A GEOMETRIC CHARACTERISATION OF TORIC VARIETIES

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ABSTRACT. We prove a conjecture of Shokurov which characterises toric varieties using log pairs.

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1. Introduction

We work over a field of characteristic zero, which is algebraically closed, unless otherwise stated.

Toric varieties appear frequently in algebraic geometry. This is surprising as the definition of a toric variety is so restrictive; X is normal and there is an open subset isomorphic to a torus such that the action of the torus on itself extends to X. On the other hand the appearance of toric varieties is very useful as many geometric problems are reduced to straightforward combinatorics. We are interested in explaining why toric varieties appear so often and to give additional criteria for their appearance.

One approach is to try to give a simple characterisation of toric varieties. We give a characterisation that only involves invariants coming from log pairs:

Definition 1.1. *Let X be a proper variety of dimension n and let (X, Δ) be a log pair. A **decomposition** of Δ is an expression of the form*

$$\sum a_i S_i \leq \Delta,$$

*where $S_i \geq 0$ are $\mathbb{Z}$-divisors and $a_i \geq 0$, $1 \leq i \leq k$. The **complexity** of this decomposition is $n+r-d$, where r is the rank of the vector space spanned by $S_1, S_2, \dots, S_k$ in the space of Weil divisors modulo algebraic equivalence and d is the sum of $a_1, a_2, \dots, a_k$.*

*The **complexity** $c = c(X, \Delta)$ of (X, Δ) is the infimum of the complexity of any decomposition of Δ.*

Note that we don't require that the divisors S_i are prime divisors (since the components of S_i might span a larger vector space). On the other hand in practice the smallest complexity is often achieved by taking $S_1, S_2, \dots, S_k$ to be prime divisors. In the special case when the coefficients of $D = \Delta = \sum S_i$ are all one, then d is the number of components of D. It is well known that for a toric pair, that is, a toric variety together with the sum of the invariant divisors, we have $d = n + r$, so that $c = 0$.

We give a characterisation of toric pairs involving the complexity:

Theorem 1.2. *Let X be a proper variety of dimension n and let (X, Δ) be a log canonical pair such that $-(K_X + \Delta)$ is nef.*

If $\sum a_i S_i$ is a decomposition of complexity c less than one then $c \geq 0$ and there is a divisor D such that (X, D) is a toric pair, where $D \geq \lfloor \Delta \rfloor$ and all but $\llcorner 2c \lrcorner$ components of D are elements of the set $\{ S_i \,|\, 1 \leq i \leq k \}$.

(1.2) is a special case of a conjecture of Shokurov, cf. [28], which is stated in the relative case. Here are two simple corollaries of (1.2):

Corollary 1.3. *Let X be a proper variety and let (X, Δ) be a log canonical pair such that $-(K_X + \Delta)$ is nef.*

Then the complexity is non-negative.

Corollary 1.4. *Let X be a proper variety of dimension n and let (X, Δ) be a log canonical pair such that $-(K_X + \Delta)$ is nef.*

If the complexity is less than one then the components of Δ span the Néron-Severi group.

One can extend (1.2) to the case of any field of characteristic zero:

Corollary 1.5. *Let k be a field of characteristic zero.*

Let X be a proper variety over k and let (X, Δ) be a log canonical pair such that $-(K_X + \Delta)$ is nef.

If $\sum a_i S_i$ is a decomposition of complexity c less than one then there is a divisor D such that (X, D) is a toric pair, where $D \geq \llcorner \Delta \lrcorner$ and all but $\llcorner 2c \lrcorner$ components of D are elements of the set $\{ S_i \,|\, 1 \leq i \leq k \}$.

We are able to prove that log pairs with small complexity have a simple birational structure:

Theorem 1.6. *Let (X, Δ) be a divisorially log terminal pair where X is a $\mathbb{Q}$-factorial projective variety.*

If $-(K_X + \Delta)$ is nef then we may find an ample divisor A and a divisor $0 \leq \Delta_0 \leq \Delta$ such that the numerical dimension of $K_X + A + \Delta_0$ is at most the complexity of (X, Δ).

In particular if $X \dashrightarrow Z$ is the maximal rationally connected fibration then the dimension of Z is at most the complexity.

Toric varieties are special as they are rational. We are able to give a rationality criterion in terms of a slightly different version of the complexity:

Definition 1.7. *Let X be a proper variety of dimension n and let (X, Δ) be a log pair. The* ***absolute complexity*** *$\gamma = \gamma(X, \Delta)$ of (X, Δ) is $n + \rho - d$, where ρ is the rank of the group of Weil divisors modulo algebraic equivalence and d is the sum of the coefficients of Δ.*

If X is $\mathbb{Q}$-factorial then ρ is the Picard number.

Theorem 1.8. *Let X be a proper variety. Suppose that (X, Δ) is log canonical and $-(K_X + \Delta)$ is nef.*

If $\gamma(X, \Delta) < \frac{3}{2}$ then there is a proper finite morphism $Y \longrightarrow X$ of degree at most two, which is étale outside a closed subset of codimension at least two, such that Y is rational.

In particular if $A_{n-1}(X)$ contains no 2-torsion then X is rational.

The condition on torsion in the class group is necessary and we give an example of this in §7. Note that most rationality criteria are used to establish irrationality. There are relatively few criteria to show rationality.

(1.2) was proved for surfaces in [19] for Picard number one (based heavily on ideas of Shokurov) and in [28] in general. Both proofs use Shokurov's theory of complements. Cheltsov, in unpublished work, proved (1.2) when X is $\mathbb{Q}$-factorial projective and the Picard number is one. The technique he uses is the basis of our proof, which we will explain below. [26] contains a proof of (1.2) for threefolds in some special cases. The method of proof is to run the MMP. [31] has a proof of (1.2) when X is a smooth projective variety, $\Delta = \sum D_i$ has global normal crossings and $K_X + \Delta$ is numerically trivial. The method of proof is quite different from the other papers and uses ideas coming from mirror symmetry and the powerful methods of Gross, Hacking, Keel and Siebert, cf. [10]. [16] contains work related to both (1.2) and (1.8).

There are some examples to show (1.2) and (1.8) are sharp. First an example to show that not every invariant divisor is a component of Δ:

Example 1.9. *Consider $(X = \mathbb{P}^2, \Delta = L_1 + L_2 + 1/2C)$ where L_1 and L_2 are two lines and C is a conic, in general position. Then (X, Δ) is divisorially log terminal, $K_X + \Delta \sim_{\mathbb{Q}} 0$ and the complexity is*

$$c = 2 + 1 - 5/2 = 1/2.$$

Note that $\lfloor \Delta \rfloor = L_1 + L_2$. Let L_3 be a third line in general position. Then $(\mathbb{P}^2, L_1 + L_2 + L_3)$ is a toric pair and two of the three invariant divisors are components of Δ but not all three.

It is also not hard to see that it is crucial that (X, Δ) is log canonical:

Example 1.10. *Take $X = \mathbb{F}_n$ the unique $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ with a curve E_∞ of self-intersection $-n$. Let $\Delta = 2E_\infty + \sum F_i$, where $F_1, F_2, \dots, F_{n+2}$ are $n+2$ fibres. Then $K_X + \Delta \sim 0$ and the complexity*

$$c = 2 + 2 - (n + 4) = -n,$$

is arbitrarily large and negative. Note that if one contracts E_∞ then the image of Δ is a boundary and the complexity is $c = 1 - n$.

One can also see that one cannot relax nef to pseudo-effective:

Example 1.11. *If we replace* Δ *by* $E_\infty + \sum F_i$ *in* (1.10) *then* (X, Δ) *is log canonical and* $-(K_X + \Delta)$ *is pseudo-effective but the complexity is again* $1 - n$.

We also have an example where X is smooth and all the coefficients are one:

Example 1.12. *Let* $Q = V(xy - zw) \subset \mathbb{P}^4$ *be a rank four quadric threefold. Pick a small resolution* $X \longrightarrow Q$ *with exceptional locus* L *isomorphic to* $\mathbb{P}^1$. *Note that any hyperplane through the vertex of* Q, *which intersects the quadric at infinity in two lines, intersects* Q *in two planes through the vertex. By adjunction the sum of three such pairs is an element of* $|-K_Q|$.

If $D = D_1 + D_2 + D_3 + D_4 + D_5 + D_6$ *is the strict transform of these six divisors then* $K_X + D \sim 0$ *and the complexity*

$$c = 3 + 2 - 6 = -1.$$

On the other hand three components of D *contain the curve* L, *so that* (X, D) *is not log canonical, even though* X *is smooth and every component of* D *has coefficient one.*

It is also easy to see that we need to work with the absolute complexity for unirationality and that (1.2) is sharp:

Example 1.13. *If* $X = E$ *is an elliptic curve and we take* $\Delta = 0$ *then* $K_E \sim 0$ *and the complexity is* 1. *On the other hand* E *is not unirational. Note that the absolute complexity is* 2.

In fact if one works over a non-algebraically field, it is easy to see that we need to allow an extension of degree two for rationality:

Example 1.14. *If* $C = V(x^2 + y^2 + z^2)$ *is a smooth conic over* $\mathbb{R}$ *without a real point then we may find a divisor* D *of degree one such that* $K_X + D \sim 0$ *so that the absolute complexity is one. On the other hand* C *is irrational but* C *becomes rational if we replace* $\mathbb{R}$ *with* $\mathbb{C}$.

We give an example in §7 to show that we need a cover of degree two to achieve rationality, cf. (1.8). This example is in some sense a geometric realisation of (1.14).

Let us turn to a description of the proof of (1.2). The first step is to replace (X, Δ) by a divisorially log terminal model (Y, Γ). This means that Y is projective, $\mathbb{Q}$-factorial and (Y, Γ) is divisorially log terminal. There is a birational contraction map $\pi\colon Y \dashrightarrow X$ and the only exceptional divisors have log discrepancy zero. If X is projective then we can take π to be a morphism and this is a standard reduction step (by a result of Hacon, see for example, [21, 3.1]). If X is not projective then there are examples which show it is not always possible to arrange for π to be a morphism.

For example, take X to be any smooth proper variety which is not projective and take Δ to be empty. Let $\pi\colon Y \dashrightarrow X$ be a divisorially log terminal model of X. As X is smooth it is kawamata log terminal and so π is small. π is not the identity morphism as Y is projective and X is not. Therefore π is not a morphism as X is $\mathbb{Q}$-factorial.

For a concrete example, consider the smooth toric threefold X on page 71 of [8]. It is not projective as it has no ample divisors. It is easy to see that if one flops an invariant curve $X \dashrightarrow Y$, corresponding to a diagonal edge of the slanted faces of the tetrahedron, then Y is projective and the induced birational map $\pi\colon Y \dashrightarrow X$ is a divisorially log terminal model.

We prove the existence of divisorially log terminal models in (2.2.5), contingent on the existence of a nef divisor M such that $K_X + \Delta + M$ is nef. This covers the case when either $K_X + \Delta$ or $-(K_X + \Delta)$ is nef and the latter is sufficient for our purposes. We check in (2.4.1) that the complexity of (Y, Γ) is at most the complexity of (X, Δ); this is straightforward since every exceptional divisor extracted by π is a component of Γ of coefficient one. Finally it is not hard to see that it is enough to work with (Y, Γ), cf. (2.3.2).

Thus we may assume that X is projective, $\mathbb{Q}$-factorial and (X, Δ) is divisorially log terminal. The next step is to proceed based on the assumption that X is a Mori dream space.

To explain this step we first describe Cheltsov's argument which applies when the Picard number is one. In this case K_X and all the components of Δ are proportional to a very ample divisor H. If we let (Y, Γ) be the cone over (X, Δ) under the embedding given by H then (Y, Γ) is log canonical and by construction every component of Γ is $\mathbb{Q}$-Cartier and passes through the vertex p of the cone.

The goal is then to prove (2.4.3), a local version of (1.2). The proof of (2.4.3) is based on the proof of [20, 18.22], which establishes that the sum of the coefficients of Γ, which is precisely the sum of the coefficients of Δ, is no more than the dimension of Y. Passing to a composition of cyclic covers, we may assume that both K_Y and every component of Γ is Cartier and in this case it suffices to check that Y is smooth. If we replace components of Γ whose coefficients sum to one by a general element of the linear system they span we can apply adjunction and induction to conclude that Y is smooth. Since the original variety is a quotient by a product of cyclic groups, it is not hard to see that the original variety Y is toric. Since the only way to get a toric variety as a cone is to start with a toric variety we see that X must be toric; indeed X is isomorphic to the exceptional divisor of the blow up of the cone at the vertex p.

Unfortunately the naive generalisation of this argument does not apply if the Picard number is not one. The problem is that the cone over a variety of Picard number at least two is not even $\mathbb{Q}$-factorial; for example the quadric cone which is the cone over $\mathbb{P}^1 \times \mathbb{P}^1$ is not $\mathbb{Q}$-factorial.

Instead of working with a cone we work with the affine variety Y associated to the Cox ring of X. X is a Mori dream space if and only if the Cox ring is finitely generated. The Cox ring is naturally graded by the class group, the group of Weil divisors modulo linear equivalence. As usual this grading corresponds to the action on Y of an algebraic group H, the spectrum of the group algebra associated to the class group, which is the product of a torus and a finite abelian group. We can recover X as the quotient of Y by H. In the case when the class group is isomorphic to $\mathbb{Z}$ (so that, in particular, the Picard number is one), Y is a cone and H is a one dimensional torus, acting in the usual way on the lines of the cone. As in the case of a cone, there is a natural log pair (Y, Γ) associated to X and every component of Γ passes through the same point p. (Y, Γ) is log canonical if and only if (X, Δ) is log canonical by [4], [9], and [18]. Mori dream spaces were introduced in the very influential paper [15]. We actually use a more sophisticated version of the Cox ring, which was introduced in [14]. It has the advantage that every Weil divisor on X corresponds to a Cartier divisor on Y, so that we don't even need to take any cyclic covers.

The main point at this step of the proof is to bound the dimension of Y. The dimension of Y is the dimension of X plus the Picard number. By assumption the sum of the coefficients of a decomposition $\sum a_i D_i$ of Δ is at least the dimension of X, minus one, plus the dimension r of the space spanned by the components $D_1, D_2, \dots, D_k$. So we have to show that $r = \rho$, that is, the components $D_1, D_2, \dots, D_k$ generate the vector space of divisors modulo linear equivalence.

We prove this result by induction on r. We start with the case that $D_1, D_2, \dots, D_k$ span the same ray of the cone of divisors. It is easy to show that the Picard number of X is one. Consider for example the case that X is a smooth projective surface and $K_X + D$ is numerically trivial. If the Picard number is not one then either there is a -1-curve Σ or a $\mathbb{P}^1$-bundle $X \longrightarrow C$. If Σ is a -1-curve then K_X is negative on Σ so that D is positive on Σ. As the components of D are proportional to each other it follows that every component of D intersects Σ. As the sum of the coefficients of D is at least three, $D \cdot \Sigma \geq 3$, which is impossible as $K_X \cdot \Sigma = -1$. If $X \longrightarrow C$ is a $\mathbb{P}^1$-bundle and Σ is a general fibre we have $D \cdot \Sigma \geq 3$ and $K_X \cdot \Sigma = -2$, which is again impossible. In the general case we run an appropriate MMP. After finitely many flips we either get a divisorial contraction or a Mori fibre space and both cases we can rule out, using a similar argument, cf. (3.3).

Otherwise we may pick two components D_1 and D_2 of D such that neither $P_1 = m_1 D_1 - m_2 D_2$ nor $P_2 = m_2 D_2 - m_1 D_1$ is pseudo-effective. In this case consider the $\mathbb{P}^1$-bundle given by the direct sum of the line bundles corresponding to P_1 and P_2. Y is a Mori dream space and the two sections corresponding to P_1 and P_2 are contractible, $Y \dashrightarrow Z$. In this case we proceed by induction on the rank r, cf. (3.2). The details of this step are in §3.

To reduce to the case when X is a Mori dream space we have to pass to a different model Y such that $-(K_Y + \Gamma)$ is ample for some kawamata log terminal pair (Y, Γ). Note that in this case $K_Y + B + \Gamma$ is numerically trivial, where $B = -(K_Y + \Gamma)$ is ample. So we look for divisors $0 \leq \Delta_0 \leq \Delta$ and ample divisors A such lthat $K_X + A + \Delta_0$ has numerical dimenson zero. In this case Y is a log terminal model of $(X, A + \Delta_0)$.

If the numerical dimension is not zero then there is a non-trivial fibration $Y \longrightarrow Z$. Not every component of D dominates Z, since otherwise the complexity of the general fibre is less than zero, cf. (4.3). On the other hand it is not hard to decrease the numerical dimension if there is a component of D which does not dominate, cf. (4.4). To finish off, we replace $A + \Delta_0$ by a convex linear combination of $A + \Delta_0$ and $M + \Delta$, where $M = -(K_X + \Delta)$, and cancel off common components of Δ_0 and exceptional divisors of $f \colon X \dashrightarrow Y$ so that the complexity of $(X, A + \Delta_0)$ is close to the complexity of (X, Δ) and f does not contract any components of Δ, cf. (4.5). The details are in §4.

To finish the proof of (1.2), we need to know that if Y is toric then so is X. The key point is to reduce to the case that $N = 0$. The first step is to pass to a model such that no centre of (X, Δ) is contained in the exceptional locus of f, cf. (5.1). We then perturb Γ so that it is more singular along at least one exceptional divisor, cf. (5.2). Taking a convex linear combination of Δ_0, a divisor supported on N and the perturbed divisor we may decrease the number of components of N and we are done by induction on the number of components of N. The details are in §5.

Now we turn to the proof of (1.8). The proof follows similar lines to the proof of (1.2). We may assume that X is projective, $\mathbb{Q}$-factorial and (X, Δ) is divisorially log terminal and by (4.2) we may assume that X is a Mori dream space. If the absolute complexity is less than two then we can conclude that the affine variety Y associated to the Cox ring of X has compound Du Val singularities, meaning that there is a surface section with Du Val singularities. If we further assume that the absolute complexity is less than $3/2$ then we can conclude that Y has a compound A_l singularity, meaning that a surface section of Y has an A_l singularity.

It follows that Y is a hypersurface in affine space $\mathbb{A}^m$ given by a polynomial q whose quadratic part has rank two. The action of H on Y extends to $\mathbb{A}^m$. The quotient of $\mathbb{A}^m$ by H is a toric variety and X is birational to the image of Y in this toric variety. If $xy \in q$, that is, xy is a monomial with non-zero coefficient in q, then it is not hard to check that there is a one dimensional torus whose general orbit intersects X in a single point. Thus X is birational to an invariant divisor so that X is rational. Otherwise after rescaling we may assume that the quadratic part of f has the form $x^2 + y^2$. If x and y have the same multidegree then we may change variable and reduce to the previous case. Otherwise there must be torsion in the class group and there is a cover $Y \longrightarrow X$ of degree two. The details are in (6.1).

In §7 we exhibit log canonical pairs (X, Δ) of absolute complexity one such that X is irrational. The idea is to start with a conic bundle of relative Picard number two over $\mathbb{P}^1 \times \mathbb{P}^1$ and take a $\mathbb{Z}_2$-quotient to achieve relative Picard number one. The key observation is that the discriminant curve, the locus of reducible fibres, makes no contribution in Kawamata's canonical bundle formula. Thus we can arrange for the discriminant curve to have arbitrarily large genus, in which case X is irrational.

We suspect that if the absolute complexity is less than two then we may always find a cover so that X is rational. In this case we have to consider the extra possibility that Y has a compound singularity of type D_l, E_6, E_7, or E_8. However we were unable to see how to proceed in this case.

2. Preliminaries

In this section we will collect some definitions and preliminary results.

2.1. **Notation and Conventions.** Let X be a proper normal variety. $\rho(X)$ is the rank of the Picard group of X. We denote the class group, the group of Weil divisors modulo linear equivalence, by $A_{n-1}(X)$. If D is an $\mathbb{R}$-Weil divisor, the sheaf $\mathcal{O}_X(D)$ is defined by

$$\mathcal{O}_X(D)(U) = \{\, f \in K(X) \,|\, (f)|_U + D|_U \geq 0 \,\},$$

so that $\mathcal{O}_X(D) = \mathcal{O}_X(\lfloor D \rfloor)$. Similarly we define $|D| = |\lfloor D \rfloor|$.

We will follow the terminology from [22]. In particular we only consider valuations ν of X whose centre on some birational model Y of X is a divisor. A *log canonical place* of a log canonical pair (X, Δ) is any valuation ν whose log discrepancy is zero.

Suppose that $f \colon X \dashrightarrow Y$ is a rational map whose domain is an open subset U whose complement has codimension at least two. In this case if D is an $\mathbb{R}$-Cartier divisor on Y we may define f^*D as the $\mathbb{R}$-Weil divisor whose restriction to U is the usual pullback.

We say a proper morphism $f \colon X \longrightarrow Y$ is a *contraction morphism* if $f_*\mathcal{O}_X = \mathcal{O}_Y$. Let $f \colon X \dashrightarrow Y$ be a proper rational map of normal quasi-projective varieties and let $p \colon W \longrightarrow X$ and $q \colon W \longrightarrow Y$ be a resolution of f and of X. We say that f is a *rational contraction* if q is a contraction morphism and the image of every p-exceptional divisor has codimension two or more in Y. We say that a prime divisor P on X is *horizontal* if the image of the generic point of P is the generic point of Y. We say that P is *vertical* if it is not horizontal.

We say that a birational map $f \colon X \dashrightarrow Y$ is a *birational contraction* if f is a rational contraction, so that every p-exceptional divisor is q-exceptional. If D is an $\mathbb{R}$-Cartier divisor on X such that $D' := f_*D$ is $\mathbb{R}$-Cartier then we say that f is *D-non-positive* (resp. *D-negative*) if we have $p^*D = q^*D' + E$ where $E \geq 0$ and E is q-exceptional (respectively E is q-exceptional and the support of E contains the strict transform of the f-exceptional divisors).

Now suppose that $f \colon X \dashrightarrow Y$ is a birational contraction of projective varieties. If X is $\mathbb{Q}$-factorial and (X, Δ) is a divisorially log terminal pair such that f is $(K_X + \Delta)$-negative, $K_Y + \Gamma$ is nef and Y is $\mathbb{Q}$-factorial, where $\Gamma = f_*\Delta$, then we say that $f \colon X \dashrightarrow Y$ is a *log terminal model* of $K_X + \Delta$. If the ring

$$R(X, K_X + \Delta) := \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(m(K_X + \Delta)))$$

is finitely generated then $X \dashrightarrow Z$ is called the *ample model* of (X, Δ), where

$$Z = \operatorname{Proj} R(X, K_X + \Delta).$$

If (X, Δ) has a log terminal model $\pi \colon X \dashrightarrow Y$ then we let

$$P = P(X, \Delta) = \pi^*(K_X + \Delta) \qquad \text{and} \qquad N = N(X, \Delta) = K_X + \Delta - \pi^*(K_X + \Delta),$$

and call the decomposition

$$K_X + \Delta = P(X, \Delta) + N(X, \Delta),$$

the *Zariski decomposition.*

Note that P is mobile, the natural inclusion

$$|mP + A| \subset |m(K_X + \Delta) + A|$$

is an equality for any fixed ample divisor A and sufficiently large integer m, $N \geq 0$ and N is exceptional for π.

Following [24] we define *the numerical dimension*

$$\kappa_\sigma(X, D) = \max_{H \in \operatorname{Pic}(X)} \{ k \in \mathbb{N} \mid \limsup_{m \to \infty} \frac{h^0(X, \mathcal{O}_X(mD + H))}{m^k} > 0 \},$$

where H is an ample divisor on X. If D is nef then this is the same as

$$\nu(X, D) = \max\{ k \in \mathbb{N} \mid H^{n-k} \cdot D^k > 0 \}.$$

Let $f \in K[x_1, x_2, \dots, x_n]$ be a polynomial. If μ is a monomial in $x_1, x_2, \dots, x_n$ then we write $\mu \in f$ if and only if the coefficient of μ in f is non-zero.

If k is a field and $\bar{k}$ is the algebraic closure of k then bars will denote extension of schemes to $\bar{k}$.

2.2. **Birational Geometry.**

Lemma 2.2.1. *Let X be a $\mathbb{Q}$-factorial projective variety and let (X, Δ) be a kawamata log terminal pair. Suppose that Δ is big and $K_X + \Delta$ is pseudo-effective. Let $\pi \colon X \dashrightarrow Z$ be the ample model and let D be a prime divisor.*

Then $K_X + \Delta - dD$ is pseudo-effective for d sufficiently small, if and only if, either D does not dominate Z or the support of D lies in the support of the stable base locus of $K_X + \Delta$.

Proof. Let $K_X + \Delta = P + N$ be the Zariski decomposition. As $K_X + \Delta$ has a log terminal model, [3, 1.2], the components of N are the prime divisors in the stable base locus of $K_X + \Delta$.

If the support of D lies in the support of the stable base locus of $K_X + \Delta$ then we may find $d > 0$ such that $dD \leq N$ and in this case

$$K_X + \Delta - dD = P + (N - dD) \geq P$$

is pseudo-effective.

Let H be the ample divisor on Z corresponding to $K_X + \Delta$. If D does not dominate Z then we can pick $d > 0$ and $H' \sim_{\mathbb{R}} H$ such that $\pi^* H' \geq dD$, so that $K_X + \Delta - dD$ is pseudo-effective.

Now suppose D dominates Z and let F be the general fibre of π. Then $P|_F = 0$. Therefore if $K_X + \Delta - dD$ is pseudo-effective then

$dD \leq N$. But then the support of D lies in the support of the stable base locus of $K_X + \Delta$. ☐

We will need a version of the MMP for log canonical pairs.

Lemma 2.2.2. *Let X be a $\mathbb{Q}$-factorial kawamata log terminal projective variety and let (X, Δ) be a log canonical pair.*

If $K_X + \Delta$ is not pseudo-effective then we may run the $(K_X + \Delta)$-MMP until we arrive at a Mori fibre space.

Proof. Pick an ample divisor A such that $K_X + A + \Delta$ is not pseudo-effective. Since X is $\mathbb{Q}$-factorial kawamata log terminal we may find a divisor $\Delta' \sim_{\mathbb{R}} A + \Delta$ such that (X, Δ') is kawamata log terminal.

In particular, [3, 1.3.3] implies that the $(K_X + \Delta')$-MMP with scaling of A always terminates with a Mori fibre space. On the other hand any run of the $(K_X + \Delta')$-MMP with scaling of A is automatically a run of the $(K_X + \Delta)$-MMP. ☐

We will need divisorially log terminal models in the case when X is proper but not necessarily projective. In this case we need to relax the requirement that we have a morphism. To emphasize this point we use the term model rather than modification.

We start by recalling the definition of a divisorially log terminal modification :

Proposition 2.2.3. *Let (X, Δ) be a log pair where X is a variety and the coefficients of Δ belong to $[0, 1]$.*

Then there is a projective birational morphism $\pi \colon Y \longrightarrow X$ such that

(1) Y is $\mathbb{Q}$-factorial,
(2) π only extracts divisors of log discrepancy at most zero,
(3) if $E = \sum E_i$ is the sum of the π-exceptional divisors and Γ is the strict transform of Δ, then $(Y, \Gamma + E)$ is divisorially log terminal and

$$K_Y + E + \Gamma = \pi^*(K_X + \Delta) + \sum_{a(E_i, X, B) < 0} a(E_i, X, B) E_i.$$

Any birational morphism $\pi \colon Y \longrightarrow X$ satisfying (1–3) is called a ***divisorially log terminal modification****.*

Proof. The construction of a divisorially log terminal modification is due to Hacon and can be found in [6], [21, 3.1], and also [2]. ☐

Definition 2.2.4. *Let (X, Δ) be a log canonical pair, where X is a proper variety.*

A ***divisorially log terminal model*** *is a divisorially log terminal pair* (Y,Γ)*, where* Y *is a projective* $\mathbb{Q}$*-factorial variety, together with a birational contraction* $\pi\colon Y \dashrightarrow X$ *such that*

$$K_Y + \Gamma = \pi^*(K_X + \Delta),$$

and the only divisors contracted by π *have log discrepancy zero with respect to* (X,Δ)*.*

Note that if $\pi\colon Y \longrightarrow X$ is a divisorially log terminal modification then π is projective but not necessarily Y. Conversely if $\pi\colon Y \dashrightarrow X$ is a divisorially log terminal model then Y is projective but π is not necessarily a morphism. See the sketch of proof in §1 which has some examples of the possible type of different behaviour.

We are only able to prove the existence of divisorially log terminal models in very special cases:

Proposition 2.2.5. *Let* (X,Δ) *be a log canonical pair where* X *is a proper variety.*

If M *is a nef divisor such that* $K_X+\Delta+M$ *is nef then we may find a divisorially log terminal model such that both* $N=\pi^*M$ *and* $K_Y+\Gamma+\lambda N$*, for some* $\lambda \geq 1$*, are nef.*

In particular if $M=\pm(K_X+\Delta)$ *then* $\pm(K_Y+\Gamma)$ *is nef.*

We will need some preliminary results, which are simple extensions of results by Shokurov, cf. Addendum 4 of [29].

Lemma 2.2.6. *Let* $m_1, m_2, \dots, m_k$ *be positive real numbers and let* m *and* r *be positive integers.*

Then there is a positive constant $\hbar$ *such that if*

$$a \in \{\sum \frac{a_i m_i}{r} \mid a_1, a_2, \dots, a_k \in \mathbb{Z}, a_i \geq -m\}$$

and $a>0$ *then* $a \geq \hbar$*.*

Proof. Clear. □

Lemma 2.2.7. *Let* (X,Δ) *be a kawamata log terminal pair where* X *is a* $\mathbb{Q}$*-factorial projective variety and* Δ *is a big* $\mathbb{R}$*-divisor.*

If M *is a nef* $\mathbb{R}$*-divisor then we may find a positive constant* $\hbar$ *with the following property:*

If $f\colon X \dashrightarrow Y$ *is any sequence of* $(K_X+\Delta)$*-flips which are* M*-trivial and* C *is any curve spanning a* $(K_Y+\Gamma)$*-extremal ray of the cone of curves of* Y *then either* $N\cdot C \geq \hbar$ *or* $N \cdot C = 0$*, where* $N=f_*M$ *and* $\Gamma = f_*\Delta$*.*

Proof. We may write

$$M = \sum m_i M_i,$$

where $m_1, m_2, \dots, m_k$ are positive real numbers and $M_1, M_2, \dots, M_k$ are $\mathbb{Q}$-Cartier divisors. Pick this decomposition minimal with this property, so that $m_1, m_2, \dots, m_k$ are independent over $\mathbb{Q}$. Pick M_i sufficiently close to M so that we may find

$$\Phi_i \sim_{\mathbb{R}} \Delta + M_i,$$

where (X, Φ_i) is kawamata log terminal. Pick a positive integer r so that rM_i is Cartier, for all indices $1 \leq i \leq k$.

We first check that all of these properties are preserved by any sequence $f\colon X \dashrightarrow Y$ of $(K_X + \Delta)$-flips which are M-trivial. By induction we are reduced to the case of one flip. If R is the corresponding $(K_X + \Delta)$-extremal ray then R is spanned by a rational curve C. As $M \cdot C = 0$ and $m_1, m_2, \dots, m_k$ are independent over $\mathbb{Q}$, we must have $M_i \cdot C = 0$. Thus N is nef and rN_i is Cartier. It is clear that

$$N = \sum m_i N_i \qquad \text{and} \qquad \Psi_i = f_*\Phi_i \sim_{\mathbb{R}} \Gamma + N_i,$$

since f is an isomorphism in codimension one. The pair (Y, Ψ_i) is kawamata log terminal as f is a $(K_X + \Phi_i)$-flip.

Thus there is no harm in assuming that f is the identity. Suppose that R is a $(K_X + \Delta)$-extremal ray. Then [17, 1] implies that R is spanned by a rational curve C such that

$$-2n \leq (K_X + \Phi_i) \cdot C \leq M_i \cdot C = \frac{a_i}{r},$$

for some integer a_i. Now apply (2.2.6) with $m = 2nr$. □

Proof of (2.2.5). As a first approximation, let $\pi\colon Y \longrightarrow X$ be a log resolution of (X, Δ) such that Y is projective. We may write

$$K_Y + \Gamma = K_Y + \widetilde{\Delta} + E = \pi^*(K_X + \Delta) + F$$

where $\widetilde{\Delta}$ is the strict transform of Δ, $E = \sum E_i$ is the sum of the exceptional divisors and $F \geq 0$ is exceptional. This model would be a divisorially log terminal model provided $F = 0$. Our goal is to contract F using the MMP, preserving the condition that N is nef.

We may write

$$K_Y + \Gamma + N = \pi^*(K_X + \Delta + M) + F.$$

Note that $K_Y + \Gamma + N$ is pseudo-effective and the diminished base locus of $K_Y + \Gamma + N$ is equal to the support of F. Pick an ample divisor A so that the support of the stable base locus of $K_Y + A + \Gamma + N$ is equal to the support of F. Then the stable base locus of $K_Y + A + \Gamma + tN$ is equal to the support of F for any $t \geq 1$.

Let

$$\lambda = \max(1, \frac{2n}{\hbar}) > 0,$$

where $\hbar$ is defined in (2.2.7). Let $f\colon Y \dashrightarrow Y'$ be a step of the $(K_Y + A+\Gamma+\lambda N)$-MMP with scaling of A. If R is the corresponding extremal ray then R is spanned by a rational curve C such that $(K_Y+A+\Gamma)\cdot C > -2n$ so that $N \cdot R = 0$. In particular f_*N is nef. If f contracts a divisor then this divisor is a component of F so that f only contracts divisors which are exceptional for π. Therefore we are free to replace Y by Y'. Note that we might lose the property that π is a morphism, when f is a flip, but we retain the property that π is a birational contraction.

Now suppose that $g\colon Y \dashrightarrow Y'$ is a sequence of flips which are N-trivial. By (2.2.7) these are all steps of the $(K_Y + A + \Gamma + \lambda N)$-MMP with scaling of A. Since this MMP always terminates, after finitely many steps we construct a model on which $F = 0$. $\square$

2.3. **Toric Geometry.** We say that X is a *toric variety* if X is a normal variety over a field k (not necessarily algebraically closed), there is a dense open subset U isomorphic to $\mathbb{G}_m^n$ such that the natural action of U on itself extends to the whole of X. (Note that this is stronger than the usual definition in the literature which only requires that U is isomorphic to $\mathbb{G}_m^n$ after passing to the algebraic closure). In the context of toric varieties, invariant always refers to the action of the torus $\mathbb{G}_m^n$. An invariant divisor is a prime divisor which is invariant under the action of the torus. We will say that a log pair (X, D) is *toric* if X is a toric variety and D is the sum of the invariant divisors.

Every toric variety has a description in terms of fans. We will use the notation of [8].

Lemma 2.3.1. *Let X be a $\mathbb{Q}$-factorial projective toric variety of dimension n and let V be a closed irreducible invariant subset. Let D be an invariant divisor (so that D is a prime divisor).*

Then we may find a divisor $B \geq 0$ on X, supported on the invariant divisors which contain neither D nor V, such that $A = B|_V$ is very ample and every element of the linear system $|A|$ lifts to X.

Proof. We may as well assume that D does not contain V. If $F \subset N_{\mathbb{R}}$ is the fan corresponding to X then D is given by a one dimensional cone ρ in F. If $P_{\mathbb{R}}$ is the quotient vector space of $N_{\mathbb{R}}$ corresponding to V then the image of ρ in $P_{\mathbb{R}}$ is either a ray or zero. Let W be the closed invariant subset of V determined by the smallest cone which contains the image.

Let $A \geq 0$ be a very ample divisor on V supported on the invariant divisors which do not contain W. A determines a continuous piecewise linear function f_A on $P_{\mathbb{R}}$, which is non-negative as $A \geq 0$. By composition we get a continuous piecewise linear function g on $N_{\mathbb{R}}$ which in turn corresponds to a divisor B supported on the invariant divisors. $B \geq 0$ as g is non-negative and the restriction to V is A, as g is the composition of the natural projection and f_A.

It is enough to lift every invariant element $A' \in |A|$. Note that, in the notation of [8], $M(\sigma) \subset M$ is naturally the space of monomials on V, where σ is the cone corresponding to V. We may find $u \in M(\sigma)$ such that

$$A' = A + (\chi^u).$$

On the other hand the zeroes and poles of χ^u, as a rational function on X, don't contain V. Note that $f' = f + u$ is the continuous piecewise linear function corresponding to A' and f' takes only non-negative values. Then $g' = g + u$ is the composition of the naturally projection and f', and so g' only take non-negative values. Hence

$$B' = B + (\chi^u) \in |B|$$

is a divisor on X which restricts to A'. □

We will need the next couple of results in the case when the groundfield is not necessarily algebraically closed:

Lemma 2.3.2. *Let k be any field. Let X and Y be two proper varieties and let (X, D) and (Y, G) be two log pairs. Let $\pi \colon X \dashrightarrow Y$ be a birational contraction and $G = \pi_* D$.*

(1) If (X, D) is toric and Y is projective then both (Y, G) and π are toric.
(2) If (Y, G) is toric, X is projective and the exceptional divisors of π are components of D that correspond to toric valuations of Y then both (X, D) and π are toric.

Proof. Suppose (X, D) is toric. If H is an ample divisor on Y then $\pi^* H$ is linearly equivalent to an invariant divisor. As $Y = \operatorname{Proj}(X, \pi^* H)$ then both (Y, G) and π are toric. This is (1).

Now suppose (Y, G) is toric, the exceptional divisors of π are components of D and correspond to toric valuations of Y. We may find a toric pair (Z, H) and a birational morphism $f \colon Z \longrightarrow Y$ whose only exceptional divisors correspond to these toric valuations. As the induced birational map $X \dashrightarrow Z$ is an isomorphism in codimension one, it is a birational contraction. Thus (2) follows from (1). □

We will need an extension of (2.3.2) to the case when X and Y are not projective, only proper. We start with:

Lemma 2.3.3. *Let (X, D) be a log pair over a field k and let bars denote extension to the algebraic closure $\bar{k}$ of k.*

Then (X, D) is toric if and only if $U = X - D$ is isomorphic to $\mathbb{G}_m^n$ and $(\bar{X}, \bar{D})$ is toric.

Proof. One direction is clear.

Otherwise if U is a torus then it acts on itself and we get a morphism

$$U \times U \longrightarrow U.$$

Now $U \times U$ is birational to $U \times X$ and so we get a rational map

$$f \colon U \times X \dashrightarrow X.$$

This induces a rational map

$$\bar{f} \colon \bar{U} \times \bar{X} \dashrightarrow \bar{X}.$$

As $\bar{X}$ is toric, $\bar{f}$ is in fact a morphism. But then f is a morphism. $\square$

Lemma 2.3.4. *Let k be any field. Let Y be a proper variety and let $\pi \colon Y \longrightarrow X$ be a birational morphism of normal varieties.*

If Y is toric then both X and π are toric.

Proof. We first prove this result using the additional hypothesis that k is algebraically closed.

Replacing Y by a toric resolution, we may assume that Y is smooth and projective. In particular π is projective. Let $U \subset Y$ be the torus. By assumption there is an action

$$U \times Y \longrightarrow Y \qquad \text{given by} \qquad (u, y) \longrightarrow u \cdot y.$$

By composition there is a morphism

$$f \colon U \times Y \longrightarrow X.$$

Since $U \times X$ is birational to $U \times Y$ there is an induced rational map

$$g \colon U \times X \dashrightarrow X.$$

We check that g is a morphism.

Suppose that y_1 and y_2 are two points of Y, with the same image in X. It suffices to check that $f(u, y_1) = f(u, y_2)$ for all points $u \in U$. As π is projective and birational and X is normal the fibres of π are connected. Then y_1 and y_2 are connected by a chain of curves C in Y which are contracted by π. As the torus U is connected the components of C and of $u \cdot C$ are numerically equivalent. But then $u \cdot y_1$ and $u \cdot y_2$ belong to the connected curve $u \cdot C$ which is contracted by π. Thus

$f(u, y_1) = f(u, y_2)$, for all $u \in U$ and so there is an induced morphism g.

It is clear that g defines an action of U on X. As π is birational $U \longrightarrow \pi(U)$ is an isomorphism. Thus X contains a torus and the natural action of the torus extends to U. Therefore X is a toric variety.

Now suppose that k is not algebraically closed. Let U be the open subset of Y isomorphic to $\mathbb{G}_m^n$. As $\bar{\pi}\colon \bar{Y} \longrightarrow \bar{X}$ is a toric morphism the restriction of $\bar{\pi}$ to $\bar{U}$ is an isomorphism, so that $\bar{\pi}(\bar{U})$ is an open subset of $\bar{X}$. But then the restriction of π to U is an isomorphism and so $\pi(U)$ is an open subset of X isomorphic to $\mathbb{G}_m^n$. It follows that X is toric by (2.3.3) and it is easy to conclude that π is toric. □

We now return to assuming that the groundfield is algebraically closed.

Lemma 2.3.5. *Let X be a proper variety and let (X, Δ) be a log canonical pair such that $-(K_X + \Delta)$ is nef. Let l be a nonnegative integer. Suppose that (Y, Γ) is a divisorially log terminal model of (X, Δ), $\pi\colon Y \dashrightarrow X$, whose existence is guaranteed by* (2.2.5).

If (Y, G) is a toric pair, where $G \geq \lfloor \Gamma \rfloor$ and at most l components of G are not in Γ, then (X, D) is a toric pair, where $D = \pi_* G \geq \lfloor \Delta \rfloor$ and at most l components of D are not in Δ.

Proof. It suffices to prove that (X, D) is a toric pair. Note that (X, D) is log canonical and

$$K_Y + G = \pi^*(K_X + D),$$

so that $K_X + D$ is $\mathbb{Q}$-linearly equivalent to zero. In particular a valuation ν is a log canonical place of (Y, G) if and only if it is a log canonical place of (X, D).

Let (Z, L) be a toric resolution of (Y, G). Then the exceptional divisors of $Z \longrightarrow Y$ have log discrepancy zero, so that the induced birational map $Z \dashrightarrow X$ is a divisorially log terminal model of (X, D). Replacing (Y, G) by (Z, L) we may assume that Y is smooth. Let $f\colon W \longrightarrow X$ be a divisorially log terminal modification of (X, D), (2.2.3), so that f is a projective morphism but W need not be projective. We may write

$$K_W + C = f^*(K_X + D),$$

where C is the strict transform of D plus the exceptionals. By (2.3.4) it suffices to prove that (W, C) is toric. As f only extracts divisors of log discrepancy zero which also have log discrepancy zero for (Y, G), possibly blowing up Y, we may assume that the induced rational map

$Y \dashrightarrow W$ is a birational contraction. Replacing (X,D) by (W,C) we may assume that X is $\mathbb{Q}$-factorial and X is kawamata log terminal.

Let $\nu_1, \nu_2, \dots, \nu_k$ be the set of valuations corresponding to the exceptional divisors of π. Then the centres of $\nu_1, \nu_2, \dots, \nu_k$ are components of G and so $\nu_1, \nu_2, \dots, \nu_k$ are log canonical places. We may find a modification $f\colon W \longrightarrow X$ such that the exceptional divisors of f are precisely the centres of $\nu_1, \nu_2, \dots, \nu_k$, where W is $\mathbb{Q}$-factorial and kawamata log terminal. Replacing (X,D) by (W,C) once again we may assume that X is isomorphic to Y in codimension one.

The result now follows by [5, Corollary 2]. $\square$

2.4. **Calculus of the complexity.** In §1 we defined the complexity $c(X,\Delta)$ and the absolute complexity $\gamma(X,\Delta)$ for any log pair (X,Δ). It is not hard to see that the infimum is achieved for the complexity as there are only finitely many partitions of the set of prime divisors contained in the support of Δ. It is immediate from the definitions that

$$c(X,\Delta) \leq \gamma(X,\Delta).$$

Lemma 2.4.1. *Let X be a proper variety and let (X,Δ) be a log canonical pair.*

If $\pi\colon Y \dashrightarrow X$ is a divisorially log terminal model,

$$K_Y + \Gamma = \pi^*(K_X + \Delta),$$

then the complexity (respectively absolute complexity) of (Y,Γ) is at most the complexity (respectively absolute complexity) of (X,Δ).

Proof. Let $\sum_{i=1}^m a_i S_i$ be a decomposition of Δ. Let R_i be the strict transform of S_i, $1 \leq i \leq m$ and let $E_1, E_2, \dots, E_k$ be the exceptional divisors.

Let

$$T_i = \begin{cases} R_i & \text{if } 1 \leq i \leq m \\ E_{i-m} & \text{if } m < i \leq m+k \end{cases}$$

and

$$b_i = \begin{cases} a_i & \text{if } 1 \leq i \leq m \\ 1 & \text{if } m < i \leq m+k. \end{cases}$$

Then $\sum b_i T_i$ is a decomposition of Γ. The sum e of the coefficients of $\sum b_i T_i$ is $d+k$. $T_1, T_2, \dots, T_{m+k}$, modulo algebraic equivalence, span a vector space of dimension at most $r+k$. Thus the complexity of the decomposition $\sum b_i T_i$ is at most

$$n + (r+k) - (d+k) = n + r - d,$$

which is the complexity of the decomposition given by $\sum a_i S_i$. Thus the complexity of (Y, Γ) is at most the complexity of (X, Δ). The absolute case is similar and easier. $\square$

Definition 2.4.2. *Let $(x \in X, \Delta)$ be the germ of a log pair. A **local decomposition** of Δ is an expression of the form*

$$\sum a_i D_i \leq \Delta,$$

*where $D_i \geq 0$ are $\mathbb{Z}$-Weil, $\mathbb{Q}$-Cartier divisors and $a_i \geq 0$, $1 \leq i \leq k$. The **local complexity** of this decomposition is $n - d$, where n is the dimension of X and d is the sum of $a_1, a_2, \dots, a_k$.*

The following lemma establishes a local version of (1.2). The proof is adapted from the proof of [20, 18.22]:

Lemma 2.4.3. *Let $(x \in X, \Delta)$ be the germ of a log canonical pair where X has dimension n and let $\sum a_i D_i \leq \Delta$ be a local decomposition. Assume that K_X and $D_1, D_2, \dots, D_k$ are Cartier.*

If $\gamma = n - \sum a_i = n - d$ is the local complexity then

(1) $\gamma \geq 0$.

(2) If $\gamma < 1$ then, possibly re-ordering $D_1, D_2, \dots, D_k$, there is an integer $m \geq n - \lfloor 2\gamma \rfloor \geq 0$ such that

$$(X, D_1 + D_2 + \cdots + D_m)$$

is log smooth, and

$$\lfloor \Delta \rfloor \leq D_1 + D_2 + \cdots + D_m.$$

(3) If $\gamma < \frac{3}{2}$ then either X is smooth at x or has a cA_l singularity at x.

Proof. We proceed by induction on n. All claims are clear for $n = 1$ and so we assume that $n \geq 2$.

Fix a log resolution $\pi \colon Y \longrightarrow X$ of (X, Δ), with exceptional divisors $E_1, E_2, \dots, E_l$. Possibly blowing up further, we may assume that no two components of the strict transform of Δ intersect. Let f be a general linear combination of $g_1, g_2, \dots, g_k$, the functions defining $D_1, D_2, \dots, D_k$. Let S be the divisor cut out by f. As S specialises to D_i, for each i, it follows that

$$\operatorname{mult}_{E_j} S \leq \operatorname{mult}_{E_j} D_i,$$

for each $1 \leq i \leq k$ and $1 \leq j \leq l$. It also follows, by Bertini, that π is a log resolution of $(X, \Delta + S)$. For any $0 \leq b_i \leq a_i$ such that $\sum b_i = b \leq 1$, it follows that the pair $(X, \Phi = bS + \sum (a_i - b_i) D_i)$ is log canonical, and the local complexity of the indicated decomposition is γ.

Suppose that $0 < b_i \neq a_i$, and $b = 1$, so that

$$\sum_{i : b_i \neq 0} a_i > 1.$$

Let $V \subset X$ be a codimension two subset. By Bertini, S is only singular along V if every component of D_i such that $b_i \neq 0$ contains V. In this case, $\Phi - S > 0$. As the pair (X, Φ) is log canonical in a neighbourhood of the generic point of V, S must be smooth at the generic point of V, using the classification of log canonical surface singularities. Thus S is R_1. By [1, 0.3] S is also S_2. Thus S is normal. In particular if $d > 1$ we may pick $b_1, b_2, \dots, b_k$ so that $b = 1$ and S is normal.

As S is Cartier and normal, X is smooth in a neighbourhood of every codimension two point of X contained in S. In particular

$$(K_X + S)|_S = K_S.$$

and the restriction of D_i to S is a divisor with integer coefficients. Therefore we may write

$$(K_X + \Phi)|_S = K_S + \Psi,$$

where $(S, \sum(a_i - b_i)D_i|_S \leq \Psi)$ is log canonical and the local complexity is at most γ, see [20, 17.2].

Now suppose that $\gamma < 1$. As $n \geq 2$ then $d > 1$ and so we may choose $b_1, b_2, \dots, b_k$ so that S is normal. By induction S is smooth. As S is Cartier X is smooth. Then $\operatorname{mult}_x \Delta \leq n$ as (X, Δ) is log canonical. In particular $\gamma \geq 0$ so that (1) holds. Further every component of $\lfloor \Delta \rfloor$ is smooth. Therefore, if $\lfloor \Delta \rfloor \neq 0$ then we can restrict to a component of $\lfloor \Delta \rfloor$ and apply induction on n to conclude that (2) holds.

Thus we may assume that $\lfloor \Delta \rfloor = 0$. If we restrict to S and apply adjunction it follows that there are $m - 1 = n - 1 - \lfloor 2\gamma \rfloor$ components $D_1, D_2, \dots, D_{m-1}$ of Δ such that $(X, D_1 + D_2 + \dots + D_{m-1} + S)$ is log smooth. It follows that the images of $g_1, g_2, \dots, g_k$ span a subspace of $T^*_x X$, the dual of the Zariski tangent space, of dimension at least $m = n - \lfloor 2\gamma \rfloor$. It is then easy to see that (2) holds.

Now suppose that $\gamma < 3/2$. If $n \geq 3$ then $d > 1$. By definition of compound singularities it suffices to prove that S has a cA_l singularity. By induction we may assume that $n = 2$ and we have to show that X has an A_l singularity. As K_X is Cartier and X is a normal surface, X is Gorenstein. As $\Delta \neq 0$ it follows that X is kawamata log terminal so that X is canonical. Thus X has du Val singularities. We may also assume that $\Delta = dD$, where $D = S$ is a prime Cartier divisor.

If $\pi \colon Y \longrightarrow X$ is the minimal desingularisation of the surface X then $K_Y = \pi^* K_X$. Let G be the strict transform of D and let $E_1, E_2, \dots, E_l$

be the exceptional divisors. Since D is Cartier, we have

$$f^*D = G + \sum m_i E_i$$

where $m_1, m_2, \dots, m_l$ are positive integers.

The log discrepancy of E_i with respect to $K_X + \Delta = K_X + dS$ is

$$1 - dm_i.$$

As (X, Δ) is log canonical and $d > \frac{1}{2}$, we must have $m_i = 1$, for all $1 \leq i \leq l$. Hence

$$\begin{aligned} 0 &= f^*D \cdot E_j \\ &= (G + \sum E_i) \cdot E_j \\ &= G \cdot E_j + \sum_{i \neq j} E_i \cdot E_j + E_j^2 \\ &= G \cdot E_j + \delta(E_j) - 2 \\ &\geq \delta(E_j) - 2, \end{aligned}$$

where $\delta(E_j)$ is the degree of the vertex corresponding to E_j in the dual graph of the resolution. It follows that every vertex in the dual graph has degree at most 2 and so X has an A_l singularity. □

Lemma 2.4.4. *Let $(x \in X, \Delta)$ be the germ of a log canonical pair where X has dimension n and let $\sum a_i D_i \leq \Delta$ be a local decomposition. Assume that K_X and $D_1, D_2, \dots, D_k$ are $\mathbb{Q}$-Cartier.*

If $\gamma = n - \sum a_i = n - d$ is the local complexity then $\gamma \geq 0$.

Proof. As K_X and $D_1, D_2, \dots, D_k$ are $\mathbb{Q}$-Cartier, there is a composition of cyclic covers, $\pi \colon Y \longrightarrow X$, which are étale in codimension one, such that K_Y and $G_1, G_2, \dots, G_k$ are Cartier, where G_i is the strict transform of D_i.

If Γ is the strict transform of Δ then

$$K_Y + \Gamma = \pi^*(K_X + \Delta).$$

If we pick a point $y \in Y$ above $x \in X$, then $(y \in Y, \Gamma)$ is the germ of a log canonical pair, $\sum a_i G_i \leq \Gamma$ is a local decomposition, with local complexity γ and the result follows by (1) of (2.4.3). □

Lemma 2.4.5. *Let (X, Δ) a log canonical pair, where X is a projective variety of dimension n. Let $\sum_{i \leq k} a_i S_i \leq \Delta$ be a decomposition.*

If $-K_X$ and $S_1, S_2, \dots, S_k$ all generate the same ray of the cone of effective divisors and this ray is also spanned by an ample divisor then $d = \sum a_i \leq n + 1$.

Proof. By assumption we may embed $X \subset \mathbb{P}^N$ such that S_i is proportional to a hyperplane section. Let Y be the cone over X with vertex y, let Γ be the cone over Δ and let D_i be the cone over S_i. Then (Y, Γ) is log canonical, $D_1, D_2, \dots, D_k$ are $\mathbb{Z}$-Weil $\mathbb{Q}$-Cartier divisors, and $\sum a_i D_i \leq \Gamma$ is a local decomposition of complexity $d - n - 1$. Thus $d \leq n + 1$ by (2.4.4). □

2.5. **Mori Dream Spaces.** Recall, cf. [15],

Definition 2.5.1. *Let X be a normal projective variety. We say that X is a* ***Mori dream space*** *if the following conditions hold:*

(1) X is $\mathbb{Q}$-factorial and $\operatorname{Pic}(X)_{\mathbb{Q}} = N^1(X)_{\mathbb{Q}}$*;*
(2) the cone of nef divisors, $\operatorname{Nef}(X)$*, is the affine hull of finitely many semi-ample divisors;*
(3) there exist finitely many small birational maps $f_i \colon X \dashrightarrow X_i$, such that each X_i satisfies (1) and (2) and the closure of the cone of movable divisors, $\operatorname{Mov}(X)$*, is the union of the cones* $f_i^* \operatorname{Nef}(X_i)$*.*

The Cox ring of a variety with finitely generated class group was originally defined in [15]; it is unique but ignores torsion in the class group. Subsequently [14] gave a refined definition which takes into account torsion in the class group. As we would like to allow torsion we will use this definition of the Cox ring.

We will need some of the basic properties of the Cox ring, cf. [14] for more details and proofs. The Cox ring is an integrally closed ring when X is normal and projective, [14, 2.2 (i)]. Perhaps the most basic observation is:

Theorem 2.5.2. *Let X be a normal $\mathbb{Q}$-factorial projective variety such that* $\operatorname{Pic}(X)_{\mathbb{Q}} = N^1(X)_{\mathbb{Q}}$.

Then X is a Mori dream space if and only if the Cox ring is finitely generated.

Proof. First note that the Cox ring, in the sense of [15], is finitely generated if and only if the Cox ring, in the sense of [14], is finitely generated, in which case this is [15, 2.9]. □

One can use this to give many examples of Mori dream spaces:

Lemma 2.5.3. *Let X be a normal projective variety.*

The following are equivalent

(1) We may find a kawamata log terminal pair (X, Δ) such that $-(K_X + \Delta)$ is ample.

(2) *We may find a kawamata log terminal pair (X, Δ) such that $-(K_X + \Delta)$ is big and nef.*
(3) *We may find a kawamata log terminal pair (X, Δ) such that $K_X + \Delta$ is numerically trivial and Δ is big.*

In particular, if X is $\mathbb{Q}$-factorial then X is a Mori dream space.

Proof. (1) clearly implies (2). If (X, Δ) is kawamata log terminal and $-(K_X+\Delta)$ is big and nef then $-(K_X+\Delta)$ is semiample. If $\pi\colon Y \longrightarrow X$ is a log resolution of (X, Δ) then we can find $B \geq 0$, $B \sim_{\mathbb{R}} -(K_X+\Delta)$, such that B does not contain the centre of any exceptional divisor, so that π^*B is equal to the strict transform of B. In particular $(X, \Delta+B)$ is kawamata log terminal. Thus (2) implies (3).

Suppose that (X, Δ) is kawamata log terminal, $K_X+\Delta$ is numerically trivial and Δ is big. We may find an ample $\mathbb{Q}$-divisor A and a divisor $B \geq 0$ such that

$$\Delta \sim_{\mathbb{R}} A + B.$$

Pick $\epsilon > 0$ such that $(X, (1-\epsilon)\Delta+\epsilon B)$ is kawamata log terminal. Then

$$-(K_X + (1 - \epsilon)\Delta + \epsilon B) \sim_{\mathbb{R}} \epsilon A,$$

is ample, so that (3) implies (1).

The last assertion is [3, 1.3.2]. □

If X is a Mori dream space then let $Y = \operatorname{Spec} R$. If D is a prime divisor on X then D determines an element $d \in R$ determined by the natural inclusion

$$\mathcal{O}_X \longrightarrow \mathcal{O}_X(D).$$

This in turn determines a Cartier divisor G on Y which is linearly equivalent to zero. The ring R is naturally graded by the class group $A_{n-1}(X)$. There is a unique closed point $p \in Y$ corresponding to the unique maximal homogeneous ideal and $p \in G$. The grading corresponds to an action of the diagonalisable group $H = \operatorname{Spec} \mathbb{C}[A_{n-1}(X)]$. X is a geometric quotient of Y by the action of H and the divisor D is naturally the image of the associated Cartier divisor G on Y.

We will need a small strengthening of [15, 2.10]:

Theorem 2.5.4. *Let X be a normal $\mathbb{Q}$-factorial projective variety of dimension n.*

Then X is toric if and only if the Cox ring is a polynomial ring generated by $n + \rho(X)$ variables, in which case the invariant divisors correspond to the coordinate hyperplanes.

Proof. If X is a toric variety then the Cox ring is the homogeneous coordinate ring of X and the Cox ring is a polynomial ring with $n+\rho(X)$

variables, which correspond to the invariant divisors on X, cf. the discussion after the proof of [14, 2.2].

Now suppose that the Cox ring is a polynomial ring. Then X is a Mori dream space. In particular its divisor class group $A_{n-1}(X)$ is a finitely generated abelian group and the Cox ring is graded by the class group. In this case X is the GIT quotient of affine space $\mathbb{A}^m$ by a diagonalisable group H, the product of a torus and a finite abelian group, [14, 2.2]. Therefore X is a toric variety. The dimension of H is the rank of the abelian group $A_{n-1}(X)$, which is $\rho(X)$ since X is $\mathbb{Q}$-factorial. Thus the dimension of the Cox ring is $n + \rho(X)$. $\square$

We will also need:

Lemma 2.5.5. *Let X be a $\mathbb{Q}$-factorial projective variety. Suppose that X is a Mori dream space and let $R = \operatorname{Cox}(X)$ be the Cox ring.*

If $Y = \operatorname{Spec} R$ then K_Y is Cartier. In particular if Y is Cohen-Macaulay then Y is Gorenstein.

Proof. Let $H = \operatorname{Spec} \mathbb{C}[A_{n-1}(X)]$. According to [14, 2.2] it suffices to check that K_Y is H-invariant. It also follows from [14, 2.2] that there is a universal H-torsor $q \colon \hat{X} \longrightarrow X$ and it suffices to prove that $K_{\hat{X}}$ is H-invariant.

The group H decomposes as a torus and a finite abelian group. The morphism q then decomposes as a torus bundle followed by an étale cover. It follows that $K_{\hat{X}} = q^* K_X$ so that $K_{\hat{X}}$ is H-invariant. $\square$

3. Local to global

Theorem 3.1. *Let X be a $\mathbb{Q}$-factorial projective variety with kawamata log terminal singularities and let (X, Δ) be a log canonical pair. Suppose that $-(K_X + \Delta)$ is nef and $\sum_{i \leq k} a_i S_i$ is a decomposition of complexity c less than one for Δ.*

If X is a Mori dream space then $c \geq 0$ and there is a divisor D such that (X, D) is a toric pair, where $D \geq \lfloor \Delta \rfloor$ and all but $\llcorner 2c \lrcorner$ components of D are elements of the set $\{ S_i \,|\, 1 \leq i \leq k \}$.

Theorem 3.2. *Let X be a $\mathbb{Q}$-factorial projective variety. Suppose that (X, Δ) is a log canonical pair such that $K_X + \Delta$ is numerically trivial. Let $\sum_{i \leq k} a_i S_i$ be a decomposition of Δ with complexity less than 1.*

If X is a Mori dream space then $S_1, S_2, \dots, S_k$ generate $A_{n-1}(X)_{\mathbb{Q}}$.

Lemma 3.3. *Let X be a $\mathbb{Q}$-factorial kawamata log terminal projective variety of dimension n and let (X, Δ) be a log canonical pair. Let $D = \sum a_i S_i \leq \Delta$ be a decomposition of Δ.*

If

(1) $K_X + \Delta$ *is numerically trivial,*
(2) $d = \sum_{i=1}^{k} a_i > n$, *and*
(3) $S_1, S_2, \dots, S_k$ *all span the same ray of the cone of effective divisors*

then the Picard number of X *is one.*

Proof. Let $\Theta = \Delta - D$. We run the $(K_X + \Theta)$-MMP with scaling of some ample divisor.

Let $f \colon X \dashrightarrow Y$ be a step of this MMP. f is D-positive and as the components of S span the same ray of the cone of effective divisors, it follows that f is S_i-positive, for every $1 \leq i \leq k$. Let $T_i = f_* S_i$.

Suppose that f is a divisorial contraction. If V is the image of the exceptional divisor E then T_i contains V. If $\Gamma = f_*\Delta$ then (Y, Γ) is log canonical and the local complexity about a point of V is negative. This is not possible by (2.4.4).

If f is a flip then $\rho(X) = \rho(Y)$ and $T_1, T_2, \dots, T_k$ all span the same ray of the cone of effective divisors. We replace X by Y in this case. (2.2.2) implies that after finitely many flips f must be a Mori fibre space. Let F be the general fibre and let Σ be the restriction of Δ to F. Then (F, Σ) is log canonical. As $S_1, S_2, \dots, S_k$ dominate Y, the sum of the coefficients of Σ is greater than n. (2.4.5) implies that F has dimension n. But then Y is a point and X has Picard number one. □

Proof of (3.2). We proceed by induction on the dimension r of the span of $S_1, S_2, \dots, S_k$ in $A_{n-1}(X)_{\mathbb{Q}}$.

Suppose $r = 1$. Let $\pi \colon Y \longrightarrow X$ be a divisorially log terminal modification of X. As X is $\mathbb{Q}$-factorial, the exceptional locus of π is a divisor. As the log discrepancy with respect to K_X of every exceptional divisor is zero it follows that the centre of every divisor is not contained in the support of Δ. If we write

$$K_Y + \Gamma = \pi^*(K_X + \Delta),$$

then (Y, Γ) is log canonical, $K_Y + \Gamma$ is numerically trivial, Y is $\mathbb{Q}$-factorial, $G = \sum a_i T_i \leq \Gamma$ is a decomposition of Γ and the divisors $T_1, T_2, \dots, T_k$ all span the same ray of the cone of effective divisors, where T_i is the strict transform of S_i. If we apply (3.3) to (Y, Γ) then we conclude that the Picard number of Y is one, so that the Picard number of X is one as well.

Otherwise, we may assume that S_1 and S_2 are linearly independent in $A_{n-1}(X)_{\mathbb{Q}}$. Pick integers m_1 and m_2 such that $m_1 S_1$ and $m_2 S_2$ are Cartier, and neither $m_1 S_1 - m_2 S_2$ nor $m_2 S_2 - m_1 S_1$ is pseudo-effective.

Consider the $\mathbb{P}^1$-bundle

$$Y = \mathbb{P}(\mathcal{O}_X(m_1S_1) \oplus \mathcal{O}_X(m_2S_2)).$$

Let $f\colon Y \longrightarrow X$ be the structure morphism. Then Y is a $\mathbb{Q}$-factorial projective variety. There are two distinguished sections, which we will call E_0 and E_∞. Set $\Gamma = f^*\Delta + E_0 + E_\infty$. Adjunction implies that (Y, Γ) is a log canonical pair, and that $K_Y + \Gamma$ is numerically trivial. Note that $\rho(Y) = \rho(X) + 1$. Finally, Y is a Mori Dream Space because the Cox ring of Y is isomorphic as a ring to the Cox ring of X with two variables adjoined, corresponding to the sections E_0 and E_∞, cf. [4, 3.2].

As both $m_1S_1 - m_2S_2$ and $m_2S_2 - m_1S_1$ are not pseudo-effective, $E_0|_{E_0}$ and $E_\infty|_{E_\infty}$ are not pseudoeffective. Thus $D = E_0 + E_\infty$ has Kodaira dimension zero. As Y is a Mori dream space we may run $g\colon Y \dashrightarrow Z$ the D-MMP and the image of D is semiample. Thus the birational map g contracts E_0 and E_∞ and no other divisors. Z is a $\mathbb{Q}$-factorial projective variety.

Note that $K_Z + \Psi$ is numerically trivial and $(Z, \Psi = g_*\Gamma)$ is log canonical. Furthermore

$$\rho(Z) = \rho(Y) - 2 = \rho(X) - 1.$$

If $T_i = f^*S_i$ then the dimension of the space spanned by $T_1, T_2, \dots, T_k$ in $A_{n-1}(Y)_{\mathbb{Q}}$ is equal to r. Let $C_i = g_*T_i$. Y is a $\mathbb{P}^1$ over X, so

$$A_{n-1}(Y) = f^*A_{n-1}(X) \oplus \mathbb{Z},$$

(see [7, 3.3] for the general case and [13, 2.3] for the case of a ruled surface), where the second factor is spanned by any section of f. As

$$E_0|_{E_0} \sim m_1S_1 - m_2S_2 \qquad \text{and} \qquad E_\infty|_{E_0} \sim 0,$$

it follows that

$$E_0 = E_\infty + f^*(m_1S_1 - m_2S_2).$$

As $A_{n-1}(Z)$ is obtained from $A_{n-1}(Y)$ by setting the classes of E_0 and E_∞ to zero, it follows that $m_1C_1 - m_2C_2$ is linearly equivalent to zero in Z, so that $C_1, C_2, \dots, C_k$ span a vector space of dimension $r - 1$ in $A_{n-1}(Z)_{\mathbb{Q}}$. Further, we can identify $A_{n-1}(Z)_{\mathbb{Q}}$ with

$$\frac{A_{n-1}(Y)_{\mathbb{Q}}}{\langle E_0, E_\infty \rangle} = \frac{A_{n-1}(X)_{\mathbb{Q}}[E_\infty]}{\langle E_\infty, E_\infty + m_1S_1 - m_2S_2 \rangle} = \frac{A_{n-1}(X)_{\mathbb{Q}}}{\langle m_1S_1 - m_2S_2 \rangle}.$$

By [25, 1.1], Z is a Mori dream space, as Y is a Mori dream space. By induction $C_1, C_2, \dots, C_k$ generate $A_{n-1}(Z)_{\mathbb{Q}}$. Hence $S_1, S_2, \dots, S_k$ span $A_{n-1}(X)_{\mathbb{Q}}$. $\square$

Proof of (3.1). Since $-(K_X+\Delta)$ is nef, and X is a Mori dream space, we can find $B \sim_{\mathbb{R}} -(K_X+\Delta)$ such that $(X, \Delta+B)$ is log canonical. By (3.2), the components of any decomposition of $(X, \Delta+B)$ with complexity less than 1 generate $A_{n-1}(X)_{\mathbb{Q}}$. The pair (X, Δ) has a decomposition with complexity less than 1, and this is a decomposition of $(X, \Delta+B)$. Thus $r=\rho$, where r is the rank of the group generated by the $S_1, S_2, \dots, S_k$ and ρ is the Picard number.

Let $Y = \operatorname{Spec} R$ where $R = \operatorname{Cox}(X)$ is the Cox ring. Then Y has dimension $n+\rho$. Let T_i be the divisor corresponding to S_i and let $\Gamma = \sum a_i T_i$. As S_i has integer coefficients, T_i is a Cartier divisor and every component $T_1, T_2, \dots, T_k$ contains the point p corresponding to the unique maximal ideal which is homogeneous, [14, 2.2]. By [18, 1.1] the pair (Y, Γ) is log canonical (note that even though the divisor Γ does not appear in the statement of [18, 1.1], it does appear implicitly, see the discussion in the second paragraph on page 2; also note that, as observed in [18, 2.5], their result applies to the Cox ring, as defined in [14]).

(2.4.3) implies that Y is smooth, $c \geq 0$ and at least $m = \dim Y - \lfloor 2c \rfloor$ components $G_1, G_2, \dots, G_m$ of $T_1, T_2, \dots, T_k$, including all such components with coefficient 1, are smooth at p and intersect transversally.

As Y is smooth and the Cox ring is graded, it follows that Y is isomorphic to affine space of dimension $n+\rho$ and $G_1, G_2, \dots, G_m$ are coordinate hyperplanes. (2.5.4) implies that X is a toric variety and the divisors $S_1, S_2, \dots, S_k$ which correspond to $G_1, G_2, \dots, G_m$ are invariant divisors. □

4. Log divisors of small numerical dimension

Proposition 4.1. *Assume* $(1.2)_{n-1}$*, that is, assume* (1.2) *when* X *has dimension* $n-1$*.*

Let (X, Δ) *be a divisorially log terminal pair where* X *is a* $\mathbb{Q}$*-factorial projective variety of dimension* n*.*

If $-(K_X+\Delta)$ *is nef then we may find an ample divisor* A *and a divisor* $0 \leq \Delta_0 \leq \Delta$ *such that* $K_X + A + \Delta_0$ *is pseudo-effective, no component of* $N(X, K_X + A + \Delta_0)$ *is a component of* Δ_0*, and the numerical dimension of* $K_X + A + \Delta_0$ *is at most the complexity of* (X, Δ)*.*

Corollary 4.2. *Assume* $(1.2)_{n-1}$*.*

Let X *be a* $\mathbb{Q}$*-factorial projective variety of dimension* n*. Suppose* (X, Δ) *is a divisorially log terminal pair such that* $-(K_X+\Delta)$ *is nef. Let* $\gamma_0 \in (0, 2)$*.*

If the absolute complexity $\gamma(X, \Delta) < \gamma_0$ then there is a log canonical pair (Y, Γ) such that $-(K_Y + \Gamma)$ is ample, $\gamma(Y, \Gamma) < \gamma_0$ and Y is a $\mathbb{Q}$-factorial projective variety birational to X.

Lemma 4.3. *Assume $(1.2)_{n-1}$.*

Let (X, Δ) be a divisorially log terminal pair where X is a $\mathbb{Q}$-factorial projective variety of dimension n. Let A be an ample divisor such that $K_X + A + \Delta$ is pseudo-effective and let $\phi \colon X \dashrightarrow Z$ be the ample model of $K_X + A + \Delta$. Assume that no component of $N = N(X, K_X + A + \Delta)$ is a component of Δ.

If the dimension of Z is greater than the complexity of (X, Δ) then we may find a component P of Δ which is vertical.

Proof. Let $f \colon X \dashrightarrow Y$ be a log terminal model of $K_X + A + \Delta$. Then there is a contraction morphism $g \colon Y \longrightarrow Z$. The divisors contracted by f are the components of N and so f does not contract any components of Δ. If $B = f_*A$ and $\Gamma = f_*\Delta$ then $(Y, B + \Gamma)$ is divisorially log terminal.

If F is the general fibre of g and Θ is the restriction of $B+\Gamma$ to F then (F, Θ) is log canonical and $K_F + \Theta$ is numerically trivial. Let $\sum a_i S_i$ be a decomposition of (X, Δ) which computes the complexity and let $d = \sum a_i$ be the sum of the coefficients. Let C_i be the restriction to F of the image of S_i. Then $\sum a_i C_i$ is a decomposition of (F, Θ), where the sum ranges over the indices i such that at least one component of S_i is horizontal. The rank s of the span of $C_1, C_2, \dots, C_k$ is at most the rank r of the span of $S_1, S_2, \dots, S_k$, the sum h of the coefficients of $C_1, C_2, \dots, C_k$ is at least the sum of the coefficients of the horizontal components of $S_1, S_2, \dots, S_k$ and $n = \dim X = \dim F + \dim Z$.

As we are assuming $(1.2)_{n-1}$, which implies $(1.3)_{n-1}$, the complexity of the pair (F, Θ) is non-negative,

$$\begin{aligned}
0 &\leq s + \dim F - h \\
&\leq r + \dim X - d + (\dim F - \dim X) + (d - h) \\
&\leq c(X, \Delta) - \dim Z + (d - h) \\
&< (d - h).
\end{aligned}$$

Thus $h < d$ so that there is an index i such that every component of S_i is vertical. In particular at least one component P of Δ is vertical. $\square$

Lemma 4.4. *Let (X, Δ) be a divisorially log terminal pair, where X is a $\mathbb{Q}$-factorial projective variety and $-(K_X + \Delta)$ is nef. Let A_0 be an ample divisor and let $0 \leq \Delta_0 \leq \Delta$ be a divisor such that $K_X + A_0 + \Delta_0$ is pseudo-effective, with numerical dimension k. Suppose Δ*

has a component P which is vertical for the ample model $\phi\colon X \dashrightarrow Z_0$ of $K_X + A_0 + \Delta_0$.

Then there is an ample divisor A_1 and a divisor $0 \leq \Delta_1 \leq \Delta$ such that $K_X + A_1 + \Delta_1$ is pseudo-effective and the numerical dimension of $K_X + A_1 + \Delta_1$ is less than k.

Proof. Set $M = -(K_X + \Delta)$. Let p be the coefficient of P in Δ. Pick λ minimal so that

$$\begin{aligned} K_X + A_1 + \Delta_1 &= K_X + \lambda A_0 + (1-\lambda)M + \lambda\Delta_0 + (1-\lambda)(\Delta - pP) \\ &= \lambda(K_X + A_0 + \Delta_0) + (1-\lambda)(K_X + M + \Delta - pP) \\ &= \lambda(K_X + A_0 + \Delta_0) - (1-\lambda)pP \end{aligned}$$

is pseudo-effective, where

$$A_1 = \lambda A_0 + (1-\lambda)M \qquad \text{and} \qquad \Delta_1 = \lambda\Delta_0 + (1-\lambda)(\Delta - pP).$$

Note that $\lambda > 0$. In particular A_1 is ample and $0 \leq \Delta_1 \leq \Delta$. On the other hand as P is vertical, (2.2.1) implies that $\lambda < 1$.

Let $A_t = (1-t)A_0 + tA_1$ and $\Delta_t = (1-t)\Delta_0 + t\Delta_1$, so that $K_X + A_t + \Delta_t$ is a convex linear combination of $K_X + A_0 + \Delta_0$ and $K_X + A_1 + \Delta_1$. In particular $K_X + A_t + \Delta_t$ is pseudo-effective. Let Z_t be the ample model of $K_X + A_t + \Delta_t$. Note that $K_X + A_t + \Delta_t$ is in the cone generated by $K_X + A_0 + \Delta_0$ and $-P$.

Suppose that P is in the stable base locus of $K_X + A_t + \Delta_t$. If

$$P_t + N_t = P(X, K_X + A_t + \Delta_t) + N(X, K_X + A_t + \Delta_t),$$

is the Zariski decomposition of $K_X + A_t + \Delta_t$ into its positive and negative parts then P is a component of N_s, $s \in [0,t]$ so that $P_t = (1-t)P_0$. In particular $Z_t = Z_0$ and so P is vertical over Z_t. If P is not in the stable base locus and $t < 1$ then (2.2.1) implies that P is vertical over Z_t, since $K_X + A_t + \Delta_t - \delta P$ is pseudo-effective, for $\delta > 0$ sufficiently small.

By [12, 3.3.2] we may find $\delta > 0$ such that $Y = Z_t$ is independent of $t \in (1-\delta, 1)$ and there is a contraction morphism $f\colon Y \longrightarrow Z_1$.

By what we just proved P is vertical for Y. On the other hand (2.2.1) implies that P is horizontal for Z_1. Thus f is not birational and so the dimension of Z_1 is less than the dimension of Y. In particular the numerical dimension of $K_X + A_1 + \Delta_1$ is less than k. □

Lemma 4.5. *Let (X, Δ) be a divisorially log terminal pair where X is a $\mathbb{Q}$-factorial projective variety and $M = -(K_X + \Delta)$ is nef. Let $\delta > 0$ be any positive real number.*

Let A be an $\mathbb{R}$-divisor and let $0 \leq \Delta_0 \leq \Delta$ (respectively $\Delta_0 \geq \alpha\Delta$ for some α, possibly negative). Suppose that $K_X + A + \Delta_0$ is pseudo-effective. Let

$$K_X + A + \Delta_0 = P + N = P(X, K_X + A + \Delta_0) + N(X, K_X + A + \Delta_0)$$

be the Zariski decomposition. We may write $N = N_0 + N_1$, where every component of N_0 is a component of Δ and no component of N_1 is a component of Δ.

Then we may find $t > 0$ and $\Delta_1 \leq \Delta \leq (1 + \delta)\Delta_1$ (respectively $\Delta_1 \geq (1 - \delta)\Delta$) such that

$$P(X, K_X + A_t + \Delta_1) = tP \qquad \text{and} \qquad N(X, K_X + A_t + \Delta_1) = tN_1,$$

where

$$A_t = tA + (1 - t)M.$$

Proof. We have

$$\begin{aligned} K_X + A + \Delta_0 &= P + N \\ K_X + M + \Delta &= 0. \end{aligned}$$

Given $t \in (0, 1]$,

$$\begin{aligned} K_X + A_t + t\Delta_0 + (1 - t)\Delta &= t(K_X + A + \Delta_0) + (1 - t)(K_X + M + \Delta) \\ &= tP + tN. \end{aligned}$$

It $t > 0$ is sufficiently small then

$$\Delta_1 = t\Delta_0 + (1 - t)\Delta - tN_0 \geq 0 \qquad \text{and} \qquad \Delta \leq (1 + \delta)\Delta_1.$$

When $0 \leq \Delta_0 \leq \Delta$, we have that $\Delta_1 \leq \Delta$. When $\Delta_0 \geq \alpha\Delta$, then $\Delta_1 \geq \Delta - t(1-\alpha)\Delta - tN_0$ and we can choose t so that $\Delta_1 \geq (1-\delta)\Delta$. $\square$

Proof of (4.1). Consider divisors of the form $K_X + A + \Delta_0$, where $0 \leq \Delta_0 \leq \Delta$ and A is ample. Let k be the minimum of the numerical dimension of pseudo-effective divisors of this form (note that we can always pick A so that $K_X + A$ is ample, so that $k \leq n$).

Pick $0 \leq \Delta_0 \leq \Delta$ and an ample divisor A such that $K_X + A + \Delta_0$ has numerical dimension k.

Suppose that k is bigger than the complexity of (X, Δ). Since the complexity is a continuous function of the coefficients of Δ, we may pick $\delta > 0$ so that the complexity of $(X, \frac{1}{1+\delta}\Delta)$ is less than k. If we replace Δ_0 by the divisor

$$\frac{1}{1 + \delta}\Delta \leq \Delta_1 \leq \Delta$$

of (4.5) we may assume that no component of Δ is a component of $N = N(X, K_X + A + \Delta_0)$ and that the complexity of $(X, A + \Delta_0)$ is less than k.

By (4.3) we may find a component P of Δ which is vertical for the ample model of $(X, A+\Delta_0)$. Then by (4.4) we can find an ample divisor A_1 and a divisor $0 \leq \Delta_1 \leq \Delta$ such that $K_X+A_1+\Delta_1$ is pseudo-effective and the numerical dimension is less than k, a contradiction. □

Proof of (4.2). Consider pseudo-effective divisors of the form $K_X+A+\Delta_0$, where $0 \leq \Delta_0 \leq \Delta$, A is ample and no component of $N(X, K_X+A+\Delta_0)$ is a component of Δ_0. Let k be the minimal numerical dimension of pseudo-effective divisors of this form. As

$$c(X, \Delta) \leq \gamma(X, \Delta) < 2,$$

by (4.1) we have $k \leq 1$ and if $k = 1$ then $c(X, \Delta) \geq 1$.

Pick $0 \leq \Delta_0 \leq \Delta$ and an ample divisor A such that $K_X + A + \Delta_0$ has numerical dimension k. By (4.5) we may assume that the absolute complexity of $(X, A+\Delta_0)$ is less than γ_0. Pick $\delta > 0$ such that $A+\delta\Delta_0$ is ample. Replacing A by $A+\delta\Delta_0$ and Δ_0 by $(1-\delta)\Delta_0$ we may assume that $(X, A + \Delta_0)$ is kawamata log terminal.

Let $\phi\colon X \dashrightarrow Z$ be the ample model of $K_X+A+\Delta_0$ and let $f\colon X \dashrightarrow Y$ be a log terminal model of $K_X+A+\Delta_0$. Then there is a contraction morphism $g\colon Y \longrightarrow Z$. If $\Gamma = f_*(\Delta_0 + A)$ then $\gamma(Y, \Gamma) < \gamma_0$.

Suppose that $k = 1$, that is, suppose Z is a curve. Then $c(X, \Delta) \geq 1$. It follows that the components of Δ span the group of Weil divisors modulo algebraic equivalence. If F is a general fibre of g and $\Theta = \Gamma|_F$ then (F, Θ) is log canonical and $K_F + \Theta$ is numerically trivial. The natural map which assigns to a divisor on Y its restriction to F has a non-trivial kernel, since F restricts to zero, so that the image of the space of all Weil divisors modulo algebraic equivalence has dimension at most $\rho(Y)-1$. Therefore the dimension r of the span of the components of Θ is at most $\rho(Y) - 1$. By minimality of k, (4.4) implies that every component of Γ dominates Z. Therefore the sum t of the coefficients of Θ is at least the sum d of the coefficients of Δ. Hence

$$\begin{aligned} c(F, \Theta) &\leq \dim F + r - t \\ &\leq (\dim Y - 1) + (\rho(Y) - 1) - d \\ &= \gamma(Y, \Gamma) - 2 \\ &< 0, \end{aligned}$$

a contradiction. Hence $k = 0$ and we may apply (2.5.3). □

5. Reduction to Mori dream spaces

Lemma 5.1. *Let X be a $\mathbb{Q}$-factorial projective variety. Suppose that (X, Δ) is a divisorially log terminal pair and $-(K_X + \Delta)$ is nef.*

Suppose that we may find a big and nef $\mathbb{Q}$-divisor divisor A and a kawamata log terminal pair (X, Δ_0) such that $K_X + \Delta_0 + A \sim_{\mathbb{R}} N \geq 0$ has numerical dimension zero.

Then we may find a divisorially log terminal modification $\pi\colon Y \longrightarrow X$ of (X, Δ), a big and nef $\mathbb{Q}$-divisor B and a kawamata log terminal pair (Y, Γ_1) such that $K_Y + \Gamma_1 + B \sim_{\mathbb{R}} L \geq 0$ has numerical dimension zero, Γ_1 and L have no common components and no non kawamata log terminal centre of (Y, Γ) is contained in the support of L, where Γ is defined by

$$K_Y + \Gamma = \pi^*(K_X + \Delta).$$

Proof. (4.5) implies that replacing Δ_0 and N we may assume that Δ and N have no common components. Pick $t > 0$ such that $\lfloor tN \rfloor = 0$. Let $\pi\colon Y \longrightarrow X$ be a divisorially log terminal modification of $(X, \Delta + tN)$, (2.2.3). Then π has finitely many exceptional divisors. If E is an exceptional divisor which is not a log canonical place of (X, Δ) then E is not a log canonical place of $(X, \Delta + \delta N)$ for $\delta > 0$ sufficiently small. Thus replacing t by $\delta > 0$ sufficiently small, we may assume that π is also a divisorially log terminal modification of (X, Δ).

If we write

$$K_Y + \Gamma = \pi^*(K_X + \Delta),$$

then (Y, Γ) is divisorially log terminal and $-(K_Y + \Gamma)$ is nef. We may also write

$$K_Y + \Gamma_0 = \pi^*(K_X + \Delta_0) \qquad \text{and} \qquad B = \pi^* A.$$

It is possible that Γ_0 has negative coefficients, but only for exceptional divisors, which are all components of Γ of coefficient one. Therefore $\Gamma_0 \geq \alpha\Gamma$ for some α, possibly negative. If $L = \pi^* N$ then $\operatorname{mult}_E L > 0$ for every exceptional divisor E whose centre V is contained in the support of N. As

$$K_Y + \Gamma_0 + B \sim_{\mathbb{R}} L,$$

by (4.5) we may find B_1 big and nef and $\Gamma_1 \geq (1 - \delta)\Gamma \geq 0$ such that

$$K_Y + \Gamma_1 + B_1 \sim_{\mathbb{R}} tL_1,$$

where $L = L_0 + L_1$ and L_1 has no common components with Γ. But then no non kawamata log terminal centre of (Y, Γ) is contained in the support of L_1, because the pair $(Y, \Gamma + \delta L_1)$ is divisorially log terminal. Now replace B by B_1 and L by L_1. $\square$

Lemma 5.2. *Let X be a $\mathbb{Q}$-factorial projective toric variety. Let $B \geq 0$ be an $\mathbb{R}$-Cartier divisor whose support contains all but one invariant divisor. Let ν be a valuation which is not toric.*

Then we may find a divisor $0 \leq B' \sim_{\mathbb{R}} B$ such that $\nu(B') > \nu(B)$ whilst $\mu(B') \leq \mu(B)$ for every toric valuation μ.

If further (X, B) is a log canonical pair such that every log canonical place is toric then we may pick B' such that (X, B') is log canonical and the only log canonical places of (X, B') are toric valuations.

Proof. We prove the first statement.

Let $\pi \colon Y \longrightarrow X$ be a birational morphism of toric varieties. As the support of B contains every invariant subset of codimension at least two, it follows that $\pi^*B \geq 0$ is an $\mathbb{R}$-Cartier divisor whose support contains all but one invariant divisor. Thus we are free to replace X by Y and B by π^*B. In particular we may assume that X is smooth. We are also free to replace B by a multiple.

Let W be the centre of ν and let V be the smallest invariant subset of X which contains W. If $W = V$ then let $\pi \colon Y \longrightarrow X$ blow up V. Replacing X by Y and repeating this procedure finitely many times we reduce to the case $W \neq V$. [30, 1.2] implies that we may find a birational morphism of toric varieties $V' \longrightarrow V$ such that the strict transform W' of W does not contain any invariant subsets of V'. Let $Y \longrightarrow X$ be a birational morphism of toric varieties, which is an isomorphism at the generic point of V, such that if U is the strict transform of V then the birational morphism $U \longrightarrow V$ factors through $V' \longrightarrow V$. [30, 1.2] implies that, replacing X by Y, we may reduce to the case when W does not contain any invariant subsets.

Note that by (2.3.1) we may find a divisor $C \geq 0$ supported on the invariant components of B not containing V such that $A = C|_V$ is very ample and we can lift elements of the linear system $|A|$.

Pick a birational morphism $\pi \colon Y \longrightarrow X$ of toric varieties such that the mobile part of π^*C is base point free. Replacing X by Y we may assume that the mobile part B_0 of C is base point free. Note that B_0 and C are the same in a neighbourhood of V. Replacing B by a multiple we may assume $B_1 = B - B_0 \geq 0$.

Let $f \colon X \longrightarrow Z$ be the contraction morphism associated to B_0. Then the restriction of f to V is an isomorphism. As W does not contain any invariant subsets, it follows that $f(W)$ does not contain any invariant subsets and so $f(W)$ does not contain the image of any invariant subvariety of X.

As B_0 is the pullback of very ample divisor from Z, we may pick $0 \leq B_0' \sim_{\mathbb{R}} B_0$ such that B_0' contains W and $\mu(B_0') \leq \mu(B_0)$ for all toric valuations μ. If $B' = B'_0 + B_1$ then $0 \leq B' \sim_{\mathbb{R}} B$, $\nu(B') > \nu(B)$, whilst $\mu(B') \leq \mu(B)$ for every toric valuation μ.

Now suppose that (X, B) is log canonical. If $B_t = tB' + (1-t)B$ then $0 \leq B_t \sim_{\mathbb{R}} B$, $\nu(B_t) > \nu(B)$ for $t > 0$, (X, B_t) is log canonical if t is sufficiently small and the only log canonical places are toric valuations. $\square$

Lemma 5.3. *Assume* $(1.2)_{n-1}$.

Let X be a $\mathbb{Q}$-factorial projective variety of dimension n. Suppose that (X, Δ) is a divisorially log terminal pair and $-(K_X + \Delta)$ is nef.

If the complexity of (X, Δ) is less than one then X is of Fano type. In particular X is a toric variety.

Proof. (4.1) implies that we may find a big and nef $\mathbb{Q}$-divisor divisor A and a kawamata log terminal pair (X, Δ_0) such that $K_X + \Delta_0 + A \sim_{\mathbb{R}} N \geq 0$ has numerical dimension zero.

By (5.1) possibly replacing X by a higher model we may assume that Δ and N have no common components and that no non kawamata log terminal centre of (X, Δ) is contained in the support of N. In particular we may pick $\epsilon > 0$ such that no non kawamata log terminal centre of $(X, \Delta + \epsilon N)$ is contained in N.

Let $\pi\colon X \dashrightarrow Y$ be a log terminal model of $(X, \Delta_0 + A)$. Set $B = \pi_* A$, $\Gamma_0 = \pi_* \Delta_0$ and $\Gamma = \pi_* \Delta$. Then $(Y, \Gamma_0 + B)$ is a kawamata log terminal pair, $\Gamma_0 + B$ is big and $K_Y + \Gamma_0 + B$ is numerically trivial so that Y is of Fano type. π does not contract any components of Δ as the only divisors contracted by π are components of N. Thus the complexity of (Y, Γ) is less than one. In particular (3.1) implies that Y is a toric variety. If π does not contract any divisors then $N = 0$, $K_X + \Delta_0 + A$ is numerically trivial and so π is an isomorphism.

Pick an exceptional divisor E of π and let ν be the corresponding valuation. If ν is a toric valuation of Y for every exceptional divisor E then X is log Fano. So we may assume that ν is not toric. E is a component of N and so ν is not a log canonical place of (Y, Γ). (5.2) implies that we may find $0 \leq \Gamma' \sim_{\mathbb{R}} \Gamma$ such that $\nu(\Gamma') > \nu(\Gamma)$, (Y, Γ') is log canonical and the only log canonical places are toric valuations. Let Δ' be the strict transform of Γ'. Then certainly (X, Δ') is log canonical outside of the support of N, since the indeterminancy locus of π is contained in the support of N.

Let $\Delta_s = s\Delta' + (1-s)\Delta$. As N contains no non kawamata log terminal centre of (X, Δ), it follows that N contains no non kawamata log terminal centre of $(X, \Delta_s + \epsilon N)$ for s sufficiently close to zero. In particular (X, Δ_s) is log canonical for s sufficiently close to zero. Replacing (Y, Γ') by $(Y, \Gamma_s = \pi_* \Delta_s)$ for s sufficiently close to zero, we may assume that (X, Δ') is log canonical and N contains no non kawamata log terminal centre of $(X, \Delta' + \epsilon N)$. We may write

$$\begin{aligned} K_X + \Delta &= \pi^*(K_Y + \Gamma) + F \\ K_X + \Delta' &= \pi^*(K_Y + \Gamma') + F' \end{aligned}$$

where F and F' are exceptional. Note that the coefficients of F' are linear functions of s and so we may pick s sufficiently close to zero so that $F - F' \leq \epsilon N$.

Let

$$E_t = t(F' - F) + (1 - t)N,$$

Decompose E_t as $E_t^+ - E_t^-$, where $E_t^{\pm} \geq 0$ and E_t^+ and E_t^- have no common components. If we put

$$\Delta_t = (1 - t)\Delta_0 + t\Delta' + E_t^- \qquad \text{and} \qquad A_t = (1 - t)A + tM$$

then (X, Δ_t) is kawamata log terminal for $t \in [0, 1)$ and we have

$$\begin{aligned} K_X + \Delta_t + A_t &= (1 - t)(K_X + \Delta_0 + A) + t(M + K_X + \Delta') + E_t^- \\ &= (1 - t)(K_X + \Delta_0 + A) + t(\Delta' - \Delta) + E_t^- \\ &\sim_{\mathbb{R}} (1 - t)N + t(F' - F) + E_t^- \\ &= E_t + E_t^- \\ &= E_t^+. \end{aligned}$$

By assumption $\operatorname{mult}_E F' < \operatorname{mult}_E F$, so that $E_t^- \neq 0$ for t sufficiently close to 1. Thus E_t^+ has fewer components than N and we are done by induction on the number of components of N. □

6. Rationality via the Cox ring

In this section the groundfield k is a not necessarily algebraically closed field of characteristic zero.

Proposition 6.1. *Let X be a $\mathbb{Q}$-factorial projective variety. Suppose that the Cox ring of X is a polynomial ring modulo a single relation Q,*

$$\operatorname{Cox}(X) = \frac{k[x_1, x_2, \dots, x_n]}{\langle Q \rangle},$$

where Q and $x_1, x_2, \dots, x_n$ are homogenous elements of $\operatorname{Cox}(X)$.

If the rank of the quadratic part of Q is at least two then there is a proper finite morphism $Y \longrightarrow X$ of degree at most two, which is étale outside a closed subset of codimension at least two, such that Y is rational.

In particular if $A_{n-1}(X)$ has no 2-torsion then X is rational.

Proof. $R = \mathrm{Cox}(X)$ is a multigraded ring, and this grading corresponds to the action of a diagonalisable group H on $\operatorname{Spec} R$. X is a GIT quotient of $\operatorname{Spec} R$ by H. The action of H extends to the polynomial ring $k[x_1, x_2, \ldots, x_n]$ and the GIT quotient of the corresponding affine space is a toric variety Z which contains X as a divisor; the relation Q is homogeneous for this action.

Let T be the torus of Z. The monomials in the coordinate ring $k[M]$ of the torus are Laurent monomials in the variables $x_1, x_2, \ldots, x_n$ of multi-degree 0 in the grading.

Suppose that $x_i x_j \in Q$ for $i \neq j$. Possibly permuting the coordinates we may assume that $x_1 x_2 \in Q$. Collecting together all of the terms divisible by x_1, we may write

$$Q = x_1(x_2 + q_0) + q_1,$$

where q_1 is a polynomial in x_2, x_3, $\ldots$, x_n. After the homogeneous change of variable,

$$x_i \longrightarrow \begin{cases} x_2 + q_0 & \text{if } i = 2 \\ x_i & \text{otherwise,} \end{cases}$$

we may write

$$Q = x_1 x_2 - q$$

where q is a polynomial in x_2, x_3, $\ldots$, x_n.

As Q is homogeneous and q is not equal to zero (since the Cox ring is an integral domain), we may find a monomial $\nu \in q$ in the variables x_2, x_3, $\ldots$, x_n, with the same multi-degree as $x_1 x_2$. If we set

$$\mu = \frac{x_1 x_2}{\nu}$$

then μ is a Laurent monomial in the variables $x_1, x_2, \ldots, x_n$ of multi-degree zero.

Therefore $\mu \in k[M] \subset k[\mathbb{Z}^n]$. As the degree of x_1 is one in μ it follows that $\mu = \mu_1$ corresponds to a primitive element m_1 of the lattice M and we may extend it to a basis $m_1, m_2, \ldots, m_p$ of M, where the first coordinate of m_i is zero, for $i > 1$. If $\mu_1, \mu_2, \ldots, \mu_p$ are the Laurent monomials corresponding to $m_1, m_2, \ldots, m_p$ of M, then μ_i, $i > 1$ are Laurent monomials in the variables x_2, x_3, $\ldots$, x_n.

Now let $G = \mathbb{G}_m$ act by t on μ and trivially on all other basis elements. Let U be the open subset of the torus where $q \neq 0$. On X we have

$$\mu = \frac{x_1 x_2}{\nu} = \frac{q}{\nu}.$$

As the RHS is invariant under the action of G and the LHS is not, it follows that the orbits of G intersect $X \cap U$ in a unique point. Thus X is birational to U/G and so X is rational.

Otherwise assume $x_i x_j \notin Q$, for all $i \neq j$. Since the quadratic part of Q has rank at least two, x_i^2, $x_j^2 \in Q$, for $i \neq j$. Possibly permuting the coordinates we may assume that x_1^2, $x_2^2 \in Q$. Rescaling we may assume that

$$Q = x_1^2 - x_2^2 + q,$$

where the quadratic part of q is a polynomial in the variables x_3, x_4, $\dots$, x_n.

As Q is homogeneous, x_1^2 and x_2^2 have the same multi-degree. If x_1 and x_2 have the same multi-degree then both $x_1 + x_2$ and $x_1 - x_2$ are homogeneous and after the change of coordinates

$$x_i \longrightarrow \begin{cases} x_1 + x_2 & \text{if } i = 1 \\ x_1 - x_2 & \text{if } i = 2 \\ x_i & \text{otherwise,} \end{cases}$$

$x_1 x_2 \in Q$ so that X is rational by what we already proved.

Otherwise x_1/x_2 is torsion of degree two. By definition of the Cox ring, there is a Weil divisor D on X such that $2D \sim 0$. D defines a proper finite morphism $Y \longrightarrow X$ of degree two, which is étale outside a closed subset of codimension at least two.

On the other hand, E lifts to a Weil divisor on Z such that $2E \sim 0$. E also defines a proper finite morphism $W \longrightarrow Z$ of degree two, which is also étale outside a closed subset of codimension at least two, and induces the original cover $Y \longrightarrow X$.

W is a toric variety. Moreover W has the same Cox ring as Z but with a grading given by setting the class of D equal to zero. This grading corresponds to the action of a diagonalisable group G on $\operatorname{Spec} R$ and W is a GIT quotient of $\operatorname{Spec} R$ by G. Y is a divisor in W, defined by the same equation as X inside Z. On W, however, x_1 and x_2 have the same multi-degree and so Y is rational by what we have already proved. $\square$

Note that Y does not necessarily have the same Cox ring as X, since Y might have more divisors than X, as happens in the example in §7.

7. An irrational example

In this section we give an example of an irrational projective threefold X, together with a log canonical pair (X, Δ) of absolute complexity 1 such that $K_X + \Delta$ numerically trivial. In particular the condition on torsion in $A_{n-1}(X)$ in (6.1) is necessary. We will construct X as a $\mathbb{Z}_2$-quotient of a conic bundle Y over $T = \mathbb{P}^1 \times \mathbb{P}^1$.

Pick bihomogeneous coordinates $([y_0 : y_1], [z_0 : z_1])$ on T. If q is a general polynomial of degree $(2d, 2d)$ in the monomials y_i^2, z_i^2 and $y_i z_i$, $i \in \{0, 1\}$, where $d > 3$, then the zero locus of q is a smooth curve D which contains none of the invariant points. The equation

$$x_0^2 - x_1^2 = q(y_0, y_1, z_0, z_1) x_2^2,$$

defines a divisor Y inside the projectivisation W of

$$\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-d, -d).$$

Y is a conic bundle $f \colon Y \longrightarrow T$ over T. Let Θ be the divisor on W given by the sum of the vanishing of x_2, together with the pullbacks of the torus invariant divisors from $\mathbb{P}^1 \times \mathbb{P}^1$, and set $\Gamma = \Theta|_Y$.

Lemma 7.1.

(1) (Y, Γ) *is log smooth,*
(2) $K_Y + \Gamma = 0$*, and*
(3) $f \colon Y \longrightarrow T$ *has relative Picard number two.*

Proof. To prove (1) and (2), by adjunction it suffices to check that $(W, Y + \Theta)$ is log smooth and $K_W + Y + \Theta = 0$.

Consider the linear series on W spanned by x_0^2, x_1^2, and $x_2^2 m$, where m ranges over all monomials in y_i^2, z_i^2 and $y_i z_i$, $i \in \{0, 1\}$, of degree $(2d, 2d)$. Rescaling x_0 and x_1 we see that Y is a general member of this linear series. On the other hand this linear series is base point free and so Y is smooth and intersects all torus invariant strata of W transversely. This gives (1) and (2).

For (3) note that the fibres of f are irreducible except over the curve D, where the fibres have two components. It follows that the relative Picard number is at most two. On the other hand, the inverse image of D consists of two prime divisors $D_1 = V(x_0 + x_1, q)$ and $D_2 = V(x_0 - x_1, q)$. This is (3). $\square$

Consider the $\mathbb{Z}_2$ action on W sending

$$(y_0, y_1, z_0, z_1, x_0, x_1, x_2) \longrightarrow (y_0, -y_1, z_0, -z_1, x_0, -x_1, x_2).$$

This action also defines an action on T and under this action both Y and D are invariant. If X is the quotient of Y and S is the quotient of

T then there is a commutative diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ T \arrow[r] & S. \end{tikzcd}$$

Note that $g\colon X \longrightarrow S$ is a conic bundle. Note also that the action on T is toric, fixing only the four torus invariant points and so S is also a toric surface with four A_1 singularities.

Let Δ be the image of Γ.

Proposition 7.2.

(1) (X, Δ) is log canonical
(2) $K_X + \Delta = 0$,
(3) the absolute complexity γ is one, and
(4) X is irrational.

Proof. Note that as $Y \longrightarrow X$ is étale in codimension one, (1) and (2) follow easily.

The action of $\mathbb{Z}_2$ switches the divisors D_1 and D_2. Thus X has relative Picard number 1 over S and so the absolute complexity of (X, Δ) is

$$\gamma = \dim X + \rho(X) - d = 3 + 3 - 5 = 1.$$

This is (3).

If X is rational then the Griffiths component of the intermediate Jacobian must be trivial. If C is the discriminant curve of the conic bundle $g\colon X \longrightarrow S$ and the Griffiths component is trivial then C is hyperelliptic, trigonal, or isomorphic to a plane quintic, cf. [27].

C is certainly not a plane quintic. On the other hand C is a smooth quotient of D by the fixed point-free action of $\mathbb{Z}_2$. It suffices to check that D has no g^1_6 (since if D has a g^1_3 it has a g^1_6). But it follows from a theorem of Martens [23] that if D has a g^1_k then $k \geq 2d$. $\square$

8. Proofs

Proof of (1.2). We proceed by induction on the dimension n of X.

(2.2.5) implies that we may find a divisorially log terminal model $\pi\colon Y \dashrightarrow X$, such that if we write

$$K_Y + \Gamma = \pi^*(K_X + \Delta),$$

then $-(K_Y + \Gamma)$ is nef. (2.4.1) implies that the complexity of (Y, Γ) is at most the complexity of (X, Δ). (2.3.2) implies that if (Y, G) is a toric pair then (X, D) is a toric pair, where $D = \pi_* G$. If $G \geq \lfloor \Gamma \rfloor$ then

$D \geq \lfloor \Delta \rfloor$ and if all but $\lfloor 2c \rfloor$ invariant divisors are components of Γ then all but $\lfloor 2c \rfloor$ invariant divisors are components of Δ. Replacing (X, Δ) by (Y, Γ) we may assume that X is a $\mathbb{Q}$-factorial projective variety and (X, Δ) is divisorially log terminal.

(5.3) implies that X is of Fano type. Thus X is a Mori dream space and we are done by (3.1). $\square$

Proof of (1.3). This is immediate from (1.2). $\square$

Proof of (1.4). If $c < 1$ then (1.2) implies that X is toric and all but one invariant divisor is a component of Δ. On the other hand the invariant divisors span the Néron-Severi group and any invariant divisor is in the span of the other invariant divisors. $\square$

Proof of (1.5). Let $\bar{k}$ be the algebraic closure of k and let bars denote extension to the algebraic closure.

Then $(\bar{X}, \bar{\Delta})$ is log canonical and $-(K_{\bar{X}} + \bar{\Delta})$ is nef. If $\sum a_i S_i$ is a decomposition of Δ of complexity less than one then $\sum a_i \bar{S}_i$ is a decomposition of $\bar{\Delta}$ of complexity less than one. (1.2) implies that there is a divisor $\bar{D}$ such that $(\bar{X}, \bar{D})$ is toric.

Let m be the number of invariant divisors. Possibly reordering, we may assume that $\bar{S}_1, \bar{S}_2, \dots, \bar{S}_{m-1}$ are invariant divisors. In particular $S_1, S_2, \dots, S_{m-1}$ are prime divisors. Consider the linear system

$$|-(K_X + \sum_{i \leq m-1} S_i)|.$$

If $\bar{D}_m$ is the last invariant divisor on $\bar{X}$ then

$$\bar{D}_m \in |-(K_{\bar{X}} + \sum_{i \leq m-1} \bar{S}_i)|.$$

Thus the linear system

$$|-(K_X + \sum_{i \leq m-1} S_i)|$$

is non-empty and we may find S_m such that $K_X + D$ is linearly equivalent to zero and $(X, D = \sum_{i \leq m} S_i)$ is log canonical.

In this case $(\bar{X}, \bar{D})$ is toric, again by (1.2). Replacing (X, Δ) by (X, D) we may assume that $\Delta = D$. In this case every component of $\bar{D}$ is an invariant divisor and $S_1, S_2, \dots, S_m$ are all prime divisors. By induction $(S_i, (D - S)|_{S_i})$ is a toric pair. In particular the strata of D are geometrically irreducible.

We may find $\bar{\pi}\colon \bar{Y} \longrightarrow \bar{X}$ a birational morphism of toric varieties such that $\bar{Y}$ is projective and there is a birational morphism to projective space, $\bar{g}\colon \bar{Y} \longrightarrow \mathbb{P}^n_{\bar{k}}$. As the strata of D are geometrically irreducible, there is a birational morphism $\pi\colon Y \longrightarrow X$ which extracts only divisors of log discrepancy zero. If we write

$$K_Y + G = \pi^*(K_X + D),$$

then G is the sum of the strict transform of D and the exceptional divisors. It is enough to prove that (Y, G) is toric by (2.3.4).

Replacing (X, D) by (Y, G) we may assume that there is a birational morphism $\bar{f}\colon \bar{X} \longrightarrow \mathbb{P}^n_{\bar{k}}$. Pulling back an invariant hyperplane, this linear system is given by a sum of invariant divisors $\sum b_j \bar{D}_j$. Consider the linear system $|\sum b_j D_j|$. This is base point free, has dimension n and separates points. Thus we get a birational map to projective space $f\colon X \longrightarrow \mathbb{P}^n_k$ such that $\bar{f}$ is toric.

In particular f only extracts divisors of log discrepancy zero. (2.3.2) implies that (X, D) is a toric pair. $\square$

Proof of (1.6). As (1.2) holds in all dimensions (4.1) implies the first statement.

Let c be the complexity of (X, Δ). Pick $\delta > 0$ such that $A + (1-\delta)\Delta_0$ is ample. Replacing Δ_0 by $(1-\delta)\Delta_0$ and A by $A + (1-\delta)\Delta_0$ we may assume that (X, Δ_0) is kawamata log terminal. Let $f\colon X \dashrightarrow Y$ be a log terminal model of $(X, A + \Delta_0)$. If $\Gamma = f_*(A + \Delta_0)$ then (Y, Γ) is kawamata log terminal, Γ is big and $K_Y + \Gamma$ is semiample. Let $f\colon Y \longrightarrow W$ be the induced model. Then $-K_Y$ is big over W and $-(K_Y + \Gamma)$ is relatively generated over W. [11, 1.2] implies that the fibres of f are rationally connected. Thus f factors through the maximal rationally connected fibration $X \dashrightarrow Z$. It follows that

$$\dim Z \leq \dim W \leq \nu(X, K_X + A + \Delta_0) \leq c. \qquad \square$$

Proof of (1.8). (2.2.5) implies that we may find a divisorially log terminal model $\pi\colon Y \dashrightarrow X$, such that if we write

$$K_Y + \Gamma = \pi^*(K_X + \Delta),$$

then $-(K_Y + \Gamma)$ is nef. (2.4.1) implies that the absolute complexity of (Y, Γ) is at most the absolute complexity of (X, Δ). Replacing (X, Δ) by (Y, Γ) we may assume that X is a $\mathbb{Q}$-factorial projective variety and (X, Δ) is divisorially log terminal.

By (4.2) we may find a divisorially log terminal pair (Y, Γ) such that $-(K_Y + \Gamma)$ is ample, the absolute complexity is less than two and Y is birational to X. Replacing (X, Δ) by (Y, Γ) we may assume that $-(K_X + \Delta)$ is ample. (2.5.3) implies that X is a Mori dream space. Pick $B \sim_{\mathbb{R}} -(K_X + \Delta)$ such that $(X, B + \Delta)$ is divisorially log terminal. Replacing (X, Δ) by $(X, B+\Delta)$ we may assume that $K_X+\Delta$ is numerically trivial.

Let $R = \mathrm{Cox}(X)$ be the Cox ring of X, $Y = \mathrm{Spec}\, R$, and Γ the divisor on Y corresponding to Δ. Then every component of Γ is Cartier and K_Y is Cartier. [18] implies that (Y, Γ) is log canonical as (X, Δ) is log canonical.

By (2.4.3) Y has a cA_l singularity at the point p. If Y is smooth then X is a toric variety and there is nothing to prove. Otherwise $\mathrm{Cox}(X)$ is a polynomial ring modulo a single relation Q, where the rank of the quadratic part of Q is at least two. Thus we may apply (6.1). $\square$

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