THREEFOLD THRESHOLDS

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Abstract. We prove that the set of accumulation points of thresholds in dimension three is equal to the set of thresholds in dimension two, excluding one.

§1 Introduction

We prove

1.1 Theorem. *The set of accumulation points of the log canonical threshold in dimension three is equal to the set of log canonical thresholds in dimension two (excluding the number one).*

In fact we prove much more, we prove a similar Theorem in all dimensions, assuming the MMP and a conjecture of Alexeev-Borisov, see §3.

It is natural to attach numerical invariants to a singular variety D which measure the complexity of the singularities. For example we can associate the multiplicity to a hypersurface singularity. Unfortunately this is rather a coarse invariant; for example consider the infinite family $y^2 + x^b = 0$, $b \geq 2$, of plane curves of multiplicity two. Clearly one needs to take account of higher order blow ups to distinguish these singularities.

One such invariant is the log canonical threshold. Suppose that D is embedded as a hypersurface in the smooth variety X. The log canonical threshold c is then the largest value of c such that $K_X + cD$ is log canonical. A similar definition pertains when X is log canonical and D is an integral $\mathbb{Q}$-Cartier divisor. If one picks a log resolution $Y \longrightarrow X$ and one writes

$$K_Y + E = \pi^* K_X + \sum a_i E_i \qquad\qquad \tilde{D} + \sum b_i E_i = \pi^* D,$$

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for appropriate rational numbers $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_k$ where E is the sum over all exceptional divisors $E_1, E_2, \ldots, E_k$ and $\tilde{D}$ is the strict transform of D, then

$$c = \min_i \frac{a_i}{b_i}.$$

If D is given as $y^2 + x^b = 0$, then the log canonical threshold $c = 1/2 + 1/b$. More generally, if D is a hypersurface of dimension n and multiplicity $a \geq n$, then the log canonical threshold $1/a \leq c \leq (n+1)/a$, so that the log canonical threshold is a subtle refinement of the multiplicity.

Shokurov conjectured that the set of log canonical thresholds in dimension n satisfies ACC (the ascending chain condition). For example, if D is the plane curve $y^a + x^b = 0$ then the log canonical threshold is $1/a + 1/b$ and it is proved in [13] (see also [22]) that the log canonical threshold of any irreducible plane curve has this form.

Now given any set that satisfies ACC, it is natural to ask for the set of accumulation points. Kollár conjectured that the set of accumulation points in dimension n is equal to the set of accumulation points in dimension $n-1$. We prove this conjecture in dimension three.

In fact the log canonical threshold is a very natural invariant; when D is a hypersurface, then the log canonical threshold is both the complex singular index of D (a number that measures the asymptotic behaviour of the integrals of vanishing cycles) and (minus) the largest root of the (reduced) Bernstein-Sato polynomial of D, see [18] for more details.

Besides being a very natural invariant of study when D is a hypersurface in X, the log canonical threshold plays an important role in inductive approaches to higher dimensional geometry. For example, one of the most important remaining issues in Mori's program is to establish the existence and termination of flips. After Shokurov's groundbreaking work on the problem of existence of four-fold flips, some of the focus of interest has turned to the problem of termination of flips; the log canonical threshold and other closely related invariants should play a crucial role in this problem.

Another reason for being interested in the log canonical threshold is its connection with questions of birational geometry and rigidity. Along these lines, [4] and [5] prove some results and state some conjectures concerning the log canonical thresholds of hypersurfaces in $\mathbb{P}^n$. In fact [26] claims that every smooth hypersurface of degree $n \geq 4$ in $\mathbb{P}^n$ is birationally superrigid and part of the proof of this fact relies on the log canonical threshold. Finally [9], [8] and [7] prove some of the conjectures in [4] and [5], and thereby establish some new cases of birational superrigidity.

In yet another direction, the log canonical threshold should be closely related to recent work of Hassett, [11]. In this paper Hassett constructs a moduli space of pairs, consisting of a curve together with a boundary, where the coefficients of the boundary are rational. A very natural question is to ask what happens as one varies the coefficients. As the coefficients are increased, some pairs transition from stable to unstable, and it is necessary to blow up the corresponding loci. In fact a pair becomes unstables as it becomes not log canonical and the critical values, at which a transition occurs, are therefore log canonical thresholds.

Furthermore the set of all log canonical thresholds seems to have a rich structure of its own. The log canonical threshold has received some attention for these reasons. Several papers establish special cases of (1.1); [16] proves that the largest log canonical threefold threshold is 41/42; [23] identifies the set of thresholds $c \geq 5/6$ for a surface D in $\mathbb{C}^3$; similarly [24] establishes that 5/6 is the largest accumulation point for all pairs and [25] identifies all the accumulation points greater than 1/2.

[16] proceeds by direct computation and Mori's explicit classification of terminal 3-fold singularities. [22] proceeds by analysing all possible exceptional divisors on a smooth surface that have log discrepancy zero with respect to some divisor $K_X + \Delta$. [23] proceeds by an explicit analysis of a defining equation for D. [9], [8] and [7] prove their results by using jet schemes, a method introduced by Mircea Mustaţă. [24] and [25] use a method first introduced by Alexeev [1], who proved that the set of all log canonical threefold thresholds satisfies ACC. This paper builds on the ideas established in these two papers, so that many of the ideas from this paper can be traced back to the work of Alexeev.

Before we give precise statements of our results, we need to give some definitions. As these definitions are rather involved, it will probably help the reader to give some of the ideas of the proof of (1.1). As observed above, many of the ideas for the argument below is due to Alexeev. However what did come as a surprise to both authors is that his argument also lends itself so well to identifying the set of accumulation points of the log canonical threshold, a fact which does not seem to have been exploited before.

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§2 Sketch of Proof of (1.1)

The definition of log canonical states that the log discrepancy of various exceptional divisors are all nonnegative. Thus in the definition of the log canonical threshold, at least one exceptional divisor has log discrepancy exactly equal to zero. A natural way to proceed then, would be to find a birational morphism $\pi\colon Y \longrightarrow X$ that extracts an exceptional divisor E of log discrepancy zero. For example, if D is the curve $y^a + x^b = 0$ then $Y \longrightarrow X$ is precisely the weighted blow up of $X = \mathbb{C}^2$, with weights (a, b), in the given coordinates x, y. In general, the existence of π is guaranteed by the MMP (minimal model program), see (6.1).

The next step is to restrict to the exceptional divisor E and apply adjunction (here we use the fact that E has log discrepancy zero). In this case the singular points of Y along E make a contribution. For example, when $D = y^a + x^b = 0$, E is a copy of $\mathbb{P}^1$ and

$$(K_Y + E)|_E = K_E + \left(\frac{a-1}{a}\right) p + \left(\frac{b-1}{b}\right) q,$$

where p corresponds to the point of index a and q to the point of index b. Similar formulae pertain in the general case; see §4 for more details. If $\tilde{D}$ denotes the strict transform of D, then $\tilde{D} \cdot E = 1$. Moreover by definition $(K_Y + c\tilde{D} + E) \cdot E = 0$ so that

$$-2 + \frac{a-1}{a} + \frac{b-1}{b} + c = 0,$$

and so

$$c = \frac{1}{a} + \frac{1}{b},$$

as claimed.

In general, then, when we restrict to E we naturally get a log canonical pair (E, Δ) where $K_E + \Delta$ is numerically trivial and the number c appears in one of the coefficients of Δ. Moreover, using the MMP, see §5, we can actually reverse this process. More often than not, it suffices to take the cone over the pair (E, Δ).

In this way, the MMP in dimension n reduces the problem of computing the log canonical threshold to the problem of working with pairs (X, Δ) of dimension $n-1$, such that $K_X + \Delta$ is log canonical and numerically trivial.

So consider the problem of trying to identify the set of accumulation points for the coefficients of such pairs (X, Δ). One reason that the dimension one case is easy is that there is only one possibility for X, X must be isomorphic to $\mathbb{P}^1$. In higher dimensions, it is not hard, running the MMP again, to reduce to the case where X has Picard number one, so that at least X is a Fano variety.

Now suppose that we had a sequence (X_i, Δ_i) of such pairs. If X_i is constant then it is easy to see that the set of coefficients of Δ have to decrease and it is easy to identify the limit of c_i. A similar argument works, if the set $\{X_i\}$ forms a bounded family.

Thus we may assume that the set $\{X_i\}$ forms an unbounded family. At this point we invoke a Theorem of Alexeev, which is conjectured to hold in all dimensions (3.7). By his Theorem, it follows that the log discrepancy of X_i must approach zero.

To see how to use this fact, suppose that in fact the log discrepancy were zero and not just approaching zero. That is suppose that we have a log canonical pair (X, Δ) which is numerically trivial but not kawamata log terminal. In this case if we extracted a divisor of log discrepancy zero and applied adjunction then we would have decreased the dimension by one as before.

Thus our aim is now clear. Generate a component of Δ with coefficient one, so that we can restrict to this component and apply adjunction.

In the more general case, we extract a divisor of smallest log discrepancy, so that there is a component of Δ whose coefficient is approaching one. If we repeat this process twice, then we get two divisors whose coefficient is approaching zero. By taking an appropriate combination of these two divisors, we generate our component of coefficient one and we can then apply induction. In practice there are some technical issues that have to be dealt with to make this argument work.

§3 Precise definitions and Statement of results

We work over an algebraically closed field of characteristic zero.

Hopefully it will have become clearer the level of generality that we need to work at to make our argument go through.

First, we are interested in two sets of numbers, those for thresholds and those when $K_X + \Delta$ is numerically trivial.

Second we have to allow Δ to have unusual coefficients, since even if we start with one integral component of coefficient c, when we restrict to a component of coefficient one and apply adjunction, we obtain more than one component, with various coefficients.

Third, we have to allow for c to be a sequence of coefficients, rather than just one coefficient. Indeed we need two divisors whose coefficient is approaching one.

Fourth, after running the MMP, we might lose information on the sequence of coefficients, as some components might well be contracted. In particular sometimes we can only determine the set of coefficients and not the full sequence.

Fifth, there are some subtleties to do with kawamata log terminal versus log canonical. On the one hand kawamata log terminal singularities are much better behaved in general. Thus we can expect sharper results in this case. On the other hand, in terms of induction we want to allow log canonical singularities, as we want to restrict to a component of coefficient one.

3.1 Definition. Let I be any subset of $(0, 1]$. I_+ denotes the set of all finite sums of elements of I less than one.

$\mathcal{C}$ denotes the set of finite sequences of elements of $(0, 1]$ whilst $\mathcal{C}^+$ denotes the subset of increasing finite sequences of elements. Note the obvious inclusion $(0, 1] \subset \mathcal{C}^+$. Define a partial order on $\mathcal{C}$ by the rule:

$x_1, x_2, \ldots, x_m < y_1, y_2, \ldots, y_n$ iff the two sequences are not equal, $m \leq n$ and $x_i \leq y_i$, for all $i \leq m$. In addition, in the case of $\mathcal{C}^+$, we require strict inequality for at least one i.

We record the following easy result.

3.2 Lemma. *Let $c^{(i)}$ be a sequence in $\mathcal{C}^+$.*

Then $c^{(i)}$ is a strictly increasing sequence iff there is a strictly increasing sequence of numbers x_i, such that for every i, there is an index j, with $x_i = c^{(i)}_j$

Proof. Clear. $\square$

Let $c^{(i)}$ be the first i terms of the sequence of reciprocals of the natural numbers. Then $c^{(i)}$ is a strictly increasing sequence in $\mathcal{C}$.

Recall that a log canonical centre is the image of an exceptional divisor of log discrepancy at most zero.

3.3 Definition. We say that the sequence $c_1, c_2, \ldots, c_m \in \mathcal{C}$ is **associated** to the boundary Δ, with coefficients $d_1, d_2, \dots, d_n$, if for every i we may write

$$d_i = \frac{m_i - 1}{m_i} + \frac{f_i}{m_i} + \sum_j \frac{k_{ij} c_j}{m_i},$$

for some $f_i \in I_+$ and natural numbers $m_1, m_2, \dots, m_n$ and k_{ij}, where for all j, there is an i such that $k_{ij} \neq 0$. We say this sequence is of **adjunction type** if in addition $m_i \neq 1$ for some i; otherwise we say c is **ordinary**.

Consider the following four conditions that can be imposed on a log canonical pair (X, Δ).

(T) X is $\mathbb{Q}$-factorial and every component of Δ contains a log canonical centre Z of (X, Δ).

(N) X is complete and $K_X + \Delta$ is numerically trivial.

(K) condition N holds, and $K_X + \Delta$ is kawamata log terminal.

(B) condition N holds, and for every j there is a big $\mathbb{Q}$-Cartier divisor whose support is contained in those components Δ_i of Δ such that $k_{ij} \neq 0$.

We can use (3.3) to build an assortment of subsets of $\mathcal{C}$.

3.4 Definition. To every log canonical pair, (X, Δ), where X is a variety of dimension n, we take any sequence c associated to Δ. We use the four letters T, K, N, B corresponding to the four conditions, with subscript n. For condition T we require that c is ordinary and for condition B we take X of dimension at most n. Calligraphic letters, $\mathcal{T}$, $\mathcal{N}$, $\mathcal{K}$, $\mathcal{B}$ indicate that we take arbitrary sequences; the letters T, N, K, B, that c is a single number.

Thus $T_3(1)$ denotes the set of threefold thresholds and, for example, $\mathcal{K}_5(I)$ denotes the set of all sequences associated to kawamata log terminal pairs (X, Δ), where X is a complete variety of dimension five and $K_X + \Delta$ is numerically trivial.

Note that (3.4) involves some choices. In particular we only allow ordinary sets for the threshold; in fact it turns out, see (6.3), that even if we allow sets of adjunction type, we get the same set of coefficients. Some authorities replace condition B by the condition that X has Picard number one, see for example [19]. In fact the corresponding sets are conjecturally equal, see (5.1) and (5.2). We choose condition B as it behaves better in terms of inductive arguments. We allow X to be of any dimension less than n, for condition B, so that the inclusion $\mathcal{B}_{n-1}(I) \subset \mathcal{B}_n(I)$ is automatic. Similar inclusions for N and K are clear.

Shokurov made the remarkable conjectures that these sets satisfy ACC. Kollár speculated that even more is true, that the set of accumulation points in dimension n should be closely related to the corresponding sets in dimension $n-1$.

3.5 Conjecture. *Let I be any subset of $(0,1]$, which contains 1.*

(1) $B_n(I) = K_n(I) = N_n(I)$.

(2) $\mathcal{T}_{n+1}(I) = \mathcal{B}_n(I)$.

(3) *I satisfies DCC iff $\mathcal{N}_n^+(I)$ satisfies ACC iff $\mathcal{B}_n(I)$ satisfies ACC.*

(4) *Suppose that the only accumulation point of I is one. Then the set of accumulation points of $N_n(I)$ is equal to $N_{n-1}(I) - \{1\}$.*

(5) *I satisfies DCC iff $\mathcal{K}_n(I)$ satisfies ACC.*

3.6 Example. Let $X = \mathbb{P}^2$ and $\Delta = L + c_1L_1 + c_2L_2 + \dots c_mL_m$, where $\sum c_i = 2$ and the lines $L, L_1, L_2, \dots, L_m$ are in general position. If we blow up any of the points of intersection of L and L_i the exceptional divisor has coefficient c_i in the log pullback Γ of Δ. Repeating this process arbitrarily often we may find a surface S and a divisor Γ where the set of coefficients of Γ is equal to $\{c_1, c_2, \dots, c_m\}$ but where each coefficient appears arbitrarily many times.

In particular $\mathcal{N}_2(1)$ does not satisfy ACC.

Recall the following very interesting conjecture, which is a weakening of a conjecture due independently to Alexeev and Borisov (the standard version of this conjecture places no restriction on the Picard number):

3.7 Conjecture. *Fix an integer n and a positive real number ϵ.*

Then the family of all varieties X such that

(1) *the dimension of X is n,*

(2) *the log discrepancy of X is at least ϵ,*

(3) $-K_X$ *is ample, and*

(4) *the Picard number of X is one,*

is bounded.

Alexeev [1] proved (3.7) in dimension two and A. Borisov and L. Borisov [3] proved (3.7) for toric varieties. Kawamata [15] proved (3.7) for Fano threefolds with Picard number one and terminal singularities and Kollár, Miyaoka, Mori and Takagi [21] proved that all Fano threefolds with canonical singularities are bounded. Just recently Alexeev and Brion [2] have produced a preprint in which they claim to prove (3.7) for spherical varieties.

We will also need the following conjecture, which is a modification of a conjecture made in [1] by Alexeev for surfaces,

3.8 Conjecture. *Fix an integer n and a positive real number ϵ.*

Then the family of all pairs $(X, \ulcorner\Delta\urcorner)$ such that

(1) *the dimension of X is n,*

(2) *the log discrepancy of the pair (X, Δ) is at least ϵ, and the coefficients of Δ are at least ϵ and at most $1-\epsilon$,*

(3) $-(K_X + \Delta)$ *is nef,*

(4) *X is rationally connected,*

is bounded.

Note that (3.8) is a much stronger version of (3.7), as any kawamata log terminal Fano variety is automatically rationally connected. Despite the fact that (3.8) seems quite a bold conjecture, we give two pieces of evidence for this conjecture; we prove that (3.8) holds if either one assumes the MMP and (3.7) and we replace (3.8.4) by the condition that $-K_X$ is big, see (10.3), or we replace (3.8.4) by the condition that X is toric, see (11.2).

3.9 Theorem. *Assume the MMP in dimension n. Then (3.5.1) and $(3.5.2)_{n-1}$ hold. Further* *assume (3.7) holds in dimension n. Then (3.5.3-4) hold. Finally assume that (3.8) holds in dimension n. Then (3.5.5) holds.*

3.10 Corollary. *(3.5) holds for $n \leq 2$. In particular (1.1) holds. Further the largest accumulation point of T_3 is $5/6$.*

In fact the set of accumulation points in dimension three is completely determined, see (4.6). Note that both the MMP and (3.7) are known to hold for toric varieties. We are therefore able to prove (3.5) in the toric case. First some convenient notation.

3.11 Definition. *We say that the pair (X, Δ) has a* **toroidal resolution** *if there is a log resolution $\pi : Y \longrightarrow X$ of the pair (X, Δ) which is toroidal with respect to some pair (X, D).*

Now we introduce some more notation which modify the definitions of condition N and T. We say that a pair (X, Δ) satisfies condition ${}^tT$ if it satisfies condition T and in addition the pair (X, Δ) has a toroidal resolution. We say that a pair (X, Δ) satisfies condition ${}^tN$ if it satisfies condition N and in addition X is toric and every exceptional valuation of log discrepancy zero is a toric valuation. With these subsets, we obtain a modified version of (3.5), which we denote by ${}^t(3.5)$.

3.12 Theorem. *${}^t(3.5)$ holds.*

A very special case of (3.12) is the result that thresholds for non-degenerate hypersurfaces satisfy ACC (a non-degenerate hypersurface automatically admits a toroidal resolution). In fact one inspiration for stating (3.12) is the preprint of [14], where it is proved that the log canonical threshold of non-degenerate hypersurfaces satisfies ACC.

It is relatively straightforward to extend the proofs of most of the results given in this paper to characteristic p, although some proofs become a little more delicate, without their exposition adding anymore insight. In particular (6.3) does not hold in characteristic p. For this reason we only state and prove our results in characteristic zero.

§4 Arithmetic of Adjunction

We put together some of the arithmetic underlying adjunction and running the MMP in this section. We start with a Proposition that we prove after a couple of Lemmas.

4.1 Proposition. *Suppose that we have a log canonical pair (X, Δ). Let Y be a component of Δ of coefficient one and define Γ on Y by adjunction,*

$$K_Y + \Gamma = (K_X + \Delta)|_Y.$$

If the sequence $c_1, c_2, \dots, c_k \in \mathcal{C}$ is associated to the pair (X, Δ) and every component of Δ intersects Y then it is also associated to the pair (Y, Γ).

We start with a useful definition and a well-known Lemma.

4.2 Definition. Let $I \subset (0, 1]$ and set

$$D(I) = \{\, a \leq 1 \,|\, a = \frac{m-1}{m} + \frac{f}{m}, m \in \mathbb{N}, f \in I_+ \,\}.$$

4.3 Lemma. *Suppose that we have a log canonical pair (X, Δ). Let Y be a component of Δ of coefficient one and define Γ on Y by adjunction,*

$$K_Y + \Gamma = (K_X + \Delta)|_Y.$$

If the coefficients of Δ belong to I then the coefficients of Γ belong to $D(I)$.

Proof. Immediate from (16.6) of [19]. □

4.4 Lemma. *Let I be any subset of $(0, 1]$, which contains 1. Then*

(1) *$D(I)_+ = D(I)$.*

(2) *$D(D(I)) = D(I)$.*

(3) *I satisfies DCC iff $D(I)$ satisfies DCC.*

Proof. Suppose that $g \in D(I)_+$. It follows that there are real numbers $f_i \in I_+$ such that

$$g = \sum \frac{n_i - 1}{n_i} + \frac{f_i}{n_i}.$$

Now all but one $n_i = 1$ else $g \geq 1$ and there is nothing to prove. Thus g has the form

$$g = \frac{n-1}{n} + \frac{f}{n}$$

for some $f \in I_+$. Hence (1).

Suppose that $h \in D(D(I))$. Then there is a real number $g \in D(I)_+$ and an integer m such that

$$h = \frac{m-1}{m} + \frac{g}{m}.$$

As $g \in D(I)_+$, $g \in D(I)$ by (1). Thus g has the form

$$g = \frac{n-1}{n} + \frac{f}{n}$$

for some $f \in I_+$. Hence

$$\begin{aligned} h = \frac{m-1}{m} + \frac{g}{m} &= \frac{m-1}{m} + \frac{(\frac{n-1}{n} + \frac{f}{n})}{m} \\ &= \frac{mn - n + n - 1}{mn} + \frac{f}{mn} \\ &= \frac{mn-1}{mn} + \frac{f}{mn} = \frac{r-1}{r} + \frac{f}{r}. \end{aligned}$$

where $r = mn$. But $f \in I_+$, so that $h \in D(I)$. Hence (2).

As $I \subset D(I)$, it is clear that if $D(I)$ satisfies DCC then so does I. Now suppose that I satisfies DCC. Let $b_1, b_2, \dots$ be a non-increasing sequence in $D(I)$. By assumption there are integers m_{ij} and elements $a_i \in I_+$ such that

$$b_i = \sum_j \frac{m_{ij} - 1}{m_{ij}} + \frac{a_i}{m_{ij}}.$$

If $b_i = 1$ then there is nothing to prove. Thus we may assume that at most one $m_{ij} \neq 1$ so that

$$b = \sum \frac{m_i - 1}{m_i} + \frac{a_i}{m_i}.$$

Thus we may assume that m_i is constant. In this case the result is clear. $\square$

Proof of (4.1). This follows easily from (4.3) and (4.4). $\square$

We end this section with a couple of useful general results.

4.5 Lemma. *Suppose that $I \subset \mathbb{R}$ satisfies DCC and $K \subset \mathbb{R}$ satisfies ACC.*

Then

$$X = \{\, c_1, c_2, \dots, c_p \in \mathcal{C} \,|\, \exists\, m_i,\ k_{ij} \in \mathbb{N},\ a \in I,\ f_i \in I^+$$
$$a + \sum_i \left(\frac{m_i - 1}{m_i} + \frac{f_i}{m_i} + \sum_j \frac{k_{ij} c_j}{m_i} \right) = k \in K,$$
$$\text{where each term } \frac{m_i - 1}{m_i} + \frac{f_i}{m_i} + \sum_j \frac{k_{ij} c_j}{m_i} \text{ of the sum is at most one}\}$$

satisfies ACC.

Proof. Suppose we have a sequence of elements $x^{(p)}$ of X. If any terms

$$\frac{m_i^{(p)} - 1}{m_i^{(p)}} + \frac{f_i^{(p)}}{m_i^{(p)}} + \frac{k_{ij}^{(p)} c_j^{(p)}}{m_i^{(p)}},$$

appearing in the sum on the left are strictly increasing then we may absorb them into $a^{(p)}$. In this case $a^{(p)}$ is strictly increasing and so at least one term of $c_i^{(p)}$ must appear elsewhere in the sum. Possibly passing to a subsequence, it follows that we may assume that the $m_i^{(p)}$ are constant. Replacing each element k of K by $m_i k$, we may assume that $m_i = 1$. Replacing I by $I_+ \cup \{0\}$ and a by $a + \sum f_i$, we may assume that each $f_i = 0$. Replacing K by all numbers of the form $k - a$, where $k \in K$ and $a \in I$ we may assume that $I = \{0\}$ so that $a = 0$. In this case the result is clear. $\square$

In general it seem hard to identify the sets defined in (3.4). The one exception to this is N_1.

4.6 Lemma.

$$N_1 = N_1(1) = \{\, a \in (0,1] \,|\, \exists m_i, k_i \in \mathbb{N},\ \sum \frac{m_i - 1}{m_i} + \frac{k_i a}{m_i} = 2$$
$$\text{where each term } \frac{m_i - 1}{m_i} + \frac{k_i a}{m_i} \text{ of the sum is at most one}\}$$

In particular the largest element of N_1, other than one, is $5/6$.

Proof. Suppose $a \in N_1$. Then there is a curve C, and a divisor Δ such that $K_C+\Delta$ is numerically trivial, where the coefficients of Δ have the form

$$\frac{m_i - 1}{m_i} + \frac{k_i c}{m_i}.$$

Assuming that $D \neq 0$ it follows that $C \simeq \mathbb{P}^1$ and so the sum of the coefficients is two. $\square$

§5 Reduction to Picard number one

We commence the proof of (3.9) in this section. The proof of (3.9) consists in showing that (3.5.1-5) holds, under suitable circumstances. We adopt the convention that (3.9.i) refers to proving that (3.5.i) holds, under the hypotheses stated in (3.9).

The first thing to prove is (3.9.1).

5.1 Lemma. *Assume that the MMP holds in dimension n.*

Let (X, Δ) be a log canonical pair satisfying condition N, where X is of dimension n. Suppose that we are given big $\mathbb{Q}$-Cartier divisors $D_1, D_2, \dots, D_k$ supported on Δ.

Then we may run a MMP $X \dashrightarrow Y$, where Y is projective with $\mathbb{Q}$-factorial log terminal singularities, and there is an extremal contraction $\pi\colon Y \longrightarrow Z$, which is not birational. Further if Γ is the log pullback of Δ then the pair (Y, Γ) satisfies condition N. Moreover the strict transforms $G_1, G_2, \dots, G_k$ of the divisors $D_1, D_2, \dots, D_k$ are big $\mathbb{Q}$-Cartier divisors.

In particular, in the definition of $\mathcal{B}_n(I)$, we may assume that X is projective, log terminal and $\mathbb{Q}$-factorial of Picard number one.

Proof. Set $\Theta = \Delta - \epsilon(\sum D_i)$, for ϵ sufficiently small. Let $f\colon W \longrightarrow X$ be a log terminal model of the pair (X, Θ) and let Γ be the log pullback of Δ. Let E be an exceptional divisor of $W \longrightarrow X$. As (X, Δ) is log canonical, the centre of E is not contained in any of the divisors D_j. Thus f^*D_j is equal to the strict transform G_j of D_j. As D_j is big, then so is G_j.

Replacing (X, Δ) by (W, Γ) we may therefore assume that X is projective and $\mathbb{Q}$-factorial. We run the K_X-MMP. It is clear that the divisors $D_1, D_2, \dots, D_k$ remain big. Hence the first statement.

Restricting to the general fibre of π, and applying (4.1) and induction on the dimension, gives the second statement. $\square$

5.2 Lemma. *Assume the MMP in dimension n.*

Let (X, Δ) be a log canonical pair satisfying condition N, where X is of dimension n. Let A be a component of Δ. Then we may run a MMP $X \dashrightarrow Y$, where Y is projective with $\mathbb{Q}$-factorial log terminal singularities, and there is an extremal contraction $\pi\colon Y \longrightarrow Z$, which is not birational. Further if Γ is the log pullback of Δ then the pair (Y, Γ) satisfies condition N and the strict transform of A dominates Z.

Moreover in the definition of $N_n(I)$ we may assume that X is projective, $\mathbb{Q}$-factorial of Picard number one and that (X, Δ) is kawamata log terminal.

Proof. Passing to a log terminal model we may assume that X is projective $\mathbb{Q}$-factorial. We run the $(K_X + \Delta - \epsilon A)$-MMP. With this choice of MMP, A is automatically positive on every ray that we contract. In particular we never contract A. We end up with a Mori fibre space $\pi\colon X \longrightarrow Z$. A dominates Z, as A is positive on the corresponding ray.

Suppose we have a log pair (X, Δ), as in the definition of $N_n(I)$. Let A be a component of Δ whose coefficient is associated to c. Suppose that we contract a component B of coefficient one, or that a component of coefficient one does not dominate Z. Then at some stage of the MMP, A must intersect B. Restricting to B and applying (4.1) we are done by induction.

Otherwise restrict to the general fibre of π, and apply (4.1) and induction on the dimension. $\square$

Proof of (3.9.1). Note that if X is $\mathbb{Q}$-factorial, of Picard number one, then any effective $\mathbb{Q}$-divisor is big and $\mathbb{Q}$-Cartier. Thus the result follows easily from (5.1) and (5.2). $\square$

§6 Some cyclic covers

The next thing to prove is (3.9.2), assuming the existence of the MMP in dimension n. (3.9.2) involves proving that two sets are equal, which can be broken down into proving that each set is a subset of the other. One inclusion is easy and well known. We begin with a proof of this case, as it sets the stage for the reverse inclusion.

6.1 Lemma. $\mathcal{T}_n(I) \subset \mathcal{B}_{n-1}(I)$.

Proof. Suppose that the pair (X, Δ) satisfies condition T. Let $\pi \colon W \longrightarrow X$ extract a divisor E of log discrepancy zero and let Y be the normalisation of a general fibre of $\pi|_E \colon E \longrightarrow \pi(E)$. Let Θ be the log pullback of Δ. We may assume that W is $\mathbb{Q}$-factorial and that π has relative Picard number one. In particular the restriction of every component of Θ to Y is ample, whence big and $\mathbb{Q}$-Cartier. Define Γ on Y by adjunction. This gives us a log pair (Y, Γ) which satisfies condition B. By (4.1) the coefficients of Γ are of the appropriate form. $\square$

6.2 Lemma. *Suppose that the pair (X, Δ) satisfies condition N, where X is $\mathbb{Q}$-factorial of Picard number one.*

Then there is a pair (Y, Γ) satisfying condition T, where Y is of dimension one more than X and the coefficients of Γ are equal to the coefficients of Δ.

In particular if $c \in \mathcal{C}$ is associated to Δ then c is associated to Γ.

Proof. Pick an embedding of X into projective space such that X is projectively normal and let (Y, Γ) be the cone over (X, Δ). As X is projectively normal, Y is normal. Suppose that p is the vertex of the cone. Note that every component of Γ contains p and that Y is $\mathbb{Q}$-factorial as X has Picard number one.

Let $W \longrightarrow Y$ blow up p, so that W is a $\mathbb{P}^1$-bundle over X and the exceptional divisor E is a copy of X. Let Θ be the strict transform of Γ. We may write

$$K_W + \Theta + E = \pi^*(K_Y + \Gamma) + aE$$

where a is by definition the log discrepancy of E. Now

$$(K_W + \Theta + E)|_E = K_X + \Delta,$$

under the natural identification, so that $a = 0$. It follows that E is a log canonical place and p is a log canonical centre. On the other hand $K_W + \Theta + E$ is log canonical by inversion of adjunction and so the pair (Y, Γ) is log canonical. Thus the pair (Y, Γ) satisfies condition T. □

Note that we cannot apply (6.2) directly, as c might not be ordinary. In characteristic zero, we can get around this by taking a cyclic cover.

6.3 Lemma. *Suppose that the pair (X, Δ) satisfies condition T and that the sequence c is associated to Δ.*

Then we may find a pair (Y, Γ) satisfying condition T, where Y has dimension at most the dimension of X, c is associated to Γ and c is ordinary. Further we may assume that every component of Γ is Cartier.

Proof. Note that condition T is local about the generic point of Z, the log canonical centre. In particular we may assume that X is affine. Cutting by hyperplanes in X we may assume that Z is a point.

Suppose the coefficient of one component A of Δ is

$$\frac{m-1}{m} + \frac{f}{m}.$$

Let $\pi : Y \longrightarrow X$ be a cyclic cover of order m which is totally ramified over the generic point of A, but which is otherwise étale in codimension one. Let Γ be the log pullback of Δ. If B is the inverse image of A, then $\pi^* A = mB$ so that the coefficient of B is equal to f. Moreover $K_Y + \Gamma$ is log canonical and q the inverse image of p is a log canonical centre of $K_Y + \Gamma$. Taking another cyclic cover of Y, which is étale in codimension one, we may assume that B is Cartier.

We are therefore done by induction on the number of components of Γ. □

Proof of (3.9.2). By (6.1) it suffices to prove $\mathcal{B}_{n-1}(I) \subset \mathcal{T}_n(I)$. Suppose we are given a pair (X, Δ) that satisfies condition B, where X has dimension $n-1$. By (5.1) we may assume that X is projective, $\mathbb{Q}$-factorial, of Picard number one. By (6.2) we may find a pair (Y, Γ) that satisfies condition T, where Y has dimension n. By (6.3) we may assume that if c is associated to the pair (X, Δ) then c is associated to (Y, Γ) and moreover c is ordinary. Thus $c \in \mathcal{T}_n(I)$. $\square$

(6.3) has another useful application.

6.4 Lemma. *Every element of $T_{n-1}(I) - \{1\}$ is an accumulation point of $T_n(I)$.*

Proof. Suppose we have a pair (X, Δ) satisfying condition T. By (6.3) we may assume that every component of Δ is Cartier. The result now follows easily from (8.21.2) of [18]. $\square$

§7 The exchange trick

The following simple trick will prove quite useful.

7.1 Lemma. *Let (X_i, Δ_i, A_i+B_i) be a sequence of triples, where (X_i, Δ_i) is a log pair satisfying condition N, A_i and B_i are Weil divisors on X_i and where each X_i is $\mathbb{Q}$-factorial, of Picard number one. Suppose that the coefficients of A_i and B_i converge.*

Then, possibly passing to a subsequence, there is a divisor Γ_i, with the following properties

(1) *The pair (X_i, Γ_i) also satisfies condition N.*
(2) *$\Delta_i - \Gamma_i$ is supported on $A_i + B_i$.*
(3) *the limits of the coefficients of A_i and B_i also converge to the same respective limits.*
(4) *Possibly switching A_i and B_i, we may assume that either the coefficient of A_i is constant, or there is a log canonical centre contained in A_i.*
(5) *If the coefficient of B_i in Δ_i is increasing (respectively decreasing) then the coefficient of B_i in Γ_i is increasing (respectively decreasing).*
(6) *If the coefficients of both A_i and B_i are either strictly increasing or strictly decreasing then the coefficient of B_i in Γ_i is not constant.*

Proof. Since X_i is $\mathbb{Q}$-factorial of Picard number one, $A_i = \lambda_i B_i$, for some λ_i. Possibly passing to a subsequence and switching A_i and B_i we may assume that $\lambda_i|a - a_i| \leq |b - b_i|$, where a_i (respectively b_i) is the coefficient of A_i (respectively B_i) and a (respectively b) is the limit of a_i (respectively b_i). Set $\Theta_t = \Delta_i + t(a - a_i)A_i - t\lambda_i(a - a_i)B_i$. When $t = 0$, we get Δ_i. If $K_X + \Theta_1$ is log canonical for $t = 1$, then set $\Gamma_i = \Theta_1$. Otherwise pick the largest value of t such that $K + \Theta_t$ is log canonical and let $\Gamma_i = \Theta_t$. The result is clear. $\square$

7.2 Lemma. *Let (X_i, Δ_i) be a sequence of pairs, where (X_i, Δ_i) is a log pair satisfying condition N, and each X_i is $\mathbb{Q}$-factorial, of Picard number one. Suppose that every coefficient of Δ_i is monotonic.*

Then, possibly passing to a tail of the sequence, there are divisors Γ_i and H_i, with the following properties

1. *The pair (X_i, Γ_i) also satisfies condition N.*
2. *H_i belongs to a base point free linear system.*
3. *The support of Γ_i is contained in the support of $\Delta_i + H_i$.*
4. *The coefficients of Γ_i and Δ_i have the same limit.*
5. *The only decreasing coefficient of Γ_i is the coefficient of H_i.*

Proof. As every coefficient of Δ_i has a limit, then, possibly passing to a tail of the sequence, we may form a divisor Θ_i on X_i, with support equal to the support of Δ_i, whose coefficients are equal to the limiting coefficients. We may write

$$\Theta_i - \Delta_i = P_i - N_i,$$

where P_i and N_i have disjoint support and both are effective. Clearly the limits of the coefficients of P_i and N_i are both zero, $\Theta_i - P_i$ is effective and the components of N_i correspond to those divisors whose coefficients are decreasing. As X_i is $\mathbb{Q}$-factorial of Picard number one, N_i is ample and so we may find $m_i > 0$, increasing, such that $m_i N_i$ is base point free. Pick H_i a general element of the corresponding linear system. Set

$$\Gamma_i = (\Theta_i - P_i) + \frac{1}{m_i} H_i.$$

The result is then clear. $\square$

7.3 Lemma. *Suppose that $\mathcal{T}^+_n(I)$ satisfies ACC.*

Let (X_i, Δ_i) be a sequence of log pairs satisfying condition N, where each X_i is $\mathbb{Q}$-factorial, of Picard number one and dimension n. Suppose that the coefficients of Δ_i form an increasing sequence of $\mathcal{C}^+$.

Then we may find a divisor Γ_i, such that (X_i, Γ_i) also satisfies condition N and exactly one coefficient of Γ_i is increasing. If further $\mathcal{B}^+_{n-1}(I)$ satisfies ACC then at most one coefficient of Δ_i is approaching one.

Proof. Suppose that there is more than one component of Δ_i whose coefficient is increasing. We apply the exchange trick to two such components. As the coefficients of Γ_i are increasing and we are assuming that $\mathcal{T}^+_n(I)$ satisfies ACC, the second possibility for (7.1.3) is ruled out. Hence the first statement follows from repeated application of (7.1) and (3.2).

If there is more than one component whose coefficient is approaching one, then we may assume that there is a component of coefficient one. Restricting to this component and applying (4.1) gives a contradiction. $\square$

7.4 Lemma. *Assume the MMP in dimension n. Suppose that both $\mathcal{B}_n(I)$ and $\mathcal{T}^+_n(I)$ satisfy ACC.*

Let (X_i, Δ_i) be a sequence of log pairs satisfying condition N, where each X_i is $\mathbb{Q}$-factorial, of Picard number one and dimension n. Suppose that the coefficient a_i of one component A_i of Δ_i is approaching one. Suppose that c_i approaches a limit c, where c_i is a sequence associated to Δ_i.

Then c belongs to $\mathcal{B}_{n-1}(I)$.

Proof. As we are assuming that $\mathcal{B}_n(I)$ satisfies ACC there is at least one component B_i of Δ_i whose coefficient b_i is decreasing.

We first apply (7.3). Thus we may assume that Δ_i has exactly one component B_i whose coefficient is decreasing, that this coefficient is approaching zero, and that B_i is chosen general in a base point free linear system.

We apply (7.1) and replace Δ_i by the sequence of divisors Γ_i so produced. Either the coefficient of A_i is one or the coefficient of B_i is zero. The coefficient of A_i must be one otherwise there is no component whose coefficient is decreasing. Hence we may assume that the coefficient of A_i is one. In this case we may restrict to A_i and apply (4.1). $\square$

§8 From Boundedness to ACC

8.1 Lemma. *Suppose we have a sequence (X_i, Δ_i) of log pairs and let c_i be the sequence associated to Δ_i.*

If $\{X_i\}$ belongs to a bounded family, then c_i is not an increasing sequence of $\mathcal{C}$.

If further each X_i is $\mathbb{Q}$-factorial of dimension n and Picard number one, c_i is a number (rather than a sequence of numbers) and the only accumulation point of I is one, then $\lim c_i \in B_{n-1}(I)$.

Proof. By assumption we may find a curve C_i belonging to the smooth locus of X_i such that $-K_{X_i} \cdot C_i \leq M$ independently of i. As C_i lies in the smooth locus of X_i the intersection number of C_i with every component of Δ_i is a positive integer. The possible set of values of $-K_{X_i} \cdot C_i$ is finite and any finite set satisfies ACC. The first statement follows by (4.5).

Now suppose that the only accumulation point of I is one and that X_i has Picard number one. It follows that the coefficient of at least one component of Δ_i is approaching one and the second statement follows by (7.4). $\square$

Proof of (3.9.3). Suppose not. Then there is an increasing sequence c_i, either in $\mathcal{N}_n^+(I)$ or $\mathcal{B}_n(I)$. Let (X_i, Δ_i) be the corresponding sequence of log pairs. We first reduce to the case of Picard number one. There are two cases. If c_i is an increasing sequence in $\mathcal{B}_n(I)$, then by (5.1) we may assume that X_i has Picard number one. Otherwise c_i belongs to $\mathcal{N}_n^+(I)$. For each i, by (3.2), we may find a component A_i of Δ_i whose coefficient is increasing. By (5.1) we may assume that X_i is $\mathbb{Q}$-factorial, of Picard number one.

As we are reduced to the case of Picard number one, then in fact we may assume that $c_i \in \mathcal{B}_n(I)$. By (8.1) we may assume that $\{X_i\}$ does not form a bounded set. As we are assuming (3.7) it follows that the log discrepancy of X_i is approaching zero. In particular we may assume that the log discrepancy is less than one. By (5.2) we reduce to the case where Δ_i contains a component A_i whose coefficient is approaching one.

Applying (8.1) a second time, we may extract $\pi_i \colon Y_i \longrightarrow X_i$ another divisor B_i whose log discrepancy is approaching zero. By induction we may assume that $\mathcal{T}_n(I)$ satisfies ACC so that we may assume that $K_{Y_i} + \Gamma_i + (1 - a_i)A_i + (1 - b_i)B_i$ is log canonical, where Γ_i is the log pullback of Δ and a_i (respectively b_i) is the coefficient of A_i (respectively B_i).

We apply (5.2) to find a sequence of flips $Y_i \dashrightarrow Y_i'$ followed by an extremal contraction $\pi' \colon Y_i' \longrightarrow X_i'$ which is not small, on which B_i' is positive. We let Γ_i' be the strict transform of Γ_i and Δ_i' the pushforward of Δ_i.

If π_i' is not birational then we restrict to the general fibre and apply (4.1) and induction on the dimension. If π' is birational and A_i is not contracted then we are reduced to the case where we have two components whose coefficient is approaching one and we apply (7.3) to derive a contradiction.

Otherwise we apply (9.2). Possibly switching the roles of A_i and B_i we may assume that $K_{Y_i} + \Theta_i$ is log canonical and π-trivial, where A_i has coefficient one in Θ_i and the coefficient of B_i is still increasing. Restricting to A_i and applying (4.1) and induction on the dimension we are done. $\square$

Proof of (3.9.4). Suppose that we have a sequence $c_i \in N_n(I)$. Let (X_i, Δ_i) be the corresponding sequence of log pairs. By (5.2) we may assume that X_i is $\mathbb{Q}$-factorial of Picard number one and that (X_i, Δ_i) is kawamata log terminal.

By (8.1) we may assume that $\{X_i\}$ does not form a bounded set. As we are assuming (3.7) it follows that the log discrepancy of X_i is approaching zero. In particular we may assume that the log discrepancy is less than one. Let $\pi_i : Y_i \longrightarrow X_i$ extract a divisor A_i of minimal log discrepancy.

By (5.2) we reduce to the case where Δ_i contains a component whose coefficient is approaching one. The only thing to check is that in this process we have not contracted every component associated to c_i. But (3.9.3) implies that there must be a component whose coefficient is decreasing, and so there must be a component whose coefficient involves c_i. Now apply (7.4). $\square$

§9 Two ray game

We employ the following simple variants of the two ray game. We introduce some convenient notation. Let Y be a normal projective variety with $\mathbb{Q}$-factorial singularities and Picard number two. Then the cone of curves of Y is spanned by two rays R and S. Now suppose that we are given a sequence of flips $g \colon Y \dashrightarrow Y'$. Then the cone of curves of Y' is also spanned by two rays R' and S'. Possibly switching the roles of R and S, we may assume that R is spanned by the first curve that is flipped in Y and that S' is spanned by the last flipping curve in Y'.

9.1 Lemma. *Let D be any divisor in Y, which is pseudo-effective. Suppose that D is negative on S. Then D is positive on R, and D' is positive on R' and negative on S', where D' denotes the strict transform of D.*

Proof. As D is negative on S, it is certainly not numerically trivial over Y. As D is pseudo-effective, but not numerically trivial, it is positive on a covering family of curves. As the cone of curves is spanned by R and S, D must be positive on R. By induction on the number of flips, it suffices to consider the case when g is a flip. As D is positive on R and g is the flip of R, D' is negative on S'. By what we have already proved it follows that D' is positive on R'. $\square$

We introduce some more notation. Suppose in addition that we have a boundary Δ so that we have a log canonical pair (Y, Δ), where $K_Y + \Delta$ is numerically trivial. Assume that both rays S and R' are contractible. Let $f\colon Y \longrightarrow X$ be the contraction of S and $f'\colon Y' \longrightarrow X'$ the contraction of R'. Assume that f and f' are both birational and not small. Let E and E' be the corresponding exceptional divisors. We may write $K_Y + \Delta = K_Y + \Gamma + aE + a'E'$, where a is the coefficient of E and a' is the coefficient of E'.

9.2 Lemma. *Suppose that (Y, Δ) is kawamata log terminal.*

Then either $K_Y + \Gamma + E + b'E'$ is trivial on S, for some $a' < b' < 1$, or $K_{Y'} + \Gamma + bE + E'$ is trivial on R', for some $a < b < 1$.

Proof. Consider the divisor $D = K_Y + \Gamma + E + E'$. Suppose that D is S-positive. As E is negative on S, it follows that E' is S-positive. Thus $K_Y + \Gamma + E + b'E'$ is trivial on S for some $b' < 1$. On the other hand $K_Y + \Gamma + E + a'E' = K_Y + \Delta + (1-a)E$ is negative on S. Thus $a' < b'$.

Thus we may assume that D is S-negative. On the other hand as D is numerically equivalent to an effective linear combination of E and E', D is certainly pseudo-effective. Thus by (9.1) D' is positive on R'. Applying the same argument as above, it follows that $K_{Y'} + \Gamma + bE + E'$ is trivial on R', for some $a < b < 1$. □

§10 Proof of (3.9.5)

To prove (3.9.5) we need the following result.

10.1 Theorem. *Let X be a projective variety.*

There is a rational map $\pi\colon X \dashrightarrow B$ with the following properties. Let U be the open subset of X where π is defined. Then $\pi|_U$ is proper and the fibres of $\pi|_U$ are rationally chain connected. Moreover B is not uniruled.

Proof. Let π be the maximally rationally connected chain fibration, whose existence is established in [20], see also [17]. By the main Theorem of [10], B is not uniruled. □

(10.1) has the following very useful consequence.

10.2 Lemma. *Let (X, Δ) be a kawamata log terminal pair, where X is a projective variety of dimension n and $-(K_X + \Delta)$ is nef. Let $\pi : X \dashrightarrow B$ be the rational map whose existence is guaranteed by (10.1).*

If the MMP holds in dimension n then every component of Δ dominates B.

Proof. Suppose not. Suppose that D is a component of Δ which does not dominate B. Suppose that we write $\Delta = \Gamma + aD$, where a is the coefficient of D in Δ. We run the $(K_X+\Delta)$-MMP with scaling of D. After a series of flips and divisorial contractions, all of which are D-positive, we arrive eventually at a Mori fibre space $f : W \longrightarrow Z$. (Note that even though the pair (X, Δ) may not remain kawamata log terminal, nevertheless $K_X + \Gamma$ does remain kawamata log terminal, and every step of the MMP is a step of the $(K_X+\Gamma)$-MMP). As every contraction of this MMP is D-positive, D is never contracted and D dominates Z. The fibres of f are Fano varieties and so in particular they are covered by rational curves. Pulling these back to X, it follows that there is a rational curve C in X which passes through the general point of X and which intersects D. Hence C is not contracted under the rational map to B. But then B is covered by rational curves, a contradiction. □

Proof of (3.9.5). Suppose not, suppose that we have an increasing sequence $c_i \in \mathcal{K}_n(I)$. Let (X_i, Δ_i) be the corresponding sequence of log pairs. Passing to a log terminal model, we may assume that X_i is $\mathbb{Q}$-factorial projective. By (10.2), restricting to the general fibre of a maximal rationally chain connected fibration $\pi_i : X_i \dashrightarrow B_i$, we may assume that X_i is in fact rationally connected.

Suppose that the log discrepancy of (X_i, Δ_i) is approaching zero. Then we could extract a divisor E_i whose coefficient was approaching one. But this would give us an increasing sequence in $\mathcal{N}_n^+(I)$, which contradicts (3.9.3).

Thus we may assume that the log discrepancy of (X_i, Δ_i) is bounded away from zero. As we are assuming (3.8), $\{(X_i, \Delta_i)\}$ forms a bounded family and the result is clear. □

We also give the following evidence for (3.8):

10.3 Theorem. *Assume the MMP in dimension n and (3.7), where we drop the condition that the Picard number is one (that is we assume the full version of the conjecture of Alexeev-Borisov).*

Then (3.8) holds, where we replace (3.8.3) and (3.8.4) by

(3)$'$ *$K_X + \Delta$ is numerically trivial.*

(4)$'$ *$-K_X$ is big.*

Note that we only need the numerically trivial case to prove (3.5) and so replacing (3) by (3)$'$ is reasonable.

10.4 Lemma. *Assume the MMP in dimension n and let (X, Δ) be a kawamata log terminal pair, where X is projective $\mathbb{Q}$-factorial, such that $K_X + \Delta$ is numerically trivial.*

Then there is a birational map $X \dashrightarrow Y$, whose inverse does not contract any divisors, such that $-K_Y$ is ample.

Proof. Pick $\epsilon > 0$ sufficiently small so that $K_X + (1+\epsilon)\Delta$ is kawamata log terminal. Suppose that $-K_X$ is not nef. As $-K_X$ is numerically equivalent to Δ, it follows that $K_X + (1+\epsilon)\Delta$ is not nef. Thus running the $(K_X + (1+\epsilon)\Delta)$-MMP we may assume that $-K_Y$ is nef. The result now follows by applying the base point free Theorem to the nef and big divisor $-K_Y$. $\square$

We now show that running this process in reverse is bounded.

10.5 Lemma. *Assume the MMP in dimension n. Suppose we are give a bounded family of log pairs (Y, Γ), where Y is projective of dimension n, and $K_Y + \Gamma$ is kawamata log terminal and numerically trivial.*

Then the family of all kawamata log terminal pairs (X, Δ), for which there is a birational map $\pi\colon X \dashrightarrow Y$, whose inverse does not contract any divisors, where Δ is the log pullback of Γ, is bounded.

Proof. We may suppose, using Noetherian induction and passing to the geometric generic fibre, that we have a single log pair (Y, Γ), and it suffices to prove in this case that there are only finitely many such pairs (X, Δ).

As $K_Y+\Gamma$ is kawamata log terminal, there are only finitely many exceptional valuations of log discrepancy at most one with respect to $K_Y+\Gamma$. As the only divisors extracted by π are of log discrepancy at most one, it only remains to prove that the set of all models $\pi\colon X \dashrightarrow Y$, which extract a set of divisors corresponding to a fixed set of valuations, is finite.

Suppose that we have two such extractions $\pi\colon X \dashrightarrow Y$ and $\pi'\colon X' \dashrightarrow Y$. By assumption X and X' are isomorphic in codimension one, over Y. Pick an ample divisor H' on X', and let H be the strict transform of H'. Note that π' is the only such model for which H' is ample. Consider performing a sequence of $(K_X+\Delta)$-flops, with respect to H. Such a sequence of flops must terminate, as we are assuming the existence of the MMP. But termination only occurs when H becomes ample, and by uniqueness we must have arrived at π'. It follows that any two models are connected by a sequence of $(K_X+\Delta)$-flops, with respect to some effective divisor.

As we are assuming the existence of the MMP, there is at least one model, for which π is a morphism. Pick an ample divisor H on Y. Then we may decompose π^*H as $A+E$, where A is ample and E is effective. Now $K_X+\Delta+\epsilon E$ is kawamata log terminal, for ϵ sufficiently small and it follows that $K_X+\Delta+\epsilon E$ is relatively ample, for ϵ positive. As we assuming the MMP in dimension n, it follows, by the results of either [12] or [28], that there is a decomposition of the relative effective cone of divisors into finitely many chambers, such that the end result of running a sequence of flops only depends on the chamber to which the effective divisor belongs. In particular there are only finitely many possible models, as required. □

Proof of (10.3). Applying (10.4) we may assume that there is a birational map $X \dashrightarrow Y$ such that $-K_Y$ is ample. The log discrepancy of K_Y is at least the log discrepancy of $K_Y+\Gamma$, where Γ is the pushforward of Δ, which is equal to the log discrepancy of $K_X+\Delta$, which is bounded away from zero, by assumption. As we are assuming the strong version of (3.7), which places no restriction on the Picard number, it follows that we may assume that (Y,Γ) forms a bounded family. Now apply (10.5). □

§11 The toric and toroidal case

In this section we prove (3.12). Most of the proof is completely straightforward. All we need to do is go through the proof of (3.9) and check that we remain in the appropriate category.

The most technical aspect of the whole proof is the following:

11.1 Lemma. *Let $\pi\colon X \longrightarrow Y$ be a toroidal and projective morphism, where we work locally about a point p on Y. Suppose that $K_X + \Delta$ is kawamata log terminal.*

Then we may run the $(K_X + \Delta)$-MMP over Y.

Proof. By assumption we may find a toric map $\tilde{\pi}\colon \tilde{X} \longrightarrow \tilde{Y}$, such that π and $\tilde{\pi}$ are formally analytic isomorphic over the point p. In particular the relative cone of curves of X/Y is naturally isomorphic to the relative cone of curves of $\tilde{X}/\tilde{Y}$.

Suppose we are given a ray R on X on which $K_X + \Delta$ is negative. Let $f\colon X \longrightarrow Z$ be the corresponding contraction. If f is a fibration, there is nothing to prove, and if the exceptional locus is a divisor, then we replace X by Z and continue in the usual way. The only problem is when f is a small contraction. In this case we need to establish the existence of the flip of f and to establish the termination of a sequence of any such flips.

Let $\tilde{f}\colon \tilde{X} \longrightarrow \tilde{Z}$ be the corresponding contraction for $\tilde{X}$. Then $\tilde{f}$ is toric and the existence and termination of toric flips is known. As an immediate consequence we get termination of flips for X over Y. To establish existence is a little more delicate. Pick an invariant divisor $\tilde{S}$ for $\tilde{X}$ such that $\tilde{S}$ is also negative on $\tilde{R}$. Then we may find an invariant boundary $\tilde{B}$ on $\tilde{X}$ such that $K_{\tilde{X}} + \tilde{S} + \tilde{B}$ is negative on $\tilde{R}$. Let S and B be the corresponding divisors on X. It is proved in [27], see also [6], that the existence of the flip of f is equivalent to finite generation of the sheaf $f_*\mathcal{F}$, where $\mathcal{F}$ is a sheaf with support on S. As the flip of $\tilde{f}$ exists, it follows that the corresponding sheaf is finitely generated on $\tilde{Z}$. Since $f_*\mathcal{F}$ is supported on the exceptional locus of $Z \longrightarrow Y$, it follows that $f_*\mathcal{F}$ is also finitely generated, whence the existence of the flip of f follows from the existence of the flip of $\tilde{f}$. □

Note that (11.1) is easy to prove over $\mathbb{C}$, if we define toroidal in this case to mean that there is an analytic isomorphism, rather than just a formal analytic isomorphism.

We will also need the following modified version of (3.8).

11.2 Proposition. *Fix an integer n and a positive real number ϵ.*

Then the family of all pairs $(X, \ulcorner\Delta\urcorner)$ such that

(1) *the dimension of X is n,*

(2) *the log discrepancy of the pair (X, Δ) is at least ϵ, and the coefficients of Δ are at least ϵ,*

(3) *$-(K_X + \Delta)$ is nef,*

(4) *X is a toric variety,*

is bounded.

Proof. Let (X, Δ) be any such log pair. Passing to a log terminal model, we may assume that X is a toric $\mathbb{Q}$-factorial projective variety. As the invariant divisors span the Picard group there is an ample divisor which is an effective linear combination of the invariant divisors. In particular as $-K_X$ is equivalent to the sum of the invariant divisors, $-K_X$ is automatically big. On the other hand any nef divisor on a toric variety is semi-ample. Thus, possibly enlarging Δ, we may assume that $K_X + \Delta$ is numerically trivial.

Since the toric MMP is known to exist, we may apply the argument appearing in the proof of (10.3). □

Proof of (3.12). ${}^t(3.5.1)$ follows just as in the proof of (3.9.1), as we may run the MMP in the toric category.

To prove ${}^t(3.5.2)$, we follow the proof of (3.9.2). By assumption we may find a pair (Y, Γ) which satisfies condition ${}^tT$, such that c is associated to the pair (Y, Γ). In particular, (Y, Γ) has a toroidal resolution $\pi \colon W \longrightarrow Y$. As condition T is local, we may assume that Y is affine. Cutting by hyperplanes, we may assume that the only log canonical centre is a point. Pick a divisor X in W of log discrepancy zero. By assumption, X is isomorphic to a toric variety. By (11.1) we may run an appropriate MMP and reduce to the case that π extracts only the divisor X. Restricting to X and applying (4.3), we get a log pair (X, Δ) where c is associated to Δ and X is toric. The condition that the pair (Y, Γ) has a toroidal resolution implies that every exceptional valuation of log discrepancy zero is toric.

To prove ${}^t(3.5.3\text{-}4)$, note that (3.7) is known for toric varieties, see [3]. As in the proof of (3.9.3-4), we reduce to the case where there is a component whose coefficient is approaching one. In the toric case, we may further assume that this component is an invariant divisor, so that when we apply adjunction we do not leave the toric category.

${}^t(3.5.5)$ follows as in the proof of (3.9.5), using (11.2). $\square$

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