FLIPS AND FLOPS

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Abstract. Flips and flops are elementary birational maps which first appear in dimension three. We give examples of how flips and flops appear in many different contexts. We describe the minimal model program and some recent progress centred around the question of termination of flips.

Contents

1. Birational Geometry

1.1. **Curves and Surfaces.** Before we start talking about flips perhaps it would help to understand the birational geometry of curves and surfaces. For the purposes of exposition we work over the complex numbers, and we will switch freely between the algebraic and holomorphic perspective.

Example 1.1. *Consider the function*

$$\phi\colon \mathbb{C}^2 \dashrightarrow \mathbb{C} \qquad \textit{defined by the rule} \qquad (x,y) \longrightarrow y/x.$$

Geometrically this is the function which assigns to every point (x,y) the slope of the line connecting $(0,0)$ to (x,y). This function is not defined where $x=0$ (the slope is infinite here). One can partially

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remedy this situation by replacing the complex plane $\mathbb{C}$ by the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. We get a function

$$\phi\colon \mathbb{C}^2 \dashrightarrow \mathbb{P}^1 \qquad \text{defined by the rule} \qquad (x, y) \longrightarrow [X : Y].$$

However ϕ is still not defined at the origin of $\mathbb{C}^2$. Geometrically this is clear, since it does not make sense to ask for the slope of the line connecting the origin to the origin. In fact, if one imagines approaching the origin along a line through the origin then ϕ is constant along any such line and picks out the slope of this line. So it is clear that we cannot extend ϕ to the whole of $\mathbb{C}^2$ continuously.

It is convenient to have some notation to handle functions which are not defined everywhere:

Definition 1.2. *Let X be an irreducible quasi-projective variety and let Y be any quasi-projective variety. Consider pairs (f, U), where $U \subset X$ is an open subset and $f : U \longrightarrow Y$ is a morphism of quasi-projective varieties. We say two pairs (f, U) and (g, V) are* ***equivalent*** *if there is an open subset $W \subset U \cap V$ such that $f|_W = g|_W$.*

A ***rational map*** *$\phi\colon X \dashrightarrow Y$ is an equivalence class of pairs (f, U).*

In fact if ϕ is represented by (f, U) and (g, V) then ϕ is also represented by $(h, U \cup V)$ where

$$h(x) = \begin{cases} f(x) & x \in U \\ g(x) & x \in V. \end{cases}$$

So there is always a largest open subset where ϕ is defined, called the **domain** of ϕ, denoted $\operatorname{dom} \phi$. The locus of points not in the domain of ϕ is called the **indeterminacy** locus.

Example 1.3. *Let C be the conic in $\mathbb{P}^2$ defined by the equation*

$$X^2 + Y^2 = Z^2.$$

Consider the rational map

$$\phi\colon C \dashrightarrow \mathbb{P}^1 \qquad \text{defined by the rule} \qquad [X : Y : Z] \longrightarrow [X : Y - Z].$$

It would seem that ϕ is not defined where both $X = 0$ and $Y = Z$ and of course $X^2 + Y^2 = Z^2$, that is, at the point $[0 : 1 : 1]$.

If one passes to the open subset $U = \mathbb{C}^2$, where $Z \neq 0$, and introduces coordinates $x = X/Z$ and $y = Y/Z$ then $C_0 = C \cap U$ is defined by the equation $x^2 + y^2 = 1$ and the map above reduces to the function

$$C_0 \dashrightarrow \mathbb{C} \qquad \text{defined by the rule} \qquad (x, y) \longrightarrow x/(y - 1),$$

which would again not seem to be defined at the point $(0, 1)$ of the curve C_0.

However note that $(Y-Z)(Y+Z) = Y^2 - Z^2 = X^2$ *on the curve* C*. Therefore, on the open set* $C - \{[0:1:1],[0:-1:1]\}$*,*

$$[X(Y+Z):(Y-Z)(Y+Z)] = [X(Y+Z):X^2] = [Y+Z:X].$$

Thus ϕ *is also the function*

$$\phi\colon C \dashrightarrow \mathbb{P}^1 \qquad \textit{defined by the rule} \qquad [X:Y:Z] \longrightarrow [Y+Z:X].$$

It is then clear that ϕ *is in fact a morphism, defined on the whole of the smooth curve* C*.*

In fact the most basic result in birational geometry is that every map from a smooth curve to a projective variety always extends to a morphism:

Lemma 1.4. *Let* $f\colon C \dashrightarrow X$ *be a rational map from a smooth curve to a projective variety. Then* f *is a morphism, that is, the domain of* f *is the whole of* C*.*

Proof. As X is a closed subset of $\mathbb{P}^n$, it suffices to show that the composition $C \dashrightarrow \mathbb{P}^n$ is a morphism. So we might as well assume that $X = \mathbb{P}^n$. C is abstractly a Riemann surface. Working locally we might as well assume that $C = \Delta$, the unit disk in the complex plane $\mathbb{C}$. We may suppose that f is defined outside of the origin and we want to extend f to a function on the whole unit disk. Let z be a coordinate on the unit disk. Then f is locally represented by a function

$$z \longrightarrow [f_0 : f_1 : \dots : f_n],$$

where each f_i is a meromorphic function of z with a possible pole at zero. It is well known that $f_i(z) = z^{m_i} g_i(z)$, where $g_i(z)$ is holomorphic and does not vanish at zero and $m_0, m_1, \dots, m_n$ are integers. Let $m = \min m_i$. Then f is locally represented by the function

$$z \longrightarrow [h_0 : h_1 : \dots : h_n],$$

where $h_i(z) = z^{-m} f_i(z)$. As $h_0, h_1, \dots, h_n$ are holomorphic functions and at least one of them does not vanish, it follows that f is a morphism. □

Note that the birational classification of curves is easy. If two curves are smooth and birational then they are isomorphic. In particular, two curves are birational if and only if their normalisations are isomorphic.

Definition 1.5. *Let* $\phi\colon X \dashrightarrow Y$ *be a rational map between two irreducible quasi-projective varieties. The* ***graph of*** ϕ*, denoted* Γ_ϕ*, is the closure in* $X \times Y$ *of the graph of the function* $f\colon U \longrightarrow Y$*, where* ϕ *is represented by the pair* (f, U)*.*

Definition 1.6. *Consider the rational function*

$$\phi\colon \mathbb{C}^2 \dashrightarrow \mathbb{C} \qquad \textit{defined by the rule} \qquad (x,y) \longrightarrow y/x,$$

which appears in (1.1)*. Then the graph* $\Gamma_\phi \subset \mathbb{C}^2 \times \mathbb{P}^1$ *is the zero locus of the polynomial* $xT = yS$*, where* (x,y) *are coordinates on* $\mathbb{C}^2$ *and* $[S:T]$ *are homogeneous coordinates on* $\mathbb{P}^1$*. Consider projection onto the first factor* $\pi\colon \Gamma_\phi \longrightarrow \mathbb{C}^2$*. Away from the origin this morphism is an isomorphism but the inverse image* E *of the origin is a copy of* $\mathbb{P}^1$*.* π *is called the* ***blow up*** *of the origin and* E *is called the* ***exceptional divisor****.*

We note that π *has a simple description in terms of toric geometry.* $\mathbb{C}^2$ *corresponds to the cone spanned by* $(0,1)$ *and* $(1,0)$*.* Γ_ϕ *is the union of the two cones spanned by* $(1,0)$ *and* $(1,1)$ *and* $(1,1)$ *and* $(0,1)$*; it is obtained in an obvious way by inserting the vector* $(1,1)$*.*

Given any smooth surface S*, we can define the blow up of a point* $p \in S$ *by using local coordinates. More generally given any smooth quasi-projective variety* X *and a smooth subvariety* V*, we may define the blow up* $\pi\colon Y \longrightarrow X$ *of* V *inside* X*.* π *is a birational morphism, which is an isomorphism outside* V*. The inverse image of* V *is a divisor* E*; the fibres of* E *over* V *are projective spaces of dimension one less than the codimension of* V *in* X *and in fact* E *is a projective bundle over* V*.* V *is called the* ***centre of*** E*.*

For example, to blow up one of the axes in $\mathbb{C}^3$*, the toric picture is again quite simple. Start with the cone spanned by* $(1,0,0)$*,* $(0,1,0)$ *and* $(0,0,1)$*, corresponding to* $\mathbb{C}^3$ *and insert the vector* $(1,1,0) = (1,0,0) + (0,1,0)$*. We get two cones one spanned by* $(1,0,0)$*,* $(1,1,0)$ *and* $(0,0,1)$ *and the other spanned by* $(0,1,0)$*,* $(1,1,0)$ *and* $(0,0,1)$*. To blow up the origin, insert the vector* $(1,1,1)$*. There are then three cones. One way to encode this data a little more efficiently is to consider the triangle (two dimensional simplex) spanned by* $(1,0,0)$*,* $(0,1,0)$ *and* $(0,0,1)$ *and consider the intersection of the corresponding cones with this triangle.*

1.2. **Strong and weak factorisation.** We have the following consequence of resolution of singularities, see [16]:

Theorem 1.7 (Resolution of indeterminancy; Hironaka)**.** *Let* $\phi\colon X \dashrightarrow Y$ *be a rational map between two quasi-projective varieties.*

If X *is smooth, then there is a sequence of blow ups* $\pi\colon W \longrightarrow X$ *along smooth centres such that the induced rational map* $\psi\colon W \longrightarrow Y$ *is a morphism.*

For surfaces we can do much better in the case of a birational map:

Theorem 1.8. *Let $\phi\colon S \dashrightarrow T$ be a birational map between two smooth quasi-projective surfaces.*

Then there is a smooth surface W and two birational morphisms $\pi\colon W \longrightarrow S$ and $\pi'\colon W \longrightarrow T$, both of which are compositions of blow ups along smooth centres.

Example 1.9. *Consider the function*

$$\phi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2 \quad \text{defined by the rule} \quad [X : Y : Z] \longrightarrow [X^{-1} : Y^{-1} : Z^{-1}].$$

Then ϕ is a birational map, an involution of $\mathbb{P}^2$. As

$$[X^{-1} : Y^{-1} : Z^{-1}] = [YZ : XZ : YZ],$$

it is not hard to see that ϕ sends the three coordinate axes to the coordinate points. But then it follows that the coordinate points $[1 : 0 : 0]$, $[0 : 1 : 0]$ and $[0 : 0 : 1]$ are part of the indeterminacy locus of ϕ. In fact, if we blow up $\pi\colon W \longrightarrow \mathbb{P}^2$ the three coordinate points, then ϕ blows down the strict transform of the three coordinate axes $\pi'\colon W \longrightarrow \mathbb{P}^2$.

Consider the standard fan for $\mathbb{P}^2$, the union of the three cones spanned by $(1,0)$, $(0,1)$ and $(-1,-1)$. Blowing up the coordinate points, corresponds to inserting the three vectors $(1,1) = (1,0) + (0,1)$, $(0,-1) = (1,0) + (-1,-1)$ and $(-1,0) = (-1,-1) + (0,1)$. The resulting fan is the fan for the toric variety W. Note that the strict transforms of the three coordinate axes are now contractible as $(1,0) = (1,1) + (0,-1)$, $(-1,-1) = (-1,0) + (0,-1)$ and $(0,1) = (1,1) + (-1,0)$.

1.3. **Flips and Flops.** It is conjectured that a result similar to (1.8) holds in all dimensions:

Conjecture 1.10 (Strong factorisation)**.** *Let $\phi\colon X \dashrightarrow Y$ be a birational map between two quasi-projective varieties.*

Then there is a quasi-projective variety W and two birational morphisms $\pi\colon W \longrightarrow X$ and $\pi'\colon W \longrightarrow Y$ which are both the composition of a sequence of blow ups of smooth centres.

Unfortunately we only know a weaker statement, see [1] and [44]:

Theorem 1.11 (Weak factorisation: Abramovich, Karu, Matsuki, Włodarczyk; Włodarczyk)**.** *Let $\phi\colon X \dashrightarrow Y$ be a birational map between two quasi-projective varieties.*

Then we may factor ϕ into a sequence of birational maps $\phi_1, \phi_2, \dots, \phi_m$, $\phi_i\colon X_i \dashrightarrow X_{i+1}$ and there are quasi-projective varieties $W_1, W_2, \dots, W_m$ and two birational morphisms $\pi_i\colon W_i \longrightarrow X_i$ and $\pi_i'\colon W_i \longrightarrow X_{i+1}$ which are both the composition of a sequence of blow ups of smooth centres.

The problem is that beginning with threefolds there are birational maps which are isomorphisms in codimension two:

Example 1.12. *Suppose we start with $\mathbb{C}^3$ and blow up both the x-axis and the y-axis. Suppose we first blow up the x-axis and then the y-axis to get $X \longrightarrow \mathbb{C}^3$. Let E_x be the exceptional divisor over the x-axis, with strict transform E'_x and let E_y be the exceptional divisor over the y-axis. The strict transform of the y-axis intersects E_x in a point. When we blow this up, we also blow up this point of E_x. So E'_x has one reducible fibre with two components and E_y is a $\mathbb{P}^1$-bundle over the y-axis. If we blow up $Y \longrightarrow \mathbb{C}^3$ in the opposite order then E_x is now a $\mathbb{P}^1$-bundle and the strict transform E'_y contains one reducible fibre. The resulting birational map $X \dashrightarrow Y$ is an isomorphism outside the extra copies of $\mathbb{P}^1$ belonging to E'_x and E'_y. On the other hand it is not an isomorphism along these curves. This is the simplest example of a flop.*

The language of fans and toric geometry is very convenient. We start with the cone spanned by $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$. Blowing up the x-axis corresponds to inserting the vector $e_2 + e_3$ and we get two cones, σ_1, spanned by e_1, $e_2 + e_3$ and e_2 and σ_2 spanned by e_1, $e_2 + e_3$ and e_3. Blowing up the y-axis we insert the vector $e_1 + e_3$, so that we subdivide σ_2 into two more cones, one spanned by e_1, $e_2 + e_3$ and $e_1 + e_3$ and the other spanned by e_3, $e_2 + e_3$, $e_1 + e_3$.

Now suppose that we reverse the order. At the first step we insert the vector $e_1 + e_3$ and we get two cones, τ_1 spanned by e_1, $e_1 + e_3$ and e_2 and τ_2 spanned by e_2, $e_1 + e_3$ and e_3. At the next step we insert the vector $e_2 + e_3$, and subdivide τ_2 into two cones, one spanned by e_2, $e_1 + e_3$ and $e_2 + e_3$ and the other spanned by e_3, $e_1 + e_3$ and $e_2 + e_3$.

In fact to prove (1.10) it suffices to prove it in the special case of toric varieties. For an interesting explanation of the difficulties in proving strong factorisation, see [17].

There is another way to construct this flop:

Example 1.13. *Let Q be the quadric cone $xz - yt = 0$ inside $\mathbb{C}^4$. If we blow up the origin we get a birational morphism $W \longrightarrow Q$ with exceptional E divisor isomorphic $\mathbb{P}^1 \times \mathbb{P}^1$. We can partially contract E, by picking one of the projection maps, $W \longrightarrow X$ and $W \longrightarrow Y$. The resulting birational map $X \dashrightarrow Y$ is the same as the flop introduced above.*

Perhaps the easiest way to see this is to use toric geometry. Q corresponds to the cone spanned by four vectors v_1, v_2, v_3 and v_4 in $\mathbb{R}^3$, belonging to the standard lattice $\mathbb{Z}^3$, any three of which span the standard lattice, such that $v_1 + v_3 = v_2 + v_4$. W corresponds to inserting the *vector $v_1 + v_3$ and subdividing the cone into four subcones. X and Y correspond to the two different ways to pair off the four maximal cones into two cones.*

One particularly nice feature of the toric description is that we can modify the picture above to get lots of examples of flips and flops. Suppose we pick any four vectors v_1, v_2, v_3 and v_4 belonging to the standard lattice which span a strongly convex cone. Then $a_1v_1 + a_3v_3 = a_2v_2 + a_4v_4$, for some positive integers a_1, a_2, a_3 and a_4. Once again we can insert the vector $a_1v_1 + a_3v_3$ and pair off the resulting cones to get two different toric threefolds X and Y which are isomorphic in codimension one.

The simplest example of a flip is when $2v_1 + v_3 = v_2 + v_4$. If we start with the wall connecting v_2 and v_4 then the flip corresponds to replacing this by the wall connecting v_1 and v_3. X has one singular point, which is a $\mathbb{Z}^2$-quotient singularity, corresponding to the cone spanned by v_2, v_3 and v_4. Indeed, $2v_1$ is an integral linear combination of these vectors but not v_1. On the other hand, Y is smooth.

Another place that flops appear naturally is in the example of a Cremona transformation of $\mathbb{P}^3$.

Example 1.14. *Consider the function*

$$\phi\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^3 \quad \textit{defined by the rule} \quad [X : Y : Z : T] \longrightarrow [X^{-1} : Y^{-1} : Z^{-1} : T^{-1}].$$

Then ϕ is a birational automorphism of $\mathbb{P}^3$. The graph of this function first blows up the four coordinate points, to get four copies of $\mathbb{P}^2$, then the six coordinate axes, to get six copies of $\mathbb{P}^1 \times \mathbb{P}^1$. The reverse map then blows down those six copies of $\mathbb{P}^1 \times \mathbb{P}^1$, but this time using the other projection and then we finally blow down the strict transforms of the four coordinate planes.

Note that if we just blow up the four coordinate points on both sides then the resulting threefolds are connected by six flops. All of this is easy to describe using toric geometry; the picture is similar to the picture above of the Cremona transformation of $\mathbb{P}^2$.

One can use flops to construct some interesting examples.

Example 1.15 (Hironaka). *Suppose we start with $X = \mathbb{P}^3$ and two conics C_1 and C_2 which intersect in two points p and q. Imagine blowing up both C_1 and C_2 but in a different order at p and q. Suppose we blow up first C_1 and then C_2 over p but first C_2 and then C_1 over q. Let $\pi\colon M \longrightarrow \mathbb{P}^3$ be the resulting birational map. Note that π does not exist in the category of varieties but that this construction does make sense in the category of complex manifolds.*

I claim that even the exceptional locus $E_1 \cup E_2$ is not a projective variety. Let l be general fibre of the exceptional divisor E_1 over C_1, let $l_1 + l_2$ be the reducible fibre over p, let m be the general fibre of E_2 over C_2 and let $m_1 + m_2$ be the reducible fibre over q. Suppose the irreducible fibre of E_2 over p is attached to l_1 and the irreducible fibre of E_1 over q is attached to m_1. Note that

$$m_1 \equiv l \equiv l_1 + l_2 \equiv m + l_2 \equiv m_1 + m_2 + l_2,$$

where $\equiv$ denotes numerical equivalence. This implies that $l_2 + m_2 \equiv 0$. If M is projective then a hyperplane class H would intersect $l_2 + m_2$ positively, a contradiction.

Note that M is related to a projective variety Y over X by an (analytic) flop. Just flop either l_2 or m_2.

Example 1.16 (Atiyah)**.** *Suppose that we start with a family of quartic surfaces in $\mathbb{P}^3$ degenerating to a quartic surface with a simple node (a singularity which in local analytic coordinates resembles $x^2+y^2+z^2 = 0$ in $\mathbb{C}^3$). It is a simple matter to find a degeneration whose total space has a singularity locally of the form $xz - yt$. In this case we can blow up this singularity in two different ways, see* (1.13)*, to get two different families of smooth K3 surfaces, which are connected by a flop.*

But now we have two distinct families of K3 surfaces, which agree outside one point. In fact even though the families are different they have isomorphic fibres. It follows that the moduli space of K3 surfaces is not Hausdorff.

Example 1.17 (Reid)**.** *Let $X_0 \subset \mathbb{C}^4$ be the smooth threefold given by the equation*

$$y^2 = ((x-a)^2 - t_1)((x-b)^2 - t_2),$$

where x, y, t_1, t_2 are coordinates on $\mathbb{C}^4$ and $a \neq b$ are constants. Let X be the closure of X_0 in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{C}^2$. Projection $\pi \colon X \longrightarrow \mathbb{C}^2$ down to $\mathbb{C}^2$ with coordinates t_1 and t_2 realises X as a family of projective curves of genus one over $\mathbb{C}^2$. If $t_1 t_2 \neq 0$ then we have a smooth curve of genus one, that is an elliptic curve. If $t_1 = 0$ and $t_2 \neq 0$ or $t_2 = 0$ and $t_1 \neq 0$ then we get a nodal rational curve (a copy of $\mathbb{P}^1$ with two points identified). If $t_1 = t_2 = 0$ then we get a pair $C_1 \cup C_2$ of copies of $\mathbb{P}^1$ joined at two points.

One can check that both C_1 and C_2 can be contracted individually to a simple node. Therefore we can flop either C_1 or C_2. Suppose that we flop C_1. Since C_1 is contracted by π this flop is over S so that the resulting threefold Y admits a morphism to $\psi \colon Y \longrightarrow \mathbb{C}^2$. We haven't changed the morphism π outside s and one can check that the fibre over $(0,0)$ of ψ is a union $D_1 \cup D_2$ of two copies of $\mathbb{P}^1$ which intersect in two *different points. Once again we can flop either of these curves. Suppose that D_2 is the strict transform of C_2 so that D_1 is the flopped curve. If we flop D_1 then we get back to X but if we flop D_2 then we get another threefold which fibres over S. Continuing in this way we get infinitely many threefolds all of which admit a morphism to S and all of which are isomorphic over the open set $S - \{s\}$. Let G be the graph whose vertices are these threefolds, where we connect two vertices by an edge if there is a flop between the two threefolds over S. Let G' be the graph whose vertices are the integers where we connect two vertices i and j if and only if $|i - j| = 1$. Then G and G' are isomorphic.*

2. Minimal model program

The idea behind the minimal model program (which we will abbreviate to MMP) is to find a particularly simple birational representative of every projective variety. For curves we have already seen that two smooth curves are birational if and only if they are isomorphic. For surfaces there are non-trivial birational maps, but by (1.8) only if there are rational curves (non-constant images of $\mathbb{P}^1$). Roughly speaking, simple means that we cannot contract any more rational curves. In practice it turns out that we don't want to contract every curve, just those curves on which the canonical divisor is negative.

Definition 2.1. *Let X be a normal projective variety. A **divisor** $D = \sum n_i D_i$ is a formal linear combination of codimension one subvarieties.*

*The **canonical divisor** K_X is the divisor associated to the zeroes and poles of any meromorphic differential form ω.*

Note that the canonical divisor is really an equivalence class of divisors.

Example 2.2. *If $X = \mathbb{P}^1$ and z is the standard coordinate on $\mathbb{C}$ then dz/z is a meromorphic differential form. It has a pole at zero and a pole at infinity, since*

$$\frac{d(1/z)}{1/z} = -\frac{dz}{z}.$$

If p represents zero and q infinity then $K_{\mathbb{P}^1} = -p - q$. If we started with dz/z^2 then $K_{\mathbb{P}^1} = -2p$ (a double pole at zero) but if we start with dz then $K_{\mathbb{P}^1} = -2q$ (a double pole at infinity). And so on. If X is an elliptic curve E then it is a one dimensional complex torus, the quotient of $\mathbb{C}$ by a lattice isomorphic to $\mathbb{Z}^2$. In this case the differential form dz descends to the torus (as it is translation invariant) and $K_E = 0$ (no zeroes or poles). If C has genus $g \geq 2$ then $\deg K_C = 2g - 2 > 0$.

For $\mathbb{P}^n$ we have $K_{\mathbb{P}^n} = -(n+1)H$, where H is the class of a hyperplane. More generally still, suppose X is a projective toric variety. Then a dense open subset of X is isomorphic to a torus $(\mathbb{C}^*)^n$. A natural holomorphic differential n-form which is invariant under the action of the torus is

$$\frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \wedge \cdots \wedge \frac{dz_n}{z_n}.$$

This form extends naturally to a meromorphic differential on the whole toric variety with simple poles along the invariant divisors. In other words,

$$K_X + \Delta \sim_{\mathbb{Q}} 0,$$

where $\Delta = \sum D_i$ is a sum of the invariant divisors. In the case of $\mathbb{P}^n$ there are $n+1$ invariant divisors corresponding to the $n+1$ coordinate hyperplanes.

One of the most useful ways to compute the canonical divisor is the adjunction formula. If M is a smooth variety and X is a smooth divisor then

$$(K_M + X)|_X = K_X.$$

For example, if X is a quartic surface in $\mathbb{P}^3$ then

$$K_X = (K_{\mathbb{P}^3} + X)|_X = (-4H + 4H)|_X = 0.$$

Together with the fact that smooth hypersurfaces of dimension at least two are simply connected this implies that X is a K3 surface.

Suppose that $T \longrightarrow S$ is the blow up of a point with exceptional divisor $E \simeq \mathbb{P}^1$. It is straightforward to check that the self-intersection $E^2 = E \cdot E = -1$. By adjunction we have

$$-2 = K_{\mathbb{P}^1} = K_E = (K_T + E)|_E = K_T \cdot E + E^2.$$

It follows that $K_T \cdot E = -1$. For obvious reasons we call any such curve a -1-curve. The idea of the MMP is to only contract curves on which the canonical divisor is negative.

Definition 2.3. *Let X be a normal projective variety and let D be a Cartier divisor (something locally defined by a single equation). We say that D is* ***nef*** *if $D \cdot C \geq 0$ for every curve $C \subset X$.*

Let us first see how the minimal model program works for surfaces.

Step 0: Start with a smooth surface S.

Step 1: Is K_S nef? If yes, then stop. S is a minimal model.

Step 2: If no, then there must be a curve C such that $K_S \cdot C < 0$. We can always choose C so that there is a contraction morphism $\pi\colon S \longrightarrow T$ which contracts C and there are three cases:

(i) $S = \mathbb{P}^2$, T is a point and C is a line.
(ii) T is a curve, S is a $\mathbb{P}^1$-bundle over T and C is a fibre.
(iii) T is a smooth surface, π is a blow up of a point on T and C is the exceptional divisor.

Step 3: If we are in case (i) or (ii), then stop. Otherwise replace S by T and go back to Step 1.

The fact that we can always find a curve C to contract is a non-trivial result, due to the Italian school of algebraic geometry. It is possible that at Step 1 there is more than one choice of π.

Example 2.4. *Suppose that we start with the blow up S of $\mathbb{P}^2$ at two different points p and q. There are three relevant curves, E and F the exceptional divisors over p and q and L, the strict transform of the line connecting p and q.*

At the first step of the K_S-MMP we are presented with three choices. We can choose to contract E, F or L, since all three of these curves are -1-curves. If we contract E, $\pi\colon S \longrightarrow T$, then at the next step we are presented with two choices of curves to contract on T. We can either contract the image of F, in which case the end product of the MMP is the original $\mathbb{P}^2$. On the other hand, there is a morphism $T \longrightarrow \mathbb{P}^1$. Every fibre is isomorphic to $\mathbb{P}^1$, L is a fibre and F is a section of this morphism. This is a possible end product of the MMP. If instead we decide to contract F, then we get almost exactly the same picture; note however that even though the two $\mathbb{P}^1$-bundles we get are isomorphic, the induced birational map between them is not an isomorphism. However if we choose to contract L then the resulting surface is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Projection to either factor $\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ are two possible other end products of the MMP.

Once again the language of toric geometry gives a convenient way to encode this picture. S corresponds to the fan with one dimensional rays spanned by $(1,0)$, $(0,1)$, $(-1,0)$, $(-1,-1)$ and $(0,-1)$. Blowing down E and F corresponds to removing the two rays $(-1,0)$ and $(0,-1)$. Blowing down E corresponds to removing $(-1,0)$ and the morphism to $\mathbb{P}^1$ corresponds to the projection of $\mathbb{R}^2$ onto the x-axis. Contracting L corresponds to removing $(-1,-1)$; the resulting fan is clearly the fan for $\mathbb{P}^1 \times \mathbb{P}^1$.

The most important feature of any algorithm is termination. Termination for surfaces is clear. Every time we contract a copy of $\mathbb{P}^1$, topologically we are replacing a copy of the sphere S^2 by a point. Consequently the second Betti number $b_2(S)$ drops by one and so the MMP terminates after at most $b_2(S)$-steps. Equivalently the Picard number drops by one at every step.

One interesting application of the MMP for surfaces is in the construction of a compactification $\overline{M}_g$ of the moduli space of curves M_g. In particular suppose we are given a family $\pi\colon S_0 \longrightarrow C_0$ of smooth projective curves over a smooth affine curve C_0. Then there is a unique projective curve C which contains C_0 as an open subset. We would like to complete S_0 to a family of curves $\pi\colon S \longrightarrow C$, which makes the following diagram commute:

$$\begin{array}{ccc} S_0 & \longrightarrow & S \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle \pi} \\ C_0 & \longrightarrow & C. \end{array}$$

Here the horizontal arrows are inclusions. We would like the fibres of π to be nodal projective curves, whose canonical divisor is ample. The first observation is that this is not in fact possible. In general we can only fill in this family after a finite cover of C_0.

Here is the general algorithm. The first step is to pick any compactification of S_0 and of the morphism π_0. The next step is to blow up S, so that the reduced fibres are curves with nodes. After this we take a cover of C and replace S by the normalisation of the fibre product. If the cover of C is sufficiently ramified along the singular fibres of π this step will eliminate the multiple fibres. The penultimate step is to run the MMP over C. This has the effect of contracting all -1-curves contained in the fibres of π. The final step is to contract all -2-curves, that is, all copies of $\mathbb{P}^1$ with self-intersection -2.

We now consider the MMP in higher dimension. There is a similar picture, except that we also encounter flips:

Definition 2.5. *Let $\pi\colon X \longrightarrow Z$ be a birational morphism. We say that π is* ***small*** *if π does not contract a divisor. We say that π is a* ***flipping contraction*** *if $-K_X$ is ample over Z and the relative Picard number is one. The* ***flip*** *of π is another small birational morphism $\psi\colon Y \longrightarrow Z$ of relative Picard number one such that K_Y is ample over Z.*

The relative Picard number is the difference in the Picard numbers. The relative Picard number is one if and only if every two curves contracted by π are numerically multiples of each other. In this case a $\mathbb{Q}$-Cartier divisor D is ample over Z if and only if $D \cdot C > 0$ for one curve C contracted by π.

Flops are defined similarly, except that now K_X and K_Y are trivial over Z and yet the induced birational map $X \dashrightarrow Y$ is not an isomorphism. The MMP in higher dimensions proceeds as follows:

Step 0: Start with a smooth projective variety X.

Step 1: Is K_X nef? If yes, then stop. X is a minimal model.

Step 2: If no, then there must be a curve C such that $K_X \cdot C < 0$. We can always choose C so that there is a contraction morphism $\pi \colon X \longrightarrow Z$ which contracts C and there are two cases:

(i) $\dim Z < \dim X$. C is contained in a fibre. The fibres F of π are Fano varieties, so that $-K_F$ is ample. π is a Mori fibre space.

(ii) $\dim Z = \dim X$. In this case π is birational and there are two sub cases:

(a) π contracts a divisor E.

(b) π is small.

Step 3: If we are in case (i), then stop. If we are in case (a) then replace X by Z and go back to (1). If we are in case (b) then replace X by the flip $X \dashrightarrow Y$ and go back to (1).

The fact that we may find C and π at step 2 is quite subtle, and is due to the work of many people, including Kawamata, Kollár, Miyaoka, Mori, Reid, Shokurov and many others. For more details see, for example, the book by Kollár and Mori, [25]. For an excellent survey of flips and flops, especially for threefolds, see [22].

Existence of terminal 3-flips was first proved by Mori, [31]. Kollár and Mori give a complete classification of all terminal flips in [24]. Shokurov proved the existence of 4-fold flips, [40]. Existence in all dimensions was proved in [14] and [15]:

Theorem 2.6 (Existence: Hacon, M$^{\mathrm{c}}$Kernan)**.** *Flips exist in all dimensions.*

Actually stating things this way is a considerable simplification; we also need the main result of [6] to finish a somewhat involved induction. The proof of (2.6) draws considerable inspiration and ideas from two sources. First, Siu's theory of multiplier ideals and his proof of deformation invariance of plurigenera, see [43], especially the recasting of these ideas in the algebraic setting [19], due to Kawamata. Second, Shokurov's theory of saturation of the restricted algebras and his proof of the existence of flips for fourfolds, [40], all of which is succinctly explained in Corti's book, [9].

We have already seen (2.4) that the end product of the MMP is not unique. For surfaces the minimal model is unique. If X is a threefold and $X \dashrightarrow Y$ is a flop then X is minimal if and only if Y is minimal, so there is often more than minimal model. In fact, Kawamata [20] proved that any two minimal models are connected by a sequence of flops.

Example 2.7. *Suppose we start with the elliptic fibration $\pi\colon X \longrightarrow S$ given in* (1.17)*. Possibly replacing S by a finite cover, we may assume that S contains no rational curves. Suppose that we run the K_X-MMP. Every step of the MMP the locus we contract is covered by rational curves. It follows that every step of the MMP is over S and the end product of the MMP is a minimal model. The MMP preserves the property that one isolated fibre is the union of two copies of $\mathbb{P}^1$ meeting in two points. It follows that X has infinitely many minimal models.*

Kawamata has similar examples of Calabi-Yau threefolds with infinitely many minimal models, [18].

If we get down to a Mori fibre space the situation is considerably more complicated, as (1.9) and (1.14) demonstrate. However Sarkisov proposed a way to use the MMP to connect any two birational Mori fibre spaces by a sequence of four types of elementary links, see [8]. The Sarkisov program was recently shown to work in all dimensions in [13].

Note that termination of the MMP is far more subtle in dimension at least three. It is clear that we cannot keep contracting divisors. As in the case of surfaces the relative Picard number drops every time we contract a divisor and is unchanged under flips and so we can only contract a divisor finitely many times. However it is far less clear which discrete invariants improve after each flip.

Conjecture 2.8. *There is no infinite sequence of flips.*

The rest of this paper will be devoted to exploring (2.8).

We know that the MMP always works for toric varieties, due to the work of Reid, [35] and Kawamata, Matsuda and Matsuki, [21]. The proof is almost entirely combinatorial.

3. Local approach to termination

We review the first approach to the termination of flips. The idea is to find an invariant of X which has three properties:

(1) The invariant takes values in an ordered set I.
(2) The invariant always increases after a flip.
(3) The set I satisfies the ascending chain condition (abbreviated to ACC).

Typically the invariant is some measure of the complexity of the singularities of X. Usually it is not hard to ensure that properties (1) and (2) hold. There are many sensible ways to measure the complexity of a singularity and flips tend to improve singularities. The most subtle part seems to be checking that (3) holds as well.

The most naive invariant of any singularity is the multiplicity. If $X \subset \mathbb{C}^{n+1}$ and X is defined by the analytic function $f(z_1, z_2, \dots, z_n)$ the **multiplicity** m of X at the origin is the smallest positive integer such that $f \in \mathfrak{m}^m = \langle z_1, z_2, \dots, z_n \rangle^m$. If we take the reciprocal of the multiplicity then the set

$$I = \{ \frac{1}{m} \,|\, m \in \mathbb{N} \},$$

is naturally ordered and clearly satisfies the ACC. Unfortunately it is hard to keep track of the behaviour of the multiplicity under flips.

The idea is to pick an invariant which is more finely-tuned to the canonical divisor:

Definition 3.1. *Let X be a normal quasi-projective variety. A **log resolution** is a projective morphism $\pi\colon Y \longrightarrow X$ such that Y and the exceptional locus is **log smooth**, that is, Y is smooth and the exceptional locus is a divisor with simple normal crossings.*

If K_X is $\mathbb{Q}$-Cartier then we may write

$$K_Y + E = \pi^* K_X + \sum a_i E_i,$$

*where $E = \sum E_i$ and a_i are rational numbers. The **log discrepancy of** E_i **with respect to** K_X is a_i. The **log discrepancy** of X is the infimum of the a_i, over all exceptional divisors on all log resolutions.*

*We say that X is **terminal**, **canonical**, **log terminal**, **log canonical** if $a > 1$, $a \geq 1$, $a > 0$ and $a \geq 0$.*

*If $V \subset X$ is a closed subset, then the **log discrepancy** of X at V is the infimum of the a_i, over all exceptional divisors whose image is V, and all log resolutions.*

*The **log discrepancy along** V is the infimum of the a_i, over all exceptional divisors whose image is contained in V, and all log resolutions.*

Let us start with some simple examples.

Example 3.2. *Let S be a smooth surface and let $p \in S$. Let $\pi\colon T \longrightarrow S$ blow up p, with exceptional divisor E. Suppose we write*

$$K_T + E = \pi^* K_S + aE.$$

If we intersect both sides with E then we get

$$-2 = K_{\mathbb{P}^1} = K_E = (K_T + E) \cdot E = (\pi^* K_S + aE) \cdot E = aE^2 = -a.$$

So $a = 2$. It is a simple matter to check that if we blow up more over the point p then every exceptional divisor has log discrepancy greater than two. So the log discrepancy of a smooth surface is 2. It is also not *hard to check that if X is not log canonical then the log discrepancy is $-\infty$ and that if X is log canonical the log discrepancy is the minimum of the log discrepancy of the exceptional divisors of any log resolution.*

If X is an affine toric variety, corresponding to the cone σ, then K_X is $\mathbb{Q}$-Cartier if the primitive generators of the one dimensional faces of $\sigma \subset \mathbb{R}^n$ lie in an affine hyperplane (this is always the case if σ is simplicial). In this case there is a linear functional $\phi\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ which takes the value 1 on this hyperplane. The log discrepancy of any toric divisor is the value of ϕ on the primitive generator of the extremal ray corresponding to this divisor. In particular X is log terminal.

For example if X is smooth of dimension n, then X corresponds to the cone spanned by the standard generators $e_1, e_2, \dots, e_n$ of the standard lattice $\mathbb{Z}^n \subset \mathbb{C}^n$. If we insert the vector $e_1 + e_2$ then the log discrepancy of the exceptional divisor of the blow up of the corresponding codimension two coordinate subspace is 2 and this is the log discrepancy of X. If we insert the sum $e_1 + e_2 + \cdots + e_n$ this corresponds to blowing up the origin. The log discrepancy of the exceptional divisor is n and this is the log discrepancy of X at the origin.

Proposition 3.3. *If $\pi\colon X \dashrightarrow Y$ is a flip, then the log discrepancy of any divisor E never goes down and always goes up if the centre of E is contained in the indeterminacy locus of π.*

Proof. See (5.11) of [21]. □

Definition 3.4 (Shokurov)**.** *Let X be a threefold with canonical singularities. The **difficulty** of X is the number of divisors of log discrepancy less than two.*

Lemma 3.5. *Let X be a threefold with canonical singularities.*

Then

1. *the difficulty is finite,*
2. *if the difficulty is zero, then X is smooth, and*
3. *the difficulty always goes down under flips.*

Proof. It is easy to check that (1) holds by direct computation on a log resolution. (2) follows from the classification of canonical threefold singularities.

Let $\phi\colon X \dashrightarrow Y$ be a flip. Let C be a flipped curve, that is, a curve contained in the indeterminacy locus of ϕ^{-1}. As the log discrepancy goes up under flips, Y is terminal about a general point of C. It follows that Y is smooth along C so that there is an exceptional divisor E with centre C of log discrepancy two. The log discrepancy of E with respect to X must be less than two, by (3.3). It follows that the difficulty decreases by at least one. (3) follows easily. □

Note that (3.5) easily implies that there is no infinite sequence of flips, starting with a threefold with canonical singularities. There have been many papers which extend Shokurov's work to higher dimensions, most especially to dimension four, for example [28], [11] and [3]. Unfortunately in higher dimensions there are infinitely many divisors of log discrepancy at most two and singular varieties of log discrepancy greater than two. It seems hard to control the situation using only the difficulty.

To remedy this situation, Shokurov has proposed some amazing conjectural properties of the log discrepancy:

Conjecture 3.6 (Shokurov)**.** *Fix a positive integer n. The set*

$$L_n = \{\, a \in \mathbb{Q} \mid a \text{ is the log discrepancy of a normal variety of dimension } n \,\},$$

satisfies the ACC.

Conjecture 3.7 (Shokurov)**.** *Let X be a quasi-projective variety. The function*

$$a\colon X \longrightarrow \mathbb{Q},$$

which sends a point x to the log discrepancy of X at x is lower semi-continuous.

Theorem 3.8 (Shokurov)**.** *Assume $(3.6)_n$ and $(3.7)_n$.*

Then every sequence of flips in dimension n terminates, that is, $(2.8)_n$ holds.

Proof. We sketch Shokurov's beautiful argument.

Suppose not, that is, suppose we are given an infinite sequence of flips $\phi_i\colon X_i \dashrightarrow X_{i+1}$. Let E_i be the locus of indeterminacy of ϕ_i. Then E_i is a closed subset of X_i.

Let a_i be the log discrepancy of X_i along E_i. Let

$$\alpha_i = \inf\{\, a_j \mid j \geq i \,\}.$$

(3.3) implies that $\alpha_i \leq \alpha_{i+1}$. As we are assuming $(3.6)_n$ it follows that α_i is eventually constant. Suppose that $\alpha_i = a$, for i sufficiently large. Then $a_i \geq a$ for all i, with equality for infinitely many i.

By assumption for each i there is a log resolution and at least one exceptional divisor F_i whose centre is contained in E_i such that the log discrepancy of F_i is a. Let d_i be the maximal dimension of the centre of any such exceptional divisor F_i. Pick d such that $d_i \leq d$ with equality for infinitely many i.

Let

$$W_i = \{\, x \in X_i \mid x \in V,\ \dim V = d, \text{ log discrepancy of } X \text{ at } V \text{ is at most } a \,\}.$$

As we are assuming $(3.7)_n$, $W_i \subset X_i$ is a closed subset. Moreover some results of Ambro [4] imply that if $V \subset W_i$ is a closed subset of dimension d then the log discrepancy of X_i at V is at most a with equality if V passes through the general point of W_i.

It is not hard to argue that as d is maximal, eventually there is an induced birational map $\phi_i \colon W_i \dashrightarrow W_{i+1}$, which by assumption is infinitely often not an isomorphism along some subvariety of dimension d. If $V \subset W_{i+1}$ is of dimension d then the log discrepancy of X_{i+1} at V is a. It follows that ϕ_i^{-1} must be an isomorphism along V, since the log discrepancy of X_i along E_i is a and log discrepancies only go up under flips, (3.3).

The only possibility is that ϕ_i must contract a subvariety of dimension d. But this cannot happen infinitely often, a contradiction. $\square$

Note that there are more general versions of (3.6) and (3.7), which involve log pairs (X, Δ) and that Shokurov proves that if one assumes these more general conjectures then any sequence of log flips terminates. For more details see [41].

Unfortunately both (3.6) and (3.7) seem to be hard conjectures. We know $(3.6)_2$ and $(3.7)_2$, by virtue of Alexeev's classification of log canonical surface singularities. We know that

$$L_3 \cap [1, \infty) = \{\, 1 + \frac{1}{r} \,|\, r \in \mathbb{N} \cup \{\infty\} \,\},$$

by virtue of the classification of terminal singularities due to Mori, [30] and Reid [36] and a result of Kawamata, see the appendix to [37]. Borisov [7] proved that (3.6) holds for toric varieties. Ambro proved [4] that $(3.7)_3$ holds and that (3.7) holds for toric varieties.

One interesting consequence of (3.7) is the following:

Conjecture 3.9 (Shokurov)**.** *Let X be a normal quasi-projective variety of dimension n.*

Then the log discrepancy of any point is at most n.

Indeed if $x \in X$, then pick a curve C which contains x and intersects the smooth locus X_0 of X. Then x is the limit of points $y \in C \cap X_0$. We have already seen that the log discrepancy of X at y is n. So if we assume $(3.7)_n$ then the log discrepancy of X at x is at most n.

Note that to prove (3.9) we may assume that the log discrepancy is greater than one, that is, we may assume that X is terminal. Even though (3.9) would appear to be much weaker than (3.7), we only know that $(3.9)_3$ holds by virtue of Mori's classification of threefold terminal singularities and a result of Markushevich, [27].

4. Global approach to termination

Instead of focusing on showing that some invariant satisfies the ACC, the global approach to termination tries to use the global geometry of X. At this point it is convenient to work with:

Definition 4.1. *A* ***log pair*** (X, Δ) *is a normal variety together with a divisor* $\Delta \geq 0$ *such that* $K_X + \Delta$ *is* $\mathbb{R}$*-Cartier.*

One can define the log discrepancy and the various flavours of log terminal, just as for the canonical divisor.

Example 4.2. *Let* X *be a toric variety and let* $\Delta = \sum D_i$ *be the sum of the invariant divisors. Then* $K_X + \Delta \sim_{\mathbb{Q}} 0$ *so that* (X, Δ) *is a log pair. We may find* $\pi \colon Y \longrightarrow X$ *a toric log resolution. Note that*

$$K_Y + \Gamma = \pi^*(K_X + \Delta),$$

where $\Gamma = \sum G_i$ *is the sum of the invariant divisors on* Y*, since both sides are zero. As* π *is toric,* Γ *contains all of the exceptional divisors with coefficient one. It follows that* (X, Δ) *is log canonical.*

We use the following finiteness result:

Theorem 4.3 (Birkar, Cascini, Hacon, M^cKernan)**.** *Let* X *be a smooth projective variety. Fix an ample divisor* A *and finitely many divisors* $B_1, B_2, \dots, B_k$ *such that* $(X, \sum B_i)$ *is log smooth.*

Then there are finitely many $1 \leq i \leq m$ *rational maps* $\phi_i \colon X \dashrightarrow Y_i$ *such that if* $(b_1, b_2, \dots, b_k) \in [0,1]^k$ *and* $\phi \colon X \dashrightarrow Y$ *is a weak log canonical model of* $K_X + A + \sum b_i B_i$ *then* $\phi = \phi_i$ *for some index* $1 \leq i \leq m$*.*

We have already remarked (2.7) that there are examples due to Reid of threefolds with infinitely many minimal models. The presence of the divisor A is therefore important in the statement of (4.3). However Shokurov [38] proves a similar result for threefolds, but now without the ample divisor A and shows that the same result holds in all dimensions if one knows the abundance conjecture, (5.7). In a related direction, Kawamata, [18] and Morrison [32] have conjectured that the number of minimal models is finite up to birational automorphisms of X, when X is Calabi-Yau and Δ is empty.

We use (4.3) to run a special MMP, known as the MMP with scaling.

Step 0: Start with a projective variety X, an ample divisor A, a divisor $B = \sum b_i B_i$, where $(X, \sum B_i)$ is log smooth and $(b_1, b_2, \dots, b_k) \in [0,1]^k$ and an ample divisor H such that $K_X + A + B + H$ is nef.

Step 1: Let

$$\lambda = \inf\{\, t \in [0,1] \mid K_X + A + B + tH \text{ is nef} \,\},$$

be the nef threshold.

Step 2: Is $\lambda = 0$? If yes, then stop.

Step 3: If no, then there must be a curve C such that $(K_X + A + B) \cdot C < 0$ and $(K_X + A + B + \lambda H) \cdot C = 0$. We can always choose C so that there is a contraction morphism $\pi \colon X \longrightarrow Z$ which contracts C and there are two cases:

(i) $\dim Z < \dim X$. C is contained in a fibre. $-(K_X + A + B)$ is ample on a fibre.

(ii) $\dim Z = \dim X$. In this case π is birational and there are two subcases:

(a) π contracts a divisor E.

(b) π is an isomorphism in codimension at least two.

Step 4: If we are in case (i), then stop. If we are in case (a) then replace X by Z and go back to (2). If we are in case (b) then replace X by the flip $X \dashrightarrow Y$ and go back to (2).

Note that if H is any ample divisor then $K_X + A + B + tH$ is ample for any t sufficiently large. So finding an ample divisor H such that $K_X + A + B + H$ is nef is never an issue. Note also that if $\lambda = 0$ then $K_X + A + B$ is nef and we have arrived at a log terminal model. The only significant difference between the MMP with scaling and the usual MMP is that we only choose to contract those curves on which $K_X + A + B + \lambda H$ is zero. With this choice, it is easy to see that we keep the condition that $K_X + A + B + \lambda H$ is nef. More to the point, every step of the MMP is a weak log canonical model of $K_X + A + (B + \lambda H)$, for some choice of $\lambda \in [0, 1]$. Finiteness of models, (4.3) and the fact that we never return to the same model, (3.3), implies that the MMP with scaling always terminates.

To run the MMP with scaling, we need the ample divisor A. If we start with $K_X + \Delta$ kawamata log terminal, we can find A ample and $B \geq 0$ such that $K_X + \Delta \sim_{\mathbb{R}} K_X + A + B$, where $K_X + B$ is kawamata log terminal if and only if Δ is big. If we start with a birational map $\pi \colon X \longrightarrow Y$ then every divisor is big over Y and so the MMP with scaling always applies if we work over Y.

For example, we may use the MMP with scaling to show that every complex manifold which is birational to a projective variety but which is not a projective variety must contain a rational curve. For example, one might modify Hironaka's example, (1.15), by starting with any smooth projective threefold X with two curves intersecting transversely at two points. It is easy to find many examples which don't contain any rational curves. But the next step involves blowing up both curves and so M contains lots of rational curves.

Shokurov [39] proved the following result assuming the full MMP and our proof is based heavily on his ideas:

Theorem 4.4 (Birkar, Cascini, Hacon, M$^{\text{c}}$Kernan)**.** *Let M be a smooth complex manifold. Suppose there is a proper birational map $\pi\colon X \longrightarrow M$ such that X is smooth and projective.*

If M does not contain a rational curve then M is projective.

Proof. Pick an ample divisor H such that $K_X + H$ is ample. We run the K_X-MMP with scaling of H. Suppose that $\pi\colon X \longrightarrow Y$ is a K_X-negative contraction. by a result of Miyaoka and Mori, [29], the locus contracted by π is covered by rational cuves. As M does not contain a rational curve, it follows that π is a morphism over M. In particular the (K_X+H)-MMP is automatically a MMP over Y. As π is birational, it follows that the MMP with scaling terminates, as observed above. At the end we have a projective variety Y such that K_Y is nef and a birational morphism $f\colon Y \longrightarrow M$. As M is smooth it follows that f is an isomorphism so that M is a projective variety. $\square$

5. Local-global approach to termination

Even though the MMP with scaling is useful, it is becoming increasingly clear that we would still like to have the full MMP, even in the special case when Δ is big. This would be useful in the construction of the moduli space of varieties of general type. One possible approach is to try to blend both the local and the global approach to termination of flips.

We have already seen that the log discrepancy always improves under flips. However the most fundamental invariant of any singularity would seem to be the multiplicity. The log canonical threshold is a more sophisticated version of the multiplicity which takes into account higher terms and is at the same time more adapted to the canonical divisor. If $X \subset \mathbb{C}^n$ is a hypersurface, then the log canonical threshold λ of X at the origin, is the largest t such that $(\mathbb{C}^n, tX)$ is log canonical in a neighbourhood of the origin. If X has multiplicity m at the origin, then we have

$$\frac{1}{m} \leq \lambda \leq \frac{n}{m}.$$

Shokurov has conjectured that the set of log canonical thresholds should satisfy the ACC:

Conjecture 5.1 (Shokurov)**.** *Fix a positive integer n and a subset $I \subset [0,1]$ which satisfies the descending chain condition (abbreviated to DCC).*

Then there is a finite set $I_0 \subset I$ such that if

(1) *X is a variety of dimension n,*
(2) *(X, Δ) is log canonical,*
(3) *every component of Δ contains a non kawamata log terminal centre of (X, Δ), and*
(4) *the coefficients of Δ belong to I,*

then the coefficients of Δ belong to I_0.

Example 5.2. *Let $X \subset \mathbb{C}^n$ be the hypersurface given by the equation*

$$x_1^{a_1} + x_2^{a_2} + \cdots + x_n^{a_n} = 0.$$

Then the log canonical threshold is

$$\min(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}, 1).$$

It is elementary to check that these numbers satisfy the ACC.

Theorem 5.3 (Special termination; Shokurov)**.** *Assume $(2.8)_{n-1}$.*

Let (X, Δ) be a projective log canonical pair of dimension n. Let $\phi_i \colon X_i \dashrightarrow X_{i+1}$ be a sequence of flips. Let $Z_{ij} \subset X_i$ be the locus where the induced birational map $X_i \dashrightarrow X_j$ is not an isomorphism. Let

$$Z_i = \bigcup_{j>i} Z_{ij}.$$

Let V_i be the locus where $K_{X_i} + \Delta_i$ is not kawamata log terminal.

Then V_i and Z_i eventually don't intersect.

Note that V_i is a closed subset of X_i, whilst Z_i is a countable union of closed subsets of X_i.

Theorem 5.4 (Birkar)**.** *Assume $(2.8)_{n-1}$ and $(5.1)_n$. Let (X, Δ) be a projective kawamata log terminal pair of dimension n.*

If there is a divisor $M \geq 0$ which is numerically equivalent to $K_X + \Delta$, then every sequence of $(K_X + \Delta)$-flips terminates.

Proof. We sketch Birkar's ingenious argument.

Let $\phi_i \colon X_i \dashrightarrow X_{i+1}$ be a sequence of $(K_X + \Delta)$-flips. Let $Z_{ij} \subset X_i$ be the locus where the induced birational map $X_i \dashrightarrow X_j$ is not an isomorphism. Let

$$Z_i = \bigcup_{j>i} Z_{ij}.$$

Let Δ_i and M_i be the strict transforms of Δ and M. Note that $K_{X_i} + \Delta_i$ is numerically equivalent to M_i. In particular $\phi_1, \phi_2, \ldots$ is also a sequence of $(K_X + \Delta + tM)$-flips for any $t \geq 0$. Let

$$\lambda_i = \sup\{\, t \in \mathbb{R} \mid K_{X_i} + \Delta_i + tM_i \text{ is log canonical along } Z_i \,\},$$

be the log canonical threshold along Z_i. Note that $\lambda_i \leq \lambda_{i+1}$, as log discrepancies only go up under flips. In particular if I is the set of all coefficients of $\Delta_i + \lambda_i M_i$, then I satisfies the DCC. As we are assuming $(5.1)_n$, it follows that $\lambda_1, \lambda_2, \dots$ is eventually constant. Suppose that $\lambda_i = \lambda$, for all $i \geq i_0$. As we are assuming $(2.8)_{n-1}$, (5.3) implies that $V_i \cap Z_i$ is eventually empty, that is, the sequence of flips is finite. $\square$

Note that we cheated a little in the proof of (5.4). Eventually $K_{X_i} + \Delta_i + \lambda_i M_i$ is not log canonical, so that strictly speaking (5.3) does not apply. In practice one can get around this by passing to a log terminal model. For more details, see [5].

To give a complete proof of termination of flips using (5.4), note that we need to do two things. Obviously we need to prove (5.1). However to complete the induction we need to deal with the case when $K_X + \Delta$ is not numerically equivalent to a divisor $M \geq 0$. This part breaks up into two separate pieces.

Definition 5.5. *Let X be a normal projective variety. We say that D is **pseudo-effective** if D is a limit of big divisors.*

Conjecture 5.6. *Suppose that $K_X + \Delta$ is kawamata log terminal.*

If $K_X + \Delta$ is pseudo-effective then there is a divisor $M \geq 0$ such that $K_X + \Delta \sim_{\mathbb{R}} M \geq 0$.

One should understand this conjecture as being part of the abundance conjecture:

Conjecture 5.7 (Abundance)**.** *Let (X, Δ) be a projective log canonical pair.*

If $K_X + \Delta$ is nef then it is semiample.

In particular (5.6) seems very hard. One way to get around this gap in our knowledge is to assume that Δ is big. In this case we have, [6] and [42]:

Theorem 5.8 (Birkar, Cascini, Hacon, M$^{\text{c}}$Kernan; Siu)**.** *Suppose that $K_X + \Delta$ is kawamata log terminal.*

If $K_X + \Delta$ is pseudo-effective and Δ is big then there is a divisor $M \geq 0$ such that $K_X + \Delta \sim_{\mathbb{R}} M \geq 0$.

Lazić [26] and Păun [34] have since given simpler proofs of (5.8). Note that the steps of the MMP preserve the property that Δ is big. The final piece of the puzzle is to deal with the case that $K_X + \Delta$ is not pseudo-effective. It seems that ideas from bend and break, [29], might prove useful in this case.

Part of the appeal of this approach to termination is that (5.1) seems far more tractable than (3.6). We know (5.1) in some highly non-trivial examples. For example, Alexeev proved $(5.1)_3$ [2], using boundedness of log del Pezzo surfaces whose log discrepancy is bounded away from zero. Further, de Fernex, Ein and Mustaţă, have proved the case when X is smooth, see [10] and the references therein.

We end with some speculation about a way to attack (5.1). We first note a reduction step due originally to Shokurov, see [33]. To prove $(5.1)_n$ we just need to prove:

Corollary 5.9. *Fix a positive integer n and a subset $I \subset [0,1]$ which satisfies DCC.*

Then there is a finite set $I_0 \subset I$ such that

(1) *X is a projective variety of dimension n,*
(2) *(X, Δ) is kawamata log terminal,*
(3) *Δ is big,*
(4) *the coefficients of Δ belong to I, and*
(5) *$K_X + \Delta$ is numerically trivial,*

then the coefficients of Δ belong to I_0.

in dimension $n-1$. To this end, consider:

Conjecture 5.10. *Fix a positive integer n.*

Then there is a constant m such that if

- *X is a projective variety of dimension n,*
- *(X, Δ) is log canonical and log smooth,*
- *$K_X + \Delta$ is big and*
- *r is a positive integer such that $r(K_X + \Delta)$ is Cartier,*

then the rational map determined by the linear system $|mr(K_X + \Delta)|$ is birational.

Note that (5.10) closely resembles some results and conjectures stated in [12]. We note that this is slightly deceptive, since (5.10) seems quite a bit harder than these conjectures. Hopefully (5.10) has a better formulation, which is more straightforward to prove and has the same consequences. Note that if we add the condition that $K_X + \Delta$ is nef then the existence of m is a result due to Kollár, [23], an effective version of the base point free theorem.

The following is standard:

Lemma 5.11. *Let X be a smooth projective variety of dimension n and let D be a Cartier divisor on X such that ϕ_D is birational.*

Then $\phi_{K_X+(2n+1)D}$ is birational.

The hope is to prove $(5.9)_n$ using:

Lemma 5.12. *Assume* $(5.10)_n$. *Let* $I \subset [0,1]$ *be a finite set and let* n *be a positive integer. Suppose that* $I \cup \{1\}$ *are linearly independent real numbers over the rationals.*

Then there is a positive real number $\epsilon > 0$ *such that if*

- X *is a projective variety of dimension* n,
- (X, Δ) *is log canonical and log smooth,*
- *the coefficients of* Δ *belong to* I, *and*
- $K_X + \Delta$ *is big*

then $K_X + (1-\epsilon)\Delta$ *is big.*

Proof. Let m be the constant given by (5.10). By simultaneous Diophantine approximation applied to the finite set I, we may pick a positive integer r with the following properties: if $a \in I$ then there is a rational number $b \geq a$ such that rb is an integer and

$$b - a < \frac{1}{2m(2n+1)r}.$$

If we set

$$t = m(2n+1)r,$$

then we may pick $\Theta \geq \Delta$ such that

$$\|\Delta - \Theta\| < \frac{1}{2t},$$

where $r\Theta$ is Cartier. By (5.11),

$$K_X + m(n+1)r(K_X + \Theta) = (t+1)(K_X + \frac{t}{t+1}\Theta),$$

defines a birational map. In particular

$$K_X + \left(1 - \frac{1}{2t}\right)\Theta,$$

is big. So we may take

$$\epsilon = \frac{1}{2m(2n+1)r}. \qquad \square$$

In practice we cannot assume that the numbers $I \cup \{1\}$ are independent over the rationals and (5.12) is a simplification of our speculative argument.

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