ is the fibre over 0 of $Z' \longrightarrow T$ and Γ is the restriction of Ψ to Y, then $(Y, \Gamma) \in \mathfrak{D}_0$. (1.8) implies that $\operatorname{vol}(Y, K_Y + \Gamma) = \operatorname{vol}(X, K_X + \Delta)$.

It follows that

$$\{\, \operatorname{vol}(X, K_X + \Delta) \,|\, (X, \Delta) \in \mathfrak{D} \,\} = \{\, \operatorname{vol}(X, K_X + \Delta) \,|\, (X, \Delta) \in \mathfrak{D}_0 \,\}.$$

Now apply (5.1). $\square$

6. Birational geometry of global quotients

Theorem 6.1 (Tsuji). *Assume* $(1.4)_{n-1}$.

Then there is a constant $C = C(n) > 2$ such that if (X, Δ) is a global quotient, where X is projective of dimension n, and $K_X + \Delta$ is big, then $\phi_{m(K_X+\Delta)}$ is birational for every integer $m \geq C + 1$ such that

$$\operatorname{vol}(X, (m-1)(K_X + \Delta)) > (Cn)^n.$$

Proof. First note that (2.2.2) implies that

$$K_X + \lceil (m-1)(K_X + \Delta) \rceil = \lfloor m(K_X + \Delta) \rfloor.$$

As we are assuming $(1.4)_{n-1}$ there is a constant $\epsilon > 0$ such that if (U, Θ) is a global quotient, where $K_U + \Theta$ is big and U is projective of dimension k at most $n-1$, then $\operatorname{vol}(U, K_U + \Theta) > \epsilon^k$. Let

$$C = 2(1+\gamma)^{n-1} \qquad \text{where} \qquad \gamma = \frac{4n}{\epsilon}.$$

By assumption there is a smooth projective variety Y of dimension n and a finite group $G \subset \operatorname{Aut}(Y)$ such that $X = Y/G$ and if $\pi\colon Y \longrightarrow X$ is the quotient morphism, then $K_Y = \pi^*(K_X + \Delta)$. As $K_X + \Delta$ is big, Y is of general type. Replacing (X, Δ) and Y by their log canonical models, which exist by [7], we lose the fact that X and Y are smooth, gain the fact that $K_X + \Delta$ and K_Y are ample, and retain the condition that $K_X + \Delta$ is kawamata log terminal and K_Y is canonical.

We check the hypotheses of (2.3.6), applied to the ample divisor $K_X + \Delta$ and the constants $\epsilon/2$ and $\gamma_0 = \frac{m-1}{C} \geq 1$. Clearly

$$\operatorname{vol}(X, \gamma_0(K_X + \Delta)) > n^n.$$

Suppose that V is a minimal non kawamata log terminal centre of a log pair $(X, \Delta + \Delta_0)$, which is log canonical at the generic point of V. Further suppose that V passes through a very general point of X, and $0 \leq \Delta_0 \sim_{\mathbb{Q}} \lambda(K_X + \Delta)$, for some rational number $\lambda \geq 1$.

If $\Gamma_0 = \pi^*\Delta_0$, then every irreducible component of $\pi^{-1}(V)$ is a non kawamata log terminal centre of (Y, Γ_0). Let V' be the normalisation of $\pi^{-1}(V)$. As $H = K_Y + \Gamma_0$ is ample, Kawamata's subadjunction formula implies that for every $\eta > 0$, there is a divisor $\Phi \geq 0$ on V' such that

$$(K_Y + \Gamma_0 + \eta H)|_{V'} = K_{V'} + \Phi.$$

Let $W \longrightarrow V'$ be a G-equivariant resolution. As V passes through a very general point of X, W is a union of irreducible varieties of general type. If $U = W/G$ is the quotient, then U is irreducible and we may find a $\mathbb{Q}$-divisor Θ such that $K_W = \psi^*(K_U + \Theta)$, where $\psi\colon W \longrightarrow U$ is the quotient map.

As (U, Θ) is a global quotient, $\operatorname{vol}(U, K_U + \Theta) > \epsilon^k$, where k is the dimension of V. Therefore

$$\begin{aligned}
|G| \operatorname{vol}(V, (K_X + \Delta + \Delta_0)|_V) &= \operatorname{vol}(V', (K_Y + \Gamma_0)|_{V'}) \\
&\geq \operatorname{vol}(V', K_{V'}) \\
&\geq \operatorname{vol}(W, K_W) \\
&= |G| \operatorname{vol}(U, K_U + \Theta) \\
&\geq |G|\epsilon^k.
\end{aligned}$$

Thus

$$\operatorname{vol}(V, (1+\lambda)(K_X + \Delta)|_V) > \epsilon^k,$$

and so

$$\operatorname{vol}(V, \lambda(K_X+\Delta)|_V) > \left(\frac{\epsilon}{2}\right)^k .$$

(2.3.6) implies that $(m-1)(K_X+\Delta)$ is potentially birational. (1) of (2.3.4) implies that $\phi_{K_X+\lceil (m-1)(K_X+\Delta)\rceil}$ is birational. □

7. Proof of (1.4) and (1.1)

Proof of (1.4). By induction on n. Assume $(1.4)_{n-1}$. By (6.1) there is a constant $C=C(n)>2$ depending only on the dimension n such that if (X,Δ) is a global quotient, where X is projective of dimension n and $K_X+\Delta$ is big, then $\phi_{m(K_X+\Delta)}$ is birational, for any $m\geq C+1$ such that

$$\operatorname{vol}(X,(m-1)(K_X+\Delta)) > (Cn)^n.$$

Note that the right hand side does not depend on m.

Fix a constant $V>n^n$ and let

$$\mathfrak{D}_V = \{\, (X,\Delta)\in\mathfrak{D} \,|\, 0<\operatorname{vol}(X,K_X+\Delta)\leq V \,\}.$$

Note that if k is a positive integer such that $\operatorname{vol}(X,k(K_X+\Delta))\leq C^nV$, then $\operatorname{vol}(X,(k+1)(K_X+\Delta))\leq 2^nC^nV$. It follows that there is a positive integer $m\geq C+1$ such that if $(X,\Delta)\in\mathfrak{D}_V$, then

$$(Cn)^n<\operatorname{vol}(X,(m-1)(K_X+\Delta))\leq 2^nC^nV,$$

so that $\phi_{m(K_X+\Delta)}$ is birational. (2) of (2.3.4) implies that $\phi_{K_X+(2n+1)m(K_X+\Delta)}$ is birational. But then (3.1) implies that $\mathfrak{D}_V$ is log birationally bounded, and so (1.9) implies that the set

$$\{\, \operatorname{vol}(X,K_X+\Delta) \,|\, (X,\Delta)\in\mathfrak{D}_V \,\},$$

satisfies the DCC, which implies that (1) and (2) of (1.4) hold in dimension n.

In particular there is a constant $\delta>0$ such that if $(X,\Delta)\in\mathfrak{D}$, and $K_X+\Delta$ is big, then $\operatorname{vol}(X,K_X+\Delta)\geq\delta$. It follows that $\phi_{M(K_X+\Delta)}$ is birational, for any

$$M>\frac{Cn}{\delta}+1,$$

and this completes the induction and the proof. □

Proof of (1.1). By (2) of (1.4) there is a constant $\delta>0$ such that if (X,Δ) is a global quotient, where X is projective of dimension n and $K_X+\Delta$ is big, then $\operatorname{vol}(X,K_X+\Delta)\geq\delta$. Let $c=\frac{1}{\delta}$.

Let Y be a projective variety of dimension n of general type. By [7], there is a log canonical model $Y \dashrightarrow Y'$. If G is the birational automorphism group of Y, then G is the automorphism group of Y'. Replacing Y by a G-equivariant resolution of Y', we may assume that

G is the automorphism group of Y. Let $\pi\colon Y \longrightarrow X = Y/G$ be the quotient of Y. Then there is a divisor Δ on X such that $K_Y = \pi^*(K_X+\Delta)$. By definition, (X,Δ) is a global quotient, X is projective and $K_X+\Delta$ is big. It follows that $\operatorname{vol}(X, K_X+\Delta) \geq \delta$. As

$$\operatorname{vol}(Y, K_Y) = |G| \operatorname{vol}(X, K_X+\Delta),$$

it follows that

$$|G| \leq c \cdot \operatorname{vol}(Y, K_Y). \qquad \square$$

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