# BOUNDEDNESS OF VARIETIES OF LOG GENERAL TYPE

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ABSTRACT. We survey recent results on the boundedness of the moduli functor of stable pairs.

CONTENTS



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## 1. Introduction

The purpose of this paper is to give an overview of recent results on the moduli of varieties of general type. We start with a gentle introduction to the subject, reviewing the case of curves and surfaces as motivation for some of the definitions. Then we switch gears a little and collect together in one place an account of boundedness of the moduli functor. None of the results here are new but we thought it would be useful to present them together, as currently they are spread over several papers. We also take this opportunity to present an alternative argument for one step of the proof. Due to constraints imposed by space, we do not give full details for many of the proofs; anyone wanting to see more details should look at the original papers.

The theory of moduli in higher dimensions is extremely rich and interesting and so we are obliged to skip many interesting topics, which are fortunately covered in the many excellent surveys and books, see for example [25], [26] and [27]. We focus on two aspects of the construction, what we need to add to get a compact moduli space and how to prove boundedness. We start with what we should add to get a compact moduli space.

The moduli space $M_g$ of smooth curves of genus $g \geq 2$ is a quasi-projective variety of dimension $3g-3$. The moduli space of stable curves $\overline{M}_g$ is a geometrically meaningful compactification of $M_g$, so that $\overline{M}_g$ is projective and $M_g$ is an open subset. Geometrically meaningful refers to the fact the added points correspond to geometric objects which are as close as possible to the original objects. In the case of $M_g$ we add stable curves $C$, connected curves of arithmetic genus $g$, with nodal singularities, such that the automorphism group is finite, or better (and equivalently), the canonical divisor $K_C$ is ample.

We adopt a similar definition of stable in higher dimensions.

**Definition 1.0.1.** *A* **semi log canonical model** $(X, B)$ *is a projective semi log canonical pair (cf. §1.4) such that* $K_X + B$ *is ample. Fix* $n \in \mathbb{N}$*,* $I \subset [0, 1]$ *and* $d \in \mathbb{R}_{>0}$*. Let* $\mathfrak{F}_{\mathrm{slc}}(n, I, d)$ *be the set of all* $n$*-dimensional semi log canonical models such that the coefficients* *of $B$ belong to $I$ (that is,* $\operatorname{coeff}(B) \subset I$*), $K_X + B$ is $\mathbb{Q}$-Cartier and $(K_X + B)^n = d$.*

We now attempt to give some motivation for the admittedly technical definition of semi log canonical models.

1.1. **Semi log canonical models.** There are in general many degenerations of the same family of varieties. Given a moduli problem properness corresponds to existence and uniqueness of the limit. Given a family of smooth curves there is a unique stable limit, as proved by Deligne and Mumford [8].

We review the construction of the stable limit. Let $f\colon X^0 \longrightarrow C^0$ be a family of smooth curves of genus $g \geq 2$ over a smooth curve $C^0 = C \setminus 0$ where $C$ is an affine curve and 0 is a closed point. By semistable reduction, after replacing $C^0$ by an appropriate base change, we may assume that there is a proper surjective morphism $f\colon X \longrightarrow C$ such that $X$ is smooth and the central fibre $X_0$ is reduced with simple normal crossings. The choice of $X_0$ is not unique, since we are free to blow up the central fibre. So we run the minimal model program over $C$, contracting $-1$-curves (that is, curves $E \cong \mathbb{P}^1$ such that $E^2 = K_X \cdot E = -1$) in the central fibre. We end with a relative minimal model $X^m \longrightarrow C$, so that $X^m$ is smooth and $K_{X^m/C}$ is nef over $C$. If we further contract all $-2$-curves, that is, curves $E \cong \mathbb{P}^1$ such that $E^2 = -2$ and $K_X \cdot E = 0$ then we obtain the relative canonical model $X^c \longrightarrow C$. The model $X^c$ is characterised by the fact that it has Gorenstein canonical (aka Du Val, aka ADE) singularities and $K_{X^c/C}$ is ample over $C$.

A key observation is that we can construct the relative canonical model directly as

$$X^c = \operatorname{Proj}_C R(X, K_X) \qquad \text{where} \qquad R(X, K_X) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X))$$

is the canonical ring; note that since $C$ is affine, $H^0(X, \mathcal{O}_X(mK_X))$ can be identified with the $\mathcal{O}_C$-module $f_*\mathcal{O}_X(mK_X)$. Observe that $X^c$ is isomorphic to $X$ over $C^0$. Since the relative canonical model is unique, it follows that the family above has a unique compactification to a family of stable curves (that is, the moduli functor of stable curves is proper).

Here is another instructive example (cf. [25]). Let $S$ be any smooth projective surface such that $K_S$ is ample. Consider the family $X = S \times \mathbb{A}^1$ and three sections $C_i$ with $i \in \{1, 2, 3\}$ which meet as transversely as possible in a point $(p, 0) \in S \times 0$. Blowing up the $C_i$ in different orders we obtain two families $X^1$ and $X^2$ which are isomorphic over $\mathbb{A}^1 \setminus 0$

but have distinct central fibres $X^1_0 \neq X^2_0$. Therefore the corresponding moduli functor is not proper. If however we only consider canonical models, then this problem does not appear since the relative canonical model

$$\operatorname{Proj} R(X^i, K_{X^i}) = \operatorname{Proj}(\bigoplus_{m\geq 0} H^0(X^i, \mathcal{O}_{X^i}(mK_{X^i}))) \cong X$$

is unique.

Properness of the moduli functor of semi log canonical models is established in [16]. The proof is similar to the argument sketched above for stable curves, except that an ad hoc argument is necessary to construct the relative canonical model, as the minimal model program for semi log canonical pairs is only known to hold in special cases.

The moduli space $M_{g,n}$ of smooth curves $C$ of genus $g$ with $n$ points $p_1, p_2, \dots, p_n$ is a natural generalisation of the moduli space of curves. It has a natural compactification $\overline{M}_{g,n}$, the moduli space of stable curves of genus $g$ with $n$ points. The points of $\overline{M}_{g,n}$ correspond to connected nodal curves with $n$ labelled points $p_1, p_2, \dots, p_n$ which are not nodes such that $K_S + \Delta$ is ample, where $\Delta$ is the sum of the labelled points. Therefore a stable pointed curve is the same as a semi log canonical model (up to ordering the labelled points), with coefficient set $I = \{1\}$.

There are many reasons to consider labelled points. $\overline{M}_{0,n}$ is a non-trivial moduli space with a very interesting geometry and yet it is given by an explicit blow up of projective space. On the other hand, allowing the coefficients of $\Delta$ to vary, so that we take different choices for the coefficient set $I$, gives a way to understand the extremely rich geometry of the moduli space of curves. For different choices of $I$ we get slightly different moduli problems and so we get different birational models of $\overline{M}_{g,n}$, [17]. Finally the normalisation of a stable curve is a stable pointed curve such that the inverse image of the nodes are labelled points. Studying stable pointed curves offers an inductive way to understand the geometry of $\overline{M}_g$.

There is a similar picture in higher dimensions. We know of the existence of a moduli space of semi log canonical models in many cases. We will sometimes refer to this space as the KSBA compactification (constructed by Kollár, Shepherd-Barron and Alexeev). If $S$ is a cubic surface in $\mathbb{P}^3$ then $K_S + \Delta = -8K_S$ is ample and log canonical, where $\Delta$ is the sum of the twenty seven lines, so that $(X, \Delta)$ is a semi log canonical model. Therefore a component of the KSBA compactification with $I = \{1\}$ gives a moduli space of cubic surfaces, [11]. If $C$ is a smooth plane curve of degree $d > 3$ then $K_S + tC$ is ample for any $t > 3/d$. Therefore a component of the KSBA compactification for suitable choice of coefficient set $I$ gives a compactification of the moduli space of plane curves of degree $d$, [10]. On the other hand, if we allow the coefficients of $\Delta$ to vary then this induces birational maps between moduli spaces and we can connect two moduli spaces by a sequence of such transformations.

If $(X, \Delta)$ is a semi log canonical pair then $X$ is in general not normal. If $\nu\colon Y \longrightarrow X$ is the normalisation then we may write

$$K_Y + \Gamma = \nu^*(K_X + \Delta).$$

The divisor $\Gamma$ is the strict transform of $\Delta$ plus the double locus taken with coefficient one. If $\Delta = 0$ then $\Gamma = 0$ if and only if $X$ is normal. The pair $(Y, \Gamma)$ is log canonical and it is a disjoint union of log canonical pairs $(Y_i, \Gamma_i)$. The pair $(X, \Delta)$ is obtained from $(Y, \Gamma)$ by an appropriate idenfication of the double locus. By a result of Kollár (cf. [24] and [25, 5.13]), if $(X, \Delta)$ is a semi log canonical model, then it can be recovered from the data of $(Y, \Gamma)$ and an involution of the double locus (that is, the components of $\Gamma$ which do not correspond to components of $\Delta$).

We have already seen that it is interesting to allow the coefficients of $\Delta$ to be fractional. It is also useful when trying to establish boundedness by induction on the dimension. For example if $(X, \Delta = S + B)$ is a log canonical pair, the coefficients of $\Delta$ are all one and $S$ is a prime divisor which is a component of $\Delta$ then by adjunction (cf. Theorem 2.2.1)

$$(K_X + B)|_S = K_S + \mathrm{Diff}_S(B - S)$$

where $(S, \mathrm{Diff}_S(B - S))$ is a log canonical pair and the coefficients of $\mathrm{Diff}_S(B - S)$ belong to

$$J = \{\, 1 - \frac{1}{n} \,|\, n \in \mathbb{N} \,\} \cup \{1\}.$$

In fact the coefficients of $\mathrm{Diff}_S(B - S)$ belong to $J$ whenever the coefficients of $B$ belong $J$. As $J$ is the smallest set containing 1 closed under taking the different, the set of coefficients $J$ is sometimes called the **standard coefficient set**.

Note that the set $J$ is not finite, however it satisfies the **descending chain condition** (or DCC condition), that is, every non increasing sequence is eventually constant. To prove boundedness it is convenient to work with any coefficient set $I \subset [0, 1]$ which satisfies the DCC.

We note that there is one aspect of the theory of moduli in higher dimensions which is quite different from the case of curves. The moduli space $\overline{M}_g$ of curves is irreducible. Moreover $M_g$ is a dense open subset. However even if we take $I = \emptyset$ and fix $d$ the KSBA moduli space might have more than one component and no point of these components corresponds to a normal surface.

1.2. **Main Theorems.** Our main result ([1, 2] for the surface case and [15] in general) is the following.

**Theorem 1.2.1.** *Fix $n \in \mathbb{N}$, a set $I \subset [0,1] \cap \mathbb{Q}$ satisfying the DCC and $d > 0$. Then the set $\mathfrak{F}_{\mathrm{slc}}(n, I, d)$ is bounded, that is, there exists a projective morphism of quasi-projective varieties $\pi\colon \mathcal{X} \longrightarrow T$ and a $\mathbb{Q}$-divsor $\mathcal{B}$ on $\mathcal{X}$ such that the set of pairs $\{ (\mathcal{X}_t, \mathcal{B}_t) \mid t \in T \}$ given by the fibres of $\pi$ is in bijection with the elements of $\mathfrak{F}_{\mathrm{slc}}(n, I, d)$.*

The above result is equivalent to:

**Theorem 1.2.2.** *Fix $n \in \mathbb{N}$, a set $I \subset [0,1] \cap \mathbb{Q}$ satisfying the DCC and $d > 0$. Then there is an integer $r = r(n, I, d)$ such that if $(X, B) \in \mathfrak{F}_{\mathrm{slc}}(n, I, d)$ then $r(K_X + B)$ is Cartier and very ample.*

*In particular, the coefficients of $B$ always belong to a finite set $I_0 \subset I$.*

One of the main results necessary to prove the previous theorem is the following, which was conjectured in [21, 1].

**Theorem 1.2.3.** *Fix $n \in \mathbb{N}$ and a set $I \subset [0,1] \cap \mathbb{Q}$ satisfying the DCC. Let*

$$\mathcal{V}(n, I) = \{ d = (K_X + B)^n \mid (X, B) \in \mathfrak{F}_{\mathrm{slc}}(n, I) \}$$

*be the set of all possible volumes of semi log canonical models of dimension $n$ with coefficients belonging to $I$.*

*Then $\mathcal{V}(n, I)$ satisfies the DCC. In particular it has a minimal element $v(n, I) > 0$.*

If $\dim X = 1$ then $X$ is a curve and

$$\mathrm{vol}(X, K_X + B) = \deg(K_X + B) = 2g - 2 + \sum b_i,$$

where $g$ is the arithmetic genus of $X$ and $B = \sum b_i B_i$. Thus, the set

$$\mathcal{V}(1, I) = \{2g - 2 + \sum b_i | b_i \in I\} \cap \mathbb{R}_{>0}$$

of possible volumes satisfies the DCC. For example, if $I$ is empty, then $v(1, \emptyset) = 2$ and if $J$ is the set of standard coefficients, then it is well known that $v(1, J) = 1/42$. Finally, if $I = \{0, 1\}$, one sees that $mK_X$ is very ample for all $m \geq 3$, as an easy consequence of Riemann Roch.

If $\dim X = 2$, and $X$ has canonical singularities then the canonical divisor is Cartier and in particular $K_X^2 \in \mathbb{N}$ so that $K_X^2 \geq 1$. By a result of Bombieri, it is also known that $mK_X$ is very ample for $m \geq 5$ [4], [5] (a similar result also follows in positive characteristic [9]). On the other hand $\mathcal{V}(2, I)$ is hard to compute and there are no explicit bounds known for $r(2, \emptyset, d)$.

If $\dim X = 3$ then there are semi log canonical models with canonical singularities of arbitrarily high index, therefore there is no integer $r > 0$ such that $rK_X$ is very ample for any 3-dimensional canonical model. Since $K_X$ is not necessarily Cartier, the volume $K_X^3$ may not be an integer and in particular it may be smaller than 1. In fact by [18] a general hypersurface $X$ of degree 46 in weighted projective space $\mathbb{P}(4, 5, 6, 7, 23)$ has volume $K_X^3 = 1/420$ and $mK_X$ is birational for $m = 23$ or $m \geq 27$. On the other hand, using Reid's Riemann-Roch formula, it is shown in [6], [7] that $K_X^3 \geq 1/1680$ and $rK_X$ is birational for $r \geq 61$ for any 3-dimensional canonical model.

1.3. **Boundedness of canonical models.** In general the problem of determining lower bounds for the volume of $K_X$ and which multiples $mK_X$ of $K_X$ that are very ample is not easy. The first general result for canonical models in arbitrary dimension is based on ideas of Tsuji ([32], [12] and [31]).

**Theorem 1.3.1.** *Fix $n \in \mathbb{N}$ and $d > 0$. Then*

1. *The set of canonical volumes $\mathcal{W}(n) = \{K_X^n\}$ where $X$ is a $n$-dimensional canonical model, is discrete. In particular the minimum $w = w(n)$ is achieved.*
2. *There exists an integer $k = k(n) > 0$ such that if $X$ is an $n$-dimensional canonical model, then $mK_X$ is birational for any $m \geq k$.*
3. *There exists an integer $r = r(n, d) > 0$ such that if $X$ is an $n$-dimensional canonical model with $K_X^n = d$, then $rK_X$ is very ample.*

Note that it is not the case that the volumes of $d$-dimensional log canonical models is discrete, in fact by examples of [23, 36], they have accumulation points from below.

*Sketch of the proof of Theorem 1.3.1.* Tsuji's idea is to first prove the following weaker version of (2):

**Claim 1.3.2.** *There exist constants $A$, $B > 0$ such that $mK_X$ is birational for any $m \geq A(K_X^n)^{-1/n} + B$.*

To prove the claim, it suffices to show that for very general points $x, y \in X$ there is an effective $\mathbb{Q}$-divisor $D$ such that

(1) $D \sim_{\mathbb{Q}} \lambda K_X$ where $\lambda < A(K_X^n)^{-1/n} + B - 1$,
(2) $\mathcal{J}(X, D)_x = \mathfrak{m}_x$ in a neighbourhood of $x \in X$ and

(3) $\mathcal{J}(X,D) \subset \mathfrak{m}_y$.

Therefore $x$ is an isolated point of the cosupport of $\mathcal{J}(X,D)$ and $y$ is contained in the cosupport of $\mathcal{J}(X,D)$. Applying Nadel vanishing we obtain $H^1(X, \omega_X^m \otimes \mathcal{J}(X,D)) = 0$ for any integer $m \geq A(K_X^n)^{-1/n} + B - 1$ and so there is a surjection

$$H^0(X, \omega_X^m) \longrightarrow H^0(X, \omega_X^m \otimes \mathcal{O}_X/\mathcal{J}(X,D)).$$

By our assumptions, $\mathcal{O}_X/\mathcal{J}(X,D) = \mathcal{F} \oplus \mathcal{G}$ where $\mathrm{Supp}(\mathcal{F}) = x$ and $y \in \mathrm{Supp}(\mathcal{G})$. From the surjection $\mathcal{F} \longrightarrow \mathcal{O}_X/\mathfrak{m}_x \cong \mathbb{C}(x)$ it easily follows that there exists a section of $\omega_X^m$ vanishing at $y$ and not vanishing at $x$. Therefore $|mK_X|$ induces a birational map.

We now explain how to produce the divisor $D$. We focus on establishing the condition $\mathcal{J}(X,D)_x = \mathfrak{m}_x$ and we ignore the condition $\mathcal{J}(X,D) \subset \mathfrak{m}_y$ since this is easier. Fix $0 < \epsilon \ll 1$. Since

$$h^0(X, \mathcal{O}_X(tK_X)) = \frac{K_X^n}{n!} t^n + o(t^n)$$

and vanishing at $x$ to order $s$ imposes at most $s^n/n! + o(s^n)$ conditions, for every $l \gg 0$ there is a section $D_l \in |lK_X|$ with

$$\mathrm{mult}_x(D_l) > l((K_X^n)^{1/n} - \epsilon).$$

If $D = \lambda D_l/l \sim_{\mathbb{Q}} \lambda K_X$ where $\lambda = \mathrm{lct}_x(X; D_l/l)$, then

$$\lambda \leq n/l((K_X^n)^{1/n} - \epsilon)$$

so that $\lambda \leq A'(K_X^n)^{-1/n} + B'$ for appropriate constants $A', B'$ depending only on $n$. By definition of $D$, we have $\mathcal{J}(X,D) \subset \mathfrak{m}_x$. Let $x \in V \subset X$ be an irreducible component of the co-support of $\mathcal{J}(X,D)$. By standard arguments (see Proposition 1.6.1), we may assume that $V$ is the only such component. If $\dim V = 0$, then $V = x$ and we are done, so suppose that $n' = \dim V > 0$.

Since $x \in X$ is very general then $V$ is of general type. Let $\nu \colon V' \longrightarrow V$ be a log resolution. Then by induction on the dimension, there exists a constant $k' = k(n-1)$ such that $\phi_{k'K_{V'}} \colon V' \dashrightarrow \mathbb{P}^M$ is birational. Let $n' = \dim V$. Pick $x' \in V'$ a general point and

$$D_{1,V'} = \frac{n'}{n'+1}(H_1 + \ldots + H_{n'+1})$$

where $H_i \in |k'K_{V'}|$ are divisors corresponding to general hyperplanes on $\mathbb{P}^M$ containing $\phi_{k'K_{V'}}(x')$. Let $D_{1,V} = \nu_* D_{1,V'}$. It is easy to see that $x'$ is an isolated non Kawamata log terminal centre of $(V, D_{1,V})$ (with a unique non Kawamata log terminal place).

Assume for simplicity that $V$ is normal. By Kawamata subadjunction, it follows that

$$(1+\lambda)K_X|_V - K_V \sim_{\mathbb{R}} (K_X + D)|_V - K_V$$

is pseudo-effective. Since $K_X$ is ample, for any $\delta > 0$, we may assume that there is an effective $\mathbb{R}$-divisor

$$D^*_{1,V} \sim_{\mathbb{R}} (1+\lambda)(\frac{n'k}{n'+1} + \delta)K_X|_V$$

such that $x'$ is an isolated non Kawamata log terminal centre of $(V, D^*_{1,V})$ (with a unique non Kawamata log terminal place). By Serre vanishing there is a divisor

$$D_1 \sim_{\mathbb{R}} (1+\lambda)(\frac{n'k}{n'+1} + \delta)K_X$$

such that $D_1|_V = D_{1,V}$. By inversion of adjunction $x'$ is a minimal non Kawamata log terminal centre of $(X, D + D_1)$. After perturbing $D' = D + D_1$ we may assume that $\mathcal{J}(X, D') = \mathfrak{m}_{x'}$ in a neighbourhood of $x' \in V \subset X$. Note that there exist constants $A'', B'' > 0$ such that

$$\lambda + (1+\lambda)(\frac{n'k}{n'+1} + \delta) \leq A''(K_X^n)^{-1/n} + B''.$$

Finally we sketch Tsuji's argument showing that Corollary 1.3.2 implies Theorem 1.3.1. Let $m_0 = \lceil A(K_X^n)^{-1/n} + B \rceil$ and $Z$ be the image of $X$ via $|m_0 K_X|$. Then $Z$ is birational to $X$. Fix any $M > 0$. If $K_X^n \geq M$ then (2) of Theorem 1.3.1 holds with $k = \lceil A(M)^{-1/n} + B \rceil$. Therefore suppose that $K_X^n < M$. In this case we have

$$\begin{aligned}
\deg(Z) &\leq m_0^n K_X^n \\
&< (A(K_X^n)^{-1/n} + B + 1)^n K_X^n \\
&\leq (A + (B+1)M)^n.
\end{aligned}$$

Therefore $X$ is birationally bounded. More precisely, using the corresponding Chow variety, we obtain a projective morphism of quasi-projective varieties $\mathcal{Z} \longrightarrow T$ such that for any $X$ as above there exists a point $t \in T$ and a birational map $X \dashrightarrow \mathcal{Z}_t$. Let $\mathcal{Z}' \longrightarrow \mathcal{Z}$ be a resolution. After decomposing $T$ (and $\mathcal{Z}$) into a disjoint union of locally closed subsets, we may assume that $\mathcal{Z}' \longrightarrow T$ is a smooth morphism. We may also assume that the subset of points $t \in T$ such that $\mathcal{Z}'_t$ is a variety of general type, is dense in $T$. By Siu's theorem on the deformation invariance of plurigenera, we may then assume that all fibres $\mathcal{Z}'_t$ are varieties of general type and that there are finitely many possible volumes

$$K_X^n = \mathrm{vol}(\mathcal{Z}_t, K_{\mathcal{Z}_t}) = \lim \frac{h^0(\mathcal{Z}_t, mK_{\mathcal{Z}_t})}{m^n/n!}.$$

This implies (1) of Theorem 1.3.1. It is also clear that (2) of Theorem 1.3.1 holds with $k = \lceil A(w(n))^{-1/n} + B \rceil$. To prove (3), assume that $d < M$. After throwing away finitely many components of $T$, we may assume that $\mathrm{vol}(\mathcal{Z}_t, K_{\mathcal{Z}_t}) = d$ for all $t \in T$. Let $\mathcal{X} \longrightarrow T$ be the relative canonical model of $\mathcal{Z}/T$ which exists by [3]. Since $K_{\mathcal{X}}$ is relatively ample, it follows that there is an integer $r$ such that $rK_{\mathcal{X}}$ is relatively very ample and hence $rK_{\mathcal{Z}_t}$ is very ample for all $t \in T$. Therefore (3) Theorem 1.3.1 also holds. $\square$

It is natural to try and generalize the above argument to the case of log pairs. Not surprisingly there are many technical difficulties. The first obvious difficulty is that it is no longer sufficient to prove the birational boundedness of varieties but we need to prove some version of birational boundedness for log pairs, cf. (1.10.2).

The basic structure of the proofs of Theorems 1.2.1, 1.2.2 and 1.2.3 is similar to that of Theorem 1.3.1. The proof can be divided into three steps (see [13], [14], [15]).

In the first step, we want to show that if we have a class $\mathfrak{D}$ of $n$-dimensional log canonical pairs which is birationally bounded and with all the coefficients belonging to a fixed DCC set $I$, then the set

$$\{\, \mathrm{vol}(X, K_X + B) \,|\, (X, B) \in \mathfrak{D} \,\}$$

also satisfies the DCC. Under suitable smoothness assumptions, we obtain a version of invariance of plurigenera for pairs. Using this, we can easily reduce to the case that $T$ is a point in the definition of a log birationally bounded family, that is, we can assume all pairs are birational to a fixed pair. Then there is a lengthy combinatorial argument, mainly using toroidal geometry calculations, to finish the argument.

In the second step, we want to prove that all log general type pairs in $\mathfrak{D}$ with volume bounded from above form a log birationally bounded family. This step is similar to the proof of Theorem 1.3.1 (unluckily many difficulties arise due to the presence of the boundary). This is done in [15] via a complicated induction which relies on the ACC for log canonical thresholds and other results. We adopt a more direct approach here, where we first prove the result for coefficient sets $I$ of the form

$$\{\, \frac{i}{p} \,|\, 0 \leq i \leq p \,\}$$

and then deduce the general case.

In the final step we deduce boundedness from log birational boundedness. This is a direct consequence of the Abundance Conjecture. In our situation, we are able to use a deformation invariance of plurigenera for pairs (proved with analytic methods by Berndtsson and Păun) to establish the required special case of the abundance conjecture.

sectionPreliminaries

1.4. **Notation and conventions.** We work over the field of complex numbers $\mathbb{C}$. A **pair** $(X, B)$ is given by a normal variety $X$ and an effective $\mathbb{R}$-divisor $B = \sum_{i=1}^{k} b_i B_i$ such that $K_X + B$ is $\mathbb{R}$-Cartier. We denote the coefficients of $B$ by $\operatorname{coeff}(B) = \{b_1, \ldots, b_k\}$. Let $q \in \mathbb{N}$, $I_0 = \{\, \frac{j}{q} \,|\, 1 \leq j \leq q \,\}$. We say that $D(I_0)$ is a **hyperstandard set of coefficients**. Observe that for any finite set of rational numbers $J_0 \subset [0, 1]$, we can find $q \in \mathbb{N}$ such that $J_0 \subset I_0$.

We let $\lfloor B \rfloor = \sum \lfloor b_i \rfloor B_i$ where $\lfloor b \rfloor$ is the greatest integer $\leq b$ and $\{B\} = B - \lfloor B \rfloor$. The **support** of $B$ is given by $\operatorname{Supp}(B) = \cup_{b_i \neq 0} B_i$. The **strata** of $(X, B)$ are the irreducible components of intersections

$$B_I = \cap_{j \in I} B_j = B_{i_1} \cap \ldots \cap B_{i_r},$$

where $I = \{\, i_1, i_2, \ldots, i_r \,\}$ is a subset of the non-zero coefficients, including the empty intersection $X = B_{\emptyset}$. If $B' = \sum b'_i B_i$ is another $\mathbb{R}$-divisor, then $B \wedge B' = \sum (b_i \wedge b'_i) B_i$ and $B \vee B' = \sum (b_i \vee b'_i) B_i$ where $b_i \wedge b'_i = \min\{b_i, b'_i\}$ and $b_i \vee b'_i = \max\{b_i, b'_i\}$.

We say the pair $(X, B)$ is a **toroidal pair** if the inclusion $U \hookrightarrow X$ of the complement $U$ of the support of $B$ is toroidal, so that locally, in the analytic topology, the inclusion is isomorphic to the inclusion of the open torus inside a toric variety, see [20].

For any proper birational morphism $\nu \colon X' \longrightarrow X$, we pick a canonical divisor $K_{X'}$ such that $\nu_* K_{X'} = K_X$ and we write

$$K_{X'} + B' = \nu^*(K_X + B) + \sum a_{E_i} E_i$$

where $B'$ is the strict transform of $B$. The numbers $a_{E_i} = a_{E_i}(X, B)$ are the **discrepancies** of $E_i$ with respect to $(X, B)$, the **discrepancy** of $(X, B)$ is $\inf\{a_E(X, B)\}$ where $E$ runs over all divisors over $X$ and the **total discrepancy** $a(X, B)$ of $(X, B)$ is the minimum of the discrepancy and coeff)$(-B)$. We say that $(X, B)$ is **Kawamata log terminal** (resp. **log canonical**, **terminal**) if $a(X, B) > -1$ (resp. $a(X, B) \geq -1$, $a_E(X, B) > 0$ for any divisor $E$ exceptional over $X$). Note that to check if a pair is either Kawamata log terminal or log canonical it suffices to check what happens on a single log resolution, that is, on a proper birational morphism $\nu \colon X' \longrightarrow X$ such that the exceptional locus is a divisor and $\nu_*^{-1} B + \operatorname{Exc}(\nu)$ has simple normal crossings. A divisor $E$ over $X$ is a **non Kawamata log terminal place** of $(X, B)$ if $a_E(X, B) \leq -1$. The image of a non Kawamata log terminal place $E$ in $X$ is a **non-Kawamata log terminal centre**. **Non log canonical places** and **centres** are defined similarly by requiring $a_E(X, B) < -1$. A pair $(X, B)$ is **divisorially log terminal** if it is log canonical and there is an open subset $U \subset X$ containing the generic points of all non Kawamata log terminal centres such that $(U, B|_U)$ has simple normal crossings. In this case, by a result of Szabó, it is known that there exists a log resolution of $(X, B)$ which is an isomorphism over $U$.

If $(X, B)$ is a log canonical pair and $D \geq 0$ is an effective $\mathbb{R}$ divisor, then we define the **log canonical threshold** of $(X, B)$ with respect to $D$ by

$$\operatorname{lct}(X, B; D) = \sup\{\, t \geq 0 \,|\, (X, B + tD) \text{ is log canonical}\,\}$$

For any closed point $x \in X$, $\operatorname{lct}_x(X, B; D)$ will denote the log canonical threshold computed on a sufficiently small open subset of $x \in X$. In particular, $(X, B + \lambda D)$ is log canonical in a neighbourhood of $x \in X$ and the non-Kawamata log terminal locus of $(X, B + \lambda D)$ contains $x$ where $\lambda = \operatorname{lct}_x(X, B; D)$.

Let $X$ be a normal variety and consider the set of all proper birational morphisms $f \colon Y \longrightarrow X$ where $Y$ is normal. We have natural maps $f_* \colon \operatorname{Div}(Y) \longrightarrow \operatorname{Div}(X)$. The space **b-divisors** is

$$\mathbf{Div}(X) = \lim_{\{Y \to X\}} \operatorname{Div}(Y).$$

Note that $f_*$ induces an isomorphism $\mathbf{Div}(Y) \cong \mathbf{Div}(X)$ and that an element $\mathbf{B} \in \mathbf{Div}(X)$ is specified by the corresponding traces $\mathbf{B}_Y$ of $\mathbf{B}$ on each birational model $Y \longrightarrow X$. If $E$ is a divisor on $Y$, then we let $\mathbf{B}(E) = \operatorname{mult}_E(\mathbf{B}_Y)$. Given a log pair $(X, B)$ and a proper birational morphism $f \colon X' \longrightarrow X$, we may write $K_{X'} + B_{X'} = f^*(K_X + B)$. We define the b-divisors $\mathbf{L}_B$ and $\mathbf{M}_B$ as follows

$$\mathbf{M}_{B,X'} = f_*^{-1} B + \operatorname{Exc}(f) \qquad \text{and} \qquad \mathbf{L}_{B,X'} = B_{X'} \vee 0.$$

A **semi log canonical pair** (SLC pair) $(X, B)$ is given by an $S_2$ variety whose singularities in codimension 1 are nodes and an effective $\mathbb{R}$-divisor $B$ none of whose components are contained in the singular locus of $X$ such that if $\nu \colon X^\nu \longrightarrow X$ is the normalisation and $K_{X^\nu} + B^\nu = \pi^*(K_X + B)$, then each component of $(X^\nu, B^\nu)$ is log canonical. A **semi log canonical model** (SLC model) is a projective SLC pair $(X, B)$ such that $K_X + B$ is ample.

If $X$ is a smooth variety and $D$ is an effective $\mathbb{R}$-divisor on $X$, then **the multiplier ideal sheaf** is defined by

$$\mathcal{J}(X, D) = \mu_*(K_{X'/X} - \lfloor \mu^* D \rfloor) \subset \mathcal{O}_X$$

where $\mu\colon X' \longrightarrow X$ is a log resolution of $(X, D)$. It is known that the definition does not depend on the choice of a log resolution and $\mathcal{J}(X, D) = \mathcal{O}_X$ if and only if $(X, D)$ is Kawamata log terminal and in fact the support of $\mathcal{O}_X/\mathcal{J}(X, D)$ (that is, the **co-support** of $\mathcal{J}(X, D)$) is the union of all non Kawamata log terminal centres of $(X, D)$. Note that

$$\operatorname{lct}(X, B; D) = \sup\{\, t \geq 0 \,|\, \mathcal{J}(X, B + tD) = \mathcal{O}_X \,\}.$$

We refer the reader to [29] for a comprehensive treatment of multiplier ideal sheaves and their properties.

Let $\pi\colon X \longrightarrow U$ be a morphism, then $\pi$ is a **contraction morphism** if and only if $\pi_*\mathcal{O}_X = \mathcal{O}_U$. If $f\colon X \longrightarrow U$ is a morphism and $(X, B)$ is a pair, then we say that $(X, B)$ is **log smooth over** $U$ if $(X, B)$ has simple normal crossings and every stratum of $(X, \operatorname{Supp}(B))$ (including $X$) is smooth over $U$.

A **birational contraction** $f\colon X \dashrightarrow Y$ is a proper birational map of normal varieties such that $f^{-1}$ has no exceptional divisors. If $p\colon W \longrightarrow X$, and $q\colon W \longrightarrow Y$ is a common resolution then $f$ is a birational contraction if and only if every $p$-exceptional divisor is $q$-exceptional. If $D$ is an $\mathbb{R}$-Cartier divisor on $X$ such that $f_*D$ is $\mathbb{R}$-Cartier on $Y$ then $f$ is $D$-**non-positive** (resp. $D$-**negative**) if $p^*D - q^*(f_*D) = E$ is effective (resp. is effective and its support contains the strict transform of the $f$ exceptional divisors). If $X \longrightarrow U$ and $Y \longrightarrow U$ are projective morphisms, $f\colon X \dashrightarrow Y$ a birational contraction over $U$ and $(X, B)$ is a log canonical pair (resp. a divisorially log terminal $\mathbb{Q}$-factorial pair) such that $f$ is $(K_X + B)$ non-positive (resp. $(K_X + B)$-negative) and $K_Y + f_*B$ is nef over $U$ (resp. $K_Y + f_*B$ is nef over $U$ and $Y$ is $\mathbb{Q}$-factorial), then $f$ is a **weak log canonical model** (resp. a **minimal model**) of $K_X + B$ over $U$. If $f\colon X \dashrightarrow Y$ is a minimal model of $K_X + B$ such that $K_Y + f_*B$ is semi-ample over $U$, then we say that $f$ is a **good minimal model** of $K_X + B$ over $U$. Recall that if $\pi\colon X \longrightarrow U$ is a projective morphism and $D$ is a $\mathbb{R}$-Cartier divisor on $X$, then $D$ is **semi-ample over** $U$ if and only if there exists a projective morphism $g\colon X \longrightarrow W$ over $U$ and an $\mathbb{R}$-divisor $A$ on $W$ which is ample over $U$ such that $g^*A \sim_{\mathbb{R}} D$.

If $D$ is an $\mathbb{R}$-divisor on a normal projective variety $X$, then $\phi_D$ denotes the rational map induced by the linear series $|\lfloor D \rfloor|$ and

$$H^0(X, \mathcal{O}_X(D)) = H^0(X, \mathcal{O}_X(\lfloor D \rfloor)).$$

If $\phi_D$ induces a birational map, then we say that $|D|$ is birational.

1.5. **Volumes.** If $X$ is a normal projective variety, $D$ is an $\mathbb{R}$-divisor and $n = \dim X$, then we define the **volume** of $D$ by

$$\operatorname{vol}(X, D) = \lim \frac{n! h^0(X, mD)}{m^n}.$$

Note that if $D$ is nef, then $\operatorname{vol}(X, D) = D^n$. By definition $D$ is **big** if $\operatorname{vol}(X, D) > 0$. It is well known that if $D$ is big then $D \sim_{\mathbb{R}} A + E$ where $E \geq 0$ and $A$ is ample. Note that the volume only depends on $[D] \in N^1(X)$, so that if $D \equiv D'$, then $\operatorname{vol}(X, D) = \operatorname{vol}(X, D')$. The induced function $\operatorname{vol}\colon N^1(X) \longrightarrow \mathbb{R}$ is continuous [28, 2.2.45].

**Lemma 1.5.1.** *Let $f\colon X \longrightarrow W$ and $g\colon Y \longrightarrow X$ be birational morphisms of normal projective varieties and $D$ be an $\mathbb{R}$-divisor on $X$. Then*

*(1)* $\operatorname{vol}(W, f_*D) \geq \operatorname{vol}(X, D)$.
*(2) If $D$ is $\mathbb{R}$-Cartier and $G$ is an $\mathbb{R}$-divisor on $Y$ such that $G - g^*D \geq 0$ is effective and $g$-exceptional, then* $\operatorname{vol}(Y, G) = \operatorname{vol}(X, D)$. *In particular if $(X, B)$ is a projective log canonical pair and $f\colon Y \longrightarrow X$ a birational morphism, then*

$$\operatorname{vol}(X, K_X + B) = \operatorname{vol}(Y, K_Y + \mathbf{L}_{B,Y}) = \operatorname{vol}(Y, K_Y + \mathbf{M}_{B,Y}).$$

*(3) If $D \geq 0$, $(W, f_*D)$ has simple normal crossings, and $L = \mathbf{L}_{f_*D,X}$, then*

$$\operatorname{vol}(X, K_X + D) = \operatorname{vol}(X, K_X + D \wedge L).$$

*(4) If $(X, B)$ is a log canonical pair and $X \dashrightarrow X'$ is a birational contraction of normal projective varieties, then*

$$\operatorname{vol}(X', K_{X'} + \mathbf{M}_{B,X'}) \geq \operatorname{vol}(X, K_X + B).$$

*If moreover $X \longrightarrow W$ and $X' \longrightarrow W$ are morphisms and the centre of every divisor in the support of $B \wedge \mathbf{L}_{f_*B,X}$ is a divisor on $X'$, then we have equality*

$$\operatorname{vol}(X', K_{X'} + \mathbf{M}_{B,X'}) = \operatorname{vol}(X, K_X + B).$$

*Proof.* If $H \sim mD$, then $f_*H \sim mf_*D$ and so $h^0(X, \mathcal{O}_X(mD)) \leq h^0(W, \mathcal{O}_W(mf_*D))$ and (1) follows easily.

(2) follows since $H^0(X, \mathcal{O}_X(mD)) \cong H^0(Y, \mathcal{O}_Y(mG))$.

To see (3), notice that the inclusion

$$H^0(X, \mathcal{O}_X(m(K_X + D))) \supset H^0(X, \mathcal{O}_X(m(K_X + D \wedge L)))$$

is clear. We have

$$K_X + L = f^*(K_W + f_*D) + E,$$

where $E \geq 0$ and $L \wedge E = 0$. Now observe that

$$\begin{aligned} H^0(X, \mathcal{O}_X(m(K_X + D))) &\subset f^* H^0(W, \mathcal{O}_W(m(K_W + f_* D))) \\ &= H^0\big(X, \mathcal{O}_X(m(K_X + L))\big), \end{aligned}$$

where we have already demonstrated the inclusion holds and the equality follows as $E \geq 0$ is exceptional. But then every section of $H^0(X, \mathcal{O}_X(m(K_X + D)))$ vanishes along $mD - mD \wedge mL$ and (3) follows.

To see (4), let $X'' \longrightarrow X$ be a resolution of the indeterminacies of $X \dashrightarrow X'$ so that $X'' \longrightarrow X'$ is also a morphism of normal projective varieties. Then by (2) and (1), it follows that

$$\operatorname{vol}(X, K_X + B) = \operatorname{vol}(X'', K_{X''} + \mathbf{M}_{B,X''}) \leq \operatorname{vol}(X', K_{X'} + \mathbf{M}_{B,X'}).$$

Suppose now that the centre of every divisor in the support of $B \wedge \mathbf{L}_{f_* B, X}$ is a divisor on $X'$ and let $B' = \mathbf{M}_{B,X'}$. It is easy to see that

$$\mathbf{M}_{B',X''} \wedge \mathbf{L}_{f'_* B', X''} = \mathbf{M}_{B,X''} \wedge \mathbf{L}_{f_* B, X''}$$

and so by (2) and (3) we have

$$\begin{aligned} \operatorname{vol}(X, K_X + B) &= \operatorname{vol}(X'', K_{X''} + \mathbf{M}_{B,X''} \wedge \mathbf{L}_{f_* B, X''}) \\ &= \operatorname{vol}(X'', K_{X''} + \mathbf{M}_{B',X''} \wedge \mathbf{L}_{f'_* B', X''}) \\ &= \operatorname{vol}(X', K_{X'} + B'). \end{aligned}$$ □

1.6. **Non Kawamata log terminal centres.** Here we collect several useful facts about non Kawamata log terminal centres.

**Proposition 1.6.1.** *Let $(X, B)$ be a log canonical pair and $(X, B_0)$ a Kawamata log terminal pair.*

*(1) If $W_1$ and $W_2$ are non Kawamata log terminal centres of $(X, B)$ and $W$ is an irreducible component of $W_1 \cap W_2$, then $W$ is a non Kawamata log terminal centre of $(X, B)$. In particular if $x \in X$ is a point such that $(X, B)$ is not Kawamata log terminal in any neighbourhood of $x \in X$, then there is a minimal non Kawamata log terminal centre $W$ of $(X, B)$ containing $x$.*
*(2) Every minimal non Kawamata log terminal centre $W$ of $(X, B)$ is normal.*
*(3) If $W$ is a minimal non Kawamata log terminal centre of $(X, B)$, then there exists a divisor $B' \geq 0$ such that for any $0 < t < 1$, $W$ is the only non Kawamata log terminal centre of $(X, tB + (1-t)B')$ and there is a unique non Kawamata log terminal place $E$ of $(X, tB + (1-t)B')$.*

*Proof.* For (1-2) see [19]. (3) follows from [22, 8.7.1]. □

**Lemma 1.6.2.** *Let $(X, B)$ be an $n$-dimensional projective log pair and $D$ a big divisor on $X$ such that* $\mathrm{vol}(X, D) > (2n)^n$. *Then there exists a family $V \longrightarrow T$ of subvarieties of $X$ such that if $x$, $y$ are two general points of $X$, then, possibly switching $x$ and $y$, we may find a divisor $0 \leq D_t \sim_{\mathbb{R}} D$ such that $(X, B + D_t)$ is not Kawamata log terminal at both $x$ and $y$, $(X, B+D_t)$ is log canonical at $x$ and there is a unique non Kawamata log terminal place of $(X, B + D_t)$ with centre $V_t$ containing $x$.*

*Proof.* Since

$$h^0(X, \mathcal{O}_X(kD)) = \frac{\mathrm{vol}(X, D)}{n!} \cdot k^n + O(k^{n-1})$$

and vanishing at a smooth point $x \in X$ to order $l$ imposes

$$\binom{n+l}{l} = \frac{l^n}{n!} + O(l^{n-1})$$

conditions, one sees that for any $s \gg 0$ there is a divisor $0 \leq G_x \sim_{\mathbb{R}} sD$ such that $\mathrm{mult}_x(G) > 2ns$. Let

$$\lambda = \sup\{\, l > 0 \,|\, (X, B + l(G_x + G_y)) \text{ is log canonical at one of } x \text{ or } y \,\} < \frac{1}{2s}.$$

If $D' = \lambda(G_x+G_y)+(1-2\lambda s)D$ then $(X, B+D')$ is not Kawamata log terminal at $x$ and $y$. Possibly switching $x$ and $y$ we may assume that $(X, B+D')$ is log canonical in a neighbourhood of $x$. Perturbing $D'$ we may assume that there is a unique non Kawamata log terminal place for $(X, B+D')$ whose centre $V$ contains $x$ (see Proposition 1.6.1). The result now follows using the Hilbert scheme. □

## 1.7. **Minimal models.**

**Theorem 1.7.1** ([3])**.** *Let $(X, B)$ be a $\mathbb{Q}$-factorial Kawamata log terminal pair and $\pi\colon X \longrightarrow U$ be a projective morphism such that either $B$ or $K_X+B$ is big over $U$ (respectively $K_X+B$ is not pseudo-effective over $U$) , then there is a good minimal model $X \dashrightarrow X'$ of $K_X+B$ over $U$ (respectively a Mori fibre space $X' \longrightarrow Z$) which is given by a finite sequence of flips and divisorial contractions for the $K_X + B$ minimal model program with scaling of an ample divisor over $U$.*

**Theorem 1.7.2.** *Let $(X, B)$ be a log pair, then*

(1) *There is a proper birational morphism $\nu\colon X' \longrightarrow X$ such that $X'$ is $\mathbb{Q}$-factorial,*

$$K_{X'} + \nu_*^{-1}B + \mathrm{Exc}(\nu) = \nu^*(K_X + B) + E$$

*where $E \leq 0$ and $(X', \nu_*^{-1}B + \mathrm{Exc}(\nu))$ is divisorially log terminal.*

(2) *If $(X,B)$ is Kawamata log terminal, then there exists a* $\mathbb{Q}$-**factorial modification**, *that is, a small proper birational morphism $\nu\colon X' \longrightarrow X$ such that $X'$ is $\mathbb{Q}$-factorial.*

(3) *If $(X,B)$ is $\mathbb{Q}$-factorial and log canonical and $W \subset \operatorname{Supp}(B)$ is a minimal non-Kawamata log terminal centre, then there exists a proper birational morphism $\nu\colon X' \longrightarrow X$ such that $\rho(X'/X) = 1$, $\operatorname{Exc}(\nu) = E$ is an irreducible divisor and*

$$K_{X'} + \nu_*^{-1}B + E = \nu^*(K_X + B).$$

*Proof.* (1) is [15, 3.3.1]. (2) is an easy consequence of (1) and (3) follows from [3, 1.4.3]. □

**Proposition 1.7.3.** *Let $(X,B)$ be an $n$-dimensional $\mathbb{Q}$-factorial divisorially log terminal pair, $0 \neq S \leq \lfloor B \rfloor$ and $\pi\colon X \longrightarrow U$ a projective morphism to a smooth variety. Let $0 \in U$ be a closed point and $r \in \mathbb{N}$ a positive integer such that $(X_0, B_0)$ is log canonical, $K_{X_0} + B_0$ is nef and $r(K_{X_0} + B_0)$ is Cartier. Fix $\epsilon < \frac{1}{2nr+1}$.*

*If $K_X + B - \epsilon S$ is not pseudo-effective, then we may run $f\colon X \dashrightarrow Y$ the $(K_X + B - \epsilon S)$ minimal model program over $U$ such that*

(1) *each step is $K_X + B$ trivial over a neighbourhood of $0 \in U$,*

(2) *there is a Mori fibre space $\psi\colon Y \longrightarrow Z$ such that $f_*S$ dominates $Z$ and $K_{X'} + f_*B \sim_{\mathbb{R}} \psi^*L$ for some $\mathbb{R}$-divisor $L$ on $Z$.*

*Proof.* [15, 5.2]. □

1.8. **DCC sets.** A set $I \subset \mathbb{R}$ is said to satisfy the **descending chain condition** (DCC) if any non increasing sequence in $I$ is eventually constant. Similarly $I$ satisfies the **ascending chain condition** (ACC) if any non decreasing sequence in $I$ is eventually constant or equivalently $-I = \{-i | i \in I\}$ satisfies the DCC. The **derived set** of $I$ is defined by

$$D(I) = \{\, \frac{r - 1 + i_i + \ldots + i_p}{r} \,|\, r \in \mathbb{N},\ i_j \in I \,\}.$$

Note that $I$ satisfies the DCC if and only if $D(I)$ satisfies the DCC.

**Lemma 1.8.1.** *Let $I \subset [0,1]$ be a DCC set and $J_0 \subset [0,1]$ a finite set, then*

$$I_1 := \{\, i \in I \,|\, \frac{m - 1 + f + ki}{m} \in J_0, \text{ where } k, m \in \mathbb{N} \text{ and } f \in D(I) \,\}$$

*is a finite set.*

*Proof.* See [15, 5.2]. □

1.9. **Good minimal models.** We will need the following results from [15, 1.2, 1.4].

**Theorem 1.9.1.** *Let $(X, B)$ be a log pair and $\pi\colon X \longrightarrow U$ a projective morphism to a smooth affine variety such that* $\operatorname{coeff}(B) \subset (0,1] \cap \mathbb{Q}$ *and $(X, B)$ is log smooth over $U$. Suppose that there is a point $0 \subset U$ such that $(X_0, B_0)$ has a good minimal model. Then $(X, B)$ has a good minimal model over $U$ and every fibre has a good minimal model.*

*Furthermore, the relative ample model of $(X, B)$ over $U$ gives the relative ample model of each fibre.*

*Proof.* [15, 1.2, 1.4]. $\square$

**Theorem 1.9.2.** *Let $(X, B)$ be a log pair such that* $\operatorname{coeff}(B) \subset (0,1]$ *and $\pi\colon X \longrightarrow U$ a projective morphism such that $(X, B)$ is log smooth over $U$. Then $h^0(X_u, \mathcal{O}_{X_u}(m(K_{X_u} + B_u)))$ is independent of $u \in U$. In particular*

$$f_*\mathcal{O}_X(m(K_X + B)) \longrightarrow H^0(X_u, \mathcal{O}_{X_u}(m(K_{X_u} + B_u)))$$

*is surjective for all $u \in U$.*

*Proof.* Notice that

$$\mathcal{O}_X(m(K_X + B)) = \mathcal{O}_X(\lfloor m(K_X + B)\rfloor) = \mathcal{O}_X(m(K_X + B_{\lfloor m\rfloor}))$$

where $B_{\lfloor m\rfloor} = \lfloor mB\rfloor/m$. The statement now follows from [15, 1.2]. $\square$

1.10. **Log birational boundedness.** We begin with the following easy:

**Lemma 1.10.1.** *Let $(X, B)$ be a projective log pair and $D$ a big $\mathbb{R}$-divisor such that for general points $x$, $y \in X$ there is an $\mathbb{R}$-divisor $0 \leq D' \sim_{\mathbb{R}} \lambda D$ for some $\lambda < 1$ such that*

1. *$x$ is an isolated non Kawamata log terminal centre of $(X, B + D')$,*
2. *$(X, B + D')$ is log canonical in a neighbourhood of $x \in X$, and*
3. *$(X, B + D')$ is not Kawamata log terminal at $y$.*

*Then $\phi_{K_X + \lceil D\rceil}$ is birational.*

*Proof.* Fix a resolution $\nu\colon X' \longrightarrow X$ of $(X, B)$. As $x$ and $y$ are general, $\nu$ is an isomorphism in a neighbourhood of $x$ and $y$. If $\phi_{K_{X'} + \lceil \nu^* D\rceil}$ is birational then so is $\phi_{K_X + \lceil D\rceil}$. Therefore, replacing $X$ by $X'$, we may assume that $X$ is smooth and $B = 0$.

As $x$ is an isolated non Kawamata log terminal centre, we have $\mathcal{J}(D') \cong \mathfrak{m}_x$ in a neighbourhood of $x \in X$. By Nadel vanishing (see [29, 9.4.8]), we have

$$H^1\big(X, \mathcal{O}_X(K_X + \lceil D\rceil) \otimes \mathcal{J}(D')\big) = 0$$

and so

$$H^0(X, \mathcal{O}_X(K_X + \lceil D \rceil)) \longrightarrow H^0(X, \mathcal{O}_X(K_X + \lceil D \rceil) \otimes \mathcal{O}_X/\mathcal{J}(D'))$$

is surjective.

But since $\mathcal{O}_X/\mathcal{J}(D')$ has a summand isomorphic to $\mathcal{O}_X/\mathfrak{m}_x$ and another summand whose support contains $y$, it follows that we may lift a section of $H^0(X, \mathcal{O}_X(K_X + \lceil D \rceil) \otimes \mathcal{O}_X/\mathcal{J}(D'))$ not vanishing at $x$ and vanishing at $y$ to a section of $H^0(X, \mathcal{O}_X(K_X + \lceil D \rceil))$ and the assertion is proven. □

**Definition 1.10.2.** *We say that a set of varieties $\mathfrak{X}$ is* **bounded** *(resp.* **birationally bounded***) if there exists a projective morphism $Z \longrightarrow T$, where $T$ is of finite type, such that for every $X \in \mathfrak{X}$, there is a closed point $t \in T$ and an isomorphism (resp. a birational map) $f\colon X \longrightarrow Z_t$. We say that a set $\mathfrak{D}$ of log pairs is* **bounded** *(resp.* **log birationally bounded***) if there is a log pair $(Z, D)$, where the coefficients of $D$ are all one, and there is a projective morphism $Z \longrightarrow T$, where $T$ is of finite type, such that for every pair $(X, B) \in \mathfrak{D}$, there is a closed point $t \in T$ and a map $f\colon Z_t \longrightarrow X$ inducing an isomorphism $(X, B_{\mathrm{red}}) \cong (Z_t, D_t)$ (resp. such that the support of $D_t$ contains the support of the strict transform of $B$ and any $f$-exceptional divisor).*

**Remark 1.10.3.** *Note that, by a standard Hilbert scheme argument, a set of varieties $\mathfrak{X}$ (resp. of pairs) is bounded if there exists a constant $C > 0$ such that for each $X \in \mathfrak{X}$ there is a very ample divisor $H$ on $X$ such that $H^{\dim X} \leq C$ (resp. $H^{\dim X} \leq C$ and $B_{\mathrm{red}} \cdot H^{\dim X - 1} \leq C$).*

**Proposition 1.10.4.** *Fix $n \in \mathbb{N}$, $A > 0$ and $\delta > 0$. The set of projective log canonical pairs $(X, B = \sum b_i B_i)$ such that*

*(1)* $\dim X = n$,
*(2)* $b_i \geq \delta$,
*(3)* $|m(K_X + B)|$ *is birational, and*
*(4)* $\operatorname{vol}(m(K_X + B)) \leq A$

*is log birationally bounded.*

*Proof.* We first reduce to the case that the rational map

$$\phi\colon = \phi_{m(K_X+B)}\colon X \dashrightarrow Z$$

is a birational morphism. To see this let $\nu\colon X' \longrightarrow X$ be a resolution of the indeterminacies of $\phi$ and $B' = \nu_*^{-1}B + \operatorname{Exc}(\nu)$, then $K_{X'} + B' - \nu^*(K_X + B)$ is an effective exceptional divisor. In particular

$$\operatorname{vol}(X', K_{X'} + B') = \operatorname{vol}(X, K_X + B) \leq A$$

and $\phi_{m(K_{X'}+B')}$ is birational. Therefore it suffices to show that the pairs $(X', B')$ are log birationally bounded. Replacing $(X, B)$ by $(X', B')$ we may assume that $\phi$ is a morphism.

Let $|\lfloor m(K_X + B)\rfloor| = |M| + E$ where $E$ is the fixed part and $|M|$ is base point free so that $M = \phi^* H$ for some very ample divisor $H$ on $Z$. Note that

$$H^n = \operatorname{vol}(Z, H) \leq \operatorname{vol}(X, m(K_X + B)) \leq A$$

and hence it suffices to show that $\phi_*(B_{\mathrm{red}}) \cdot H^{n-1}$ is bounded. Let $B_0 = \phi_*^{-1}\phi_* B_{\mathrm{red}}$ and $L = 2(2n+1)H$. By Lemma 1.10.6

$$|K_X + (n+1)\lfloor m(K_X + B)\rfloor| \neq \emptyset$$

and since $B_0 \leq \frac{1}{\delta}B$ it follows that there is an effective $\mathbb{R}$-divisor $C$ such that

$$B_0 + C \sim_{\mathbb{R}} \frac{m(n+1)+1}{\delta}(K_X + B). \tag{1}$$

We have that

$$\begin{aligned}
\phi_* B_{\mathrm{red}} \cdot L^{n-1} &= B_0 \cdot (2(2n+1)M)^{n-1} \\
&\leq 2^n \operatorname{vol}(X, K_X + B_0 + 2(2n+1)M) \\
&\leq 2^n \operatorname{vol}(X, K_X + \frac{m(n+1)+1}{\delta}(K_X + B) + 2(2n+1)m(K_X + B)) \\
&\leq 2^n (1 + (\frac{n+2}{\delta} + 4n + 2))^n \operatorname{vol}(X, m(K_X + B)) \\
&\leq 2^n (1 + (\frac{n+2}{\delta} + 4n + 2))^n A,
\end{aligned}$$

where the second inequality follows from Lemma 1.10.5 and the third from equation (1). $\square$

**Lemma 1.10.5.** *Let $X$ be an $n$-dimensional normal projective variety, $M$ a Cartier divisor such that $|M|$ is base point free and $\phi_M$ is birational. If $L = 2(2n+1)M$, and $D$ is a reduced divisor, then*

$$D \cdot L^{n-1} \leq 2^n \operatorname{vol}(X, K_X + D + L).$$

*Proof.* Let $\nu\colon X' \longrightarrow X$ be a proper birational morphism. Since

$$\operatorname{vol}(X', K_{X'} + \nu_*^{-1}D + \nu^* L) \leq \operatorname{vol}(X, K_X + D + L),$$

we may assume that $X$ and $D$ are smooth (in particular the components of $D$ are disjoint). Let $\phi = \phi_M \colon X \longrightarrow Z$ be the induced birational morphism. Since

$$D \cdot L^{n-1} = \phi_*^{-1}\phi_* D \cdot L^{n-1},$$

we may assume that no component of $D$ is contracted by $\phi$ and we may replace $D$ by $\phi_*^{-1}\phi_* D$. Thus we can write $M \sim_{\mathbb{Q}} A + B$ where

$A$ is ample, $B \geq 0$ and $B$ and $D$ have no common components. In particular $K_X + D + \delta B$ is divisorially log terminal for some $\delta > 0$ and so, by Kawamata-Viehweg vanishing,

$$H^i(X, \mathcal{O}_X(K_X + E + pM)) = 0$$

for all $i > 0$, $p > 0$ and reduced divisors $E$ such that $0 \leq E \leq D$. In particular $H^i(X, \mathcal{O}_D(K_D + pM|_D)) = 0$ for all $i > 0$, $p > 0$. Therefore there are surjective homomorphisms

$$H^0(X, \mathcal{O}_X(K_X + D + (2n+1)M)) \longrightarrow H^0(D, \mathcal{O}_D(K_D + (2n+1)M|_D)).$$

By Lemma 1.10.6, $|K_D + (2n+1)M|_D|$ is non-empty and so the general section of $H^0(X, \mathcal{O}_X(K_X + D + (2n+1)M))$ does not vanish along any component of $D$. It is also easy to see that $|2K_X + D + 2L|$ is non empty. Consider the commutative diagram

$$\begin{tikzcd}
\mathcal{O}_X(K_X + mL + D) \arrow[r] \arrow[d] & \mathcal{O}_D(K_D + mL|_D) \arrow[d] \\
\mathcal{O}_X((2m-1)(K_X + L + D)) \arrow[r] & \mathcal{O}_D((2m-1)(K_D + L|_D))
\end{tikzcd}$$

whose vertical maps are induced by a general section of

$$(m-1)(2K_X + L + 2D) = 2(m-1)(K_X + (2n+1)M + D).$$

Since

$$|2K_X + 2D + L| = |2(K_X + D + (2n+1)M)|$$

is non empty and $H^1(X, \mathcal{O}_X(K_X + mL)) = 0$, it follows that

$$\begin{aligned}
&h^0(X, \mathcal{O}_X((2m-1)(K_X + L + D))) - h^0(X, \mathcal{O}_X((2m-3)(K_X + L + D))) \\
&\geq h^0(X, \mathcal{O}_X((2m-1)(K_X + L + D))) - h^0(X, \mathcal{O}_X((2m-2)(K_X + L + D) + K_X + L)) \\
&= \dim \operatorname{Im} \big(H^0(X, \mathcal{O}_X((2m-1)(K_X + L + D))) \longrightarrow H^0(D, \mathcal{O}_D((2m-1)(K_D + L|_D)))\big) \\
&\geq h^0(D, \mathcal{O}_D((K_D + mL|_D))).
\end{aligned}$$

The leading coefficient of $m^n$ in

$$h^0(X, \mathcal{O}_X((2m-1)(K_X + L + D)))$$

is $2^n \operatorname{vol}(K_X + L + D)/n!$ and, by the vanishing observed above,

$$h^0(D, \mathcal{O}_D(K_D + mL|_D)) = \chi(D, \mathcal{O}_D(K_D + mL|_D))$$

is a polynomial of degree $n-1$ whose leading coefficient is $D \cdot L^{n-1}/(n-1)!$. Comparing these coefficients, one sees that

$$\operatorname{vol}(X, K_X + L + D) \cdot \frac{2^n}{n!} \geq \frac{D \cdot L^{n-1}}{n!}. \qquad \square$$

**Lemma 1.10.6.** *Let $X$ be an $n$-dimensional smooth projective variety and $M$ a Cartier divisor such that $|M|$ is base point free and $\phi_{|M|}$ is generically finite (resp. birational). Then there is an open subset $U \subset X$ such that for any $x \in U$ (resp. $x$, $y \in U$) and any $t \geq n+1$ (resp. $t \geq 2n+1$) there is a section $H^0(X, \mathcal{O}_X(K_X + tM))$ not vanishining at $x$ (resp vanishing at $y$ and not vanishing at $x$).*

*Proof.* Let $\phi = \phi_M \colon X \longrightarrow Z$ be the induced morphism so that $M = \phi^* H$ where $H$ is very ample. Let $U \subset X$ be the open subset on which $\phi$ is finite (resp. an isomorphism). We may pick a divisor $G \sim_{\mathbb{Q}} \lambda H$ such that $\lambda < n+1$ (resp. $\lambda < 2n+1$) and $\phi(x)$ (resp. $\phi(x)$ and $\phi(y)$) are isolated points in the cosupport of $\mathcal{J}(G)$ (this can be achieved by letting $G = \frac{n}{n+1}\sum_{i=1}^{n+1} H_i$ where $H_i \in |H|$ are general hyperplanes containing $\phi(x)$ (resp. letting $G = \frac{n}{n+1}(H_0 + \sum_{i=1}^{n}(H_i + H_i'))$ where $H_i \in |H|$ are general hyperplanes containing $\phi(x)$, $H_i' \in |H|$ are general hyperplanes containing $\phi(y)$ and $H_0$ is a general hyperplane containing $\phi(x)$ and $\phi(y)$)).

By Nadel vanishing, $H^1(X, \mathcal{O}_X(K_X + tM) \otimes \mathcal{J}(\phi^* G)) = 0$ and so there is a surjection

$$H^0(X, \mathcal{O}_X(K_X + tM)) \longrightarrow H^0(X, \mathcal{O}_X(K_X + tM) \otimes \mathcal{O}_X/\mathcal{J}(\phi^* G)).$$

Since $x$ (resp. $x$, $y$) is an isolated component of the support of $\mathcal{O}_X/\mathcal{J}(\phi^* G)$, it follows that there is a section $H^0(X, \mathcal{O}_X(K_X + tM))$ not vanishining at $x$ (resp. vanishing at $y$ and not vanishing at $x$). $\square$

## 2. Pairs with hyperstandard coefficients

2.1. **The DCC for volumes of log birationally bounded pairs.** In this section, we will show that for pairs which are log birationally bounded, if the coefficients satisfy the DCC, then the volumes satisfy the DCC.

Let $(Z, D)$ be a simple normal crossings pair with the obvious toroidal structure. Let $\mathbf{B}$ be an effective $b$-divisor. If $Y_1, Y_2, \dots, Y_m$ are finitely many toroidal models over $Z$, and $Z' \longrightarrow Z$ is a proper birational morphism factoring through each $Y_i$, then we define the corresponding **cut** $(Z', \mathbf{B}')$ of $(Z, \mathbf{B})$ to be

$$\mathbf{B}' = \mathbf{B} \wedge \mathbf{M}_{\Theta} \qquad \text{where} \qquad B_i = \mathbf{L}_{\mathbf{B}_{Y_i}, Z'} \quad \text{and} \quad \Theta = \wedge_{i=1}^{m} B_i.$$

Notice that the coefficients of $\mathbf{B}'$ belong to the set $I' = I \cup \operatorname{coeff}(\mathbf{B}'_{Z'})$, which also satisfies the DCC. If $Z$ is projective then it follows easily from Lemma 1.5.1 that $\operatorname{vol}(Z', K_{Z'} + \mathbf{B}'_{Z'}) = \operatorname{vol}(Z', K_{Z'} + \mathbf{B}_{Z'})$. We say that $(Z', \mathbf{B}')$ is a **reduction** of $(Z, \mathbf{B})$ if it is obtained by a finite sequence $(Z_i, \mathbf{B}_i)$, $i = 0, 1, \dots, k$, where $(Z_0, \mathbf{B}_0) = (Z, \mathbf{B})$, $(Z_i, \mathbf{B}_i)$ is a cut of $(Z_{i-1}, \mathbf{B}_{i-1})$ for $i = 1, \dots, k$ and $(Z_k, \mathbf{B}_k) = (Z', \mathbf{B}')$.

**Lemma 2.1.1.** *Let $(Z, D)$ be a simple normal crossings pair and $\mathbf{B}$ a b-divisor whose coefficients are in a DCC set contained in $[0, 1]$ such that* $\operatorname{Supp}(\mathbf{B}_Z) \subset \operatorname{Supp}(D)$, *then there exists a reduction $(Z', \mathbf{B}')$ of $(Z, \mathbf{B})$ that*

$$\mathbf{B}' \geq \mathbf{L}_{\mathbf{B}'_{Z'}}.$$

*Proof.* If $\mathbf{B} \geq \mathbf{L}_{\mathbf{B}_Z}$, then there is nothing to prove. Suppose now that for any divisorial valuation $\nu$ such that $\mathbf{B}(\nu) < \mathbf{L}_{\mathbf{B}_Z}(\nu)$, the centre of $\nu$ is not contained in any strata of $\lfloor \mathbf{B}_Z \rfloor$. Let $Z' \longrightarrow Z$ be a finite sequence of blow ups along strata of $\{\mathbf{B}_Z\}$ such that $(Z', \{\mathbf{L}_{\mathbf{B}_Z, Z'}\})$ is terminal and let $\mathbf{B}' = \mathbf{B} \wedge \mathbf{M}_{\mathbf{L}_{\mathbf{B}_Z, Z'}}$, that is, $\mathbf{B}'$ is the cut of $\mathbf{B}$ with respect to $Z' \longrightarrow Z$. Let $\nu$ be a valuation such that $\mathbf{B}'(\nu) < \mathbf{L}_{\mathbf{B}'_{Z'}}(\nu)$. Since $\mathbf{B}'_{Z'} = \mathbf{L}_{\mathbf{B}'_{Z'}, Z'}$, the centre of $\nu$ is not a divisor on $Z'$. But then

$$\mathbf{B}(\nu) = \mathbf{B}'(\nu) < \mathbf{L}_{\mathbf{B}'_{Z'}}(\nu) \leq \mathbf{L}_{\mathbf{B}_Z}(\nu)$$

and so the centre of $\nu$ is not contained in any strata of $\lfloor \mathbf{B}'_{Z'} \rfloor$ (as the strata of $\lfloor \mathbf{B}'_{Z'} \rfloor$ map to strata of $\lfloor \mathbf{B}_Z \rfloor$). But then, since $(Z', \{\mathbf{B}'_{Z'}\})$ is terminal $\mathbf{L}_{\mathbf{B}'_{Z'}}(\nu) = 0$ which is impossible as $\mathbf{B} \geq 0$.

We may therefore assume that there is a divisorial valuation $\nu$ with centre contained in a stratum of $\lfloor \mathbf{B}_Z \rfloor$. Let $k = k(Z, \mathbf{B})$ be the maximal codimension of such a stratum. We will prove the statement by induction on $k \geq 1$. It suffices to show that there is a cut $(Z', \mathbf{B}')$ of $(Z, \mathbf{B})$ such that $k(Z', \mathbf{B}') < k(Z, \mathbf{B})$. Since $\lfloor \mathbf{B}_Z \rfloor$ has finitely many strata, we may work locally around each stratum. We may therefore assume that $Z = \mathbb{C}^n$ and

$$\mathbf{B}_Z = E_1 + \ldots + E_k + \sum_{i=k+1}^{n} a_i E_i$$

where $0 \leq a_i < 1$ and the $E_i$ are the coordinate hyperplanes. The divisors $E_1, E_2, \dots, E_n$ correspond to vectors $e_1, e_2, \dots, e_n$ and the valuations we consider correspond to the non zero integral prime vectors with non negative coefficients. Let $E$ be a toric valuation corresponding to a vector $\sum_{j=1}^n b_j e_j$ for $b_j \in \mathbb{N}$. By standard toric geometry the coefficient of $E$ in $\mathbf{L}_{\mathbf{B}_Z}$ is

$$\left(1 - \sum_{j=k+1}^{n} (1 - a_j) b_j\right) \vee 0. \tag{2}$$

By equation 2, there is a finite set $V_0 \subset \mathbb{N}^{n-k}$, such that if $(b_{k+1}, \dots, b_n) \notin V_0$, then any divisor $E$ corresponding to a vector

$$(*, ..., *, b_{k+1}, \dots, b_n)$$

where the first $k$ entries are arbitrary, satisfies $\mathbf{L}_{\mathbf{B}_Z}(E) = 0$.

In what follows below, by abuse of notation, we will use $\sigma$ to denote both an integral prime vector in $\mathbb{N}^n$ and the corresponding toric valuation. Fix $v \in V_0$, among all valuations of the form $\sigma = (*, ..., *, v)$, we consider $\sigma = \sigma(v)$ such that $\mathbf{B}(\sigma)$ is minimal (this is possible since the coefficients of $\mathbf{B}$ belong to a DCC set). We pick a toroidal log resolution $Z' \longrightarrow Z$ such that for any $v \in V_0$, and any $\sigma = \sigma(v)$ as above, the induced rational map $Z' \longrightarrow Y_\sigma$ is a morphism (where $Y_\sigma \longrightarrow Z$ is the toroidal morphism with exceptional divisor $E_\sigma$ such that $\rho(Y_\sigma/Z) = 1$). Let $(Z', \mathbf{B}')$ be the cut corresponding to $\{Y_\sigma\}_{v \in V_0}$.

We claim that $k(Z', \mathbf{B}') < k(Z, \mathbf{B}) = k$. Suppose to the contrary that there exists a stratum of $\mathbf{B}'_{Z'}$ of codimension $k$ containing the centre of a valuation $\nu$ such that $\mathbf{B}'(\nu) < \mathbf{L}_{\mathbf{B}'_{Z'}}(\nu)$. Clearly $\nu$ is exceptional over $Z'$ and $\nu$ is a toric valuation corresponding to a vector $\tau = (*, \dots, *, b_{k+1}, \dots, b_n)$ for some $v = (b_{k+1}, \dots, b_n) \in V_0$. Let $\sigma = \sigma(v)$ be the valuation defined above. Then $Z' \longrightarrow Z$ factors through $Y_\sigma$. The toric morphism $Y_\sigma \longrightarrow Z$ corresponds to subdividing the cone given by the basis vectors $e_1, e_2, \dots, e_n$ in to $m \leq n$ cones spanned by $\sigma$ and $e_1, \dots, e_{l-1}, e_{l+1}, \dots, e_n$. Since $\tau$ belongs to one of these cones, we may write

$$\tau = \lambda\sigma + \sum_{i \neq l} \lambda_i e_i, \qquad \lambda, \lambda_i \in \mathbb{Q}_{\geq 0}.$$

Since $\lfloor \mathbf{B}'_{Z'} \rfloor$ has a codimension $k$ stratum containing the centre of $\tau$, the same is true for $(\mathbf{L}_{\mathbf{B}})_{Y_\sigma}$. Since $\mathbf{L}_{\mathbf{B}_Z}(\sigma) < 1$, it follows that $b_l \neq 0$ for some $k+1 \leq l \leq n$. Therefore $\tau$ belongs to a cone spanned by $\sigma$ and $\{e_i\}_{i \neq l}$ where $k+1 \leq l \leq n$. Since the last $n-k$ coordinates of $\sigma$ and $\tau$ are identical, it follows that $\lambda = 1$ and hence $\tau \geq \sigma$ (in the sense that $\tau - \sigma = \sum_{i \neq l} \lambda_i e_i$ where $\lambda_i \geq 0$).

It then follows that

$$\begin{aligned}
\mathbf{L}_{\mathbf{B}'_{Z'}}(\tau) &\leq \mathbf{L}_{\mathbf{B}_{Y_\sigma}}(\tau) \\
&\leq \mathbf{L}_{\mathbf{B}_{Y_\sigma}}(\sigma) \\
&\leq \mathbf{B}(\sigma) \\
&\leq \mathbf{B}(\tau) \\
&= \mathbf{B}'(\tau),
\end{aligned}$$

where the first inequality follows from the definition of $\mathbf{B}'$, the second as $\tau \leq \sigma$, the third as $\sigma$ is a divisor on $\mathbf{B}_{Y_\sigma}$, the fourth by our choice of $\sigma$, and the fifth by the definition of $\mathbf{B}'$ and the fact that $\tau$ is exceptional over $Z'$. $\square$

The following theorem is [13, 5.1].

**Theorem 2.1.2.** *Fix a set $I \subset [0,1]$ which satisfies the DCC and $(Z, D)$ a simple normal crossing pair where $D$ is reduced. Consider the set $\mathfrak{D}$ of all projective simple normal crossings pairs $(X, B)$ such that* $\operatorname{coeff}(B) \subset I$*, $f\colon X \longrightarrow Z$ is a birational morphism and $f_*B \leq D$. Then the set*

$$\{\, \operatorname{vol}(X, K_X + B) \,|\, (X, B) \in \mathfrak{D} \,\}$$

*satisfies the DCC.*

*Proof.* Suppose that $(X_i, B_i)$ is an infinite sequence of pairs in $\mathfrak{D}$ such that $\operatorname{vol}(X_i, K_{X_i} + B_i)$ is a strictly decreasing sequence. Let $f_i\colon X_i \longrightarrow Z$ be the induced morphisms such that $\operatorname{Supp}(f_{i*}B_i) \subset D$. By Lemma 1.5.1, we know that

$$\operatorname{vol}(X_i, K_{X_i} + B_i) \leq \operatorname{vol}(Z, K_Z + f_{i*}B_i)$$

and if this inequality is strict then

$$\operatorname{mult}_E(K_{X_i} + B_i) < \operatorname{mult}_E f_i^*(K_Z + f_{i*}B_i)$$

for some divisor $E$ contained in $\operatorname{Exc}(f_i)$. In this case $E$ must define a toroidal valuation with respect to the toroidal structure associated to $(Z, D)$. Let $f'_i\colon X'_i \longrightarrow Z$ be a toroidal morphism such that if $E$ is a divisor on $X_i$ corresponding to a toroidal valuation for $(Z, D)$, then $E$ is a divisor on $X'_i$. Let $B'_i$ be the strict transform of $B_i$ plus the $X'_i \dashrightarrow X_i$ exceptional divisors. By Lemma 1.5.1, we have

$$\operatorname{vol}(X_i, K_{X_i} + B_i) = \operatorname{vol}(X'_i, K_{X'_i} + B'_i)$$

and the coefficients of $B'_i$ are contained in $I \cup \{1\}$. Replacing $(X_i, B_i)$ by $(X'_i, B'_i)$, we may assume that $f_i\colon (X_i, B_i) \longrightarrow (Z, D)$ are toroidal morphisms.

For each pair $(X_i, B_i)$, consider the $b$-divisors $\mathbf{M}_{B_i}$. Since there are only countably many toroidal valuations over $(X, D)$, by a standard diagonalisation argument, after passing to a subsequence, we may assume that for any toroidal divisor $E$ over $X$, the sequence $\mathbf{M}_{B_i}(E)$ is eventually non decreasing. Note that for any non-toroidal divisor $E$ exceptional over $Z$, we have $\mathbf{M}_{B_i}(E) = 1$. Therefore the limit $\lim \mathbf{M}_{B_i}(E)$ exists for any divisor $E$ over $Z$. We let $\mathbf{B} = \lim \mathbf{M}_{B_i}$ so that $\mathbf{B}(E) = \lim \mathbf{M}_{B_i}(E)$ for any divisor $E$ over $Z$.

By Lemma 2.1.1 there is a reduction $(\mathbf{B}', Z')$ of $(\mathbf{B}, Z)$ such that $\mathbf{B}' \geq \mathbf{L}_{\mathbf{B}'_{Z'}}$ and

$$\operatorname{vol}(Z', K_{Z'} + \mathbf{B}'_{Z'}) = \operatorname{vol}(Z', K_{Z'} + \mathbf{B}_{Z'}).$$

Since for any divisor $E$ over $Z$, the sequence $\mathbf{B}_i(E)$, $i = 1, \dots$ is eventually non decreasing, we may assume that $\mathbf{M}_{B_i,Z'} \leq \mathbf{B}_{Z'}$. Therefore, we have

$$\begin{aligned}\operatorname{vol}(X_i, K_{X_i} + B_i) &\leq \operatorname{vol}(Z', K_{Z'} + \mathbf{M}_{B_i,Z'}) \\ &\leq \operatorname{vol}(Z', K_{Z'} + \mathbf{B}_{Z'}) \\ &= \operatorname{vol}(Z', K_{Z'} + \mathbf{B}'_{Z'}).\end{aligned}$$

If $\operatorname{vol}(X_i, K_{X_i} + B_i)$, $i = 1, \dots$ is strictly decreasing then there exists a constant $\epsilon > 0$, such that for any $j > i + 1$,

$$\begin{aligned}\operatorname{vol}(X_j, K_{X_j} + B_j) &< \operatorname{vol}(X_{i+1}, K_{X_{i+1}} + B_{i+1}) \\ &\leq \operatorname{vol}(Z', K_{Z'} + (1-\epsilon)\mathbf{B}'_{Z'}).\end{aligned}$$

Let $Z'' \longrightarrow Z'$ be a toroidal morphism which extracts all divisors $E$ with $a(E, Z', (1-\epsilon)\mathbf{B}'_{Z'}) < 0$. Then

$$\mathbf{M}_{B_j,Z''} \geq \mathbf{L}_{(1-\epsilon)\mathbf{B}'_{Z'},Z''}$$

for all $j \gg 0$, which then implies

$$\mathbf{M}_{B_j,X_j} \geq \mathbf{L}_{(1-\epsilon)\mathbf{B}'_{Z'},X_j}$$

as for any exceptional divisor $E$ of $X_j/Z''$, we know that $\mathbf{L}_{(1-\epsilon)\mathbf{B}'_{Z'}}(E) = 0$. But then

$$\begin{aligned}\operatorname{vol}(X_j, K_{X_j} + B_j) &\geq \operatorname{vol}(X_j, K_{X_j} + \mathbf{L}_{(1-\epsilon)\mathbf{B}'_{Z'},X_j}) \\ &= \operatorname{vol}(Z'', K_{Z''} + \mathbf{L}_{(1-\epsilon)\mathbf{B}'_{Z'},Z''})\end{aligned}$$

which is the required contradiction. $\square$

**Corollary 2.1.3.** [13, 1.9] *Fix a set $I \subset [0,1]$ which satisfies the DCC and a log birationally bounded set $\mathfrak{B}_0$ of log canonical pairs $(X, B)$ such that* $\operatorname{coeff}(B) \subset I$*. Then the set*

$$\{\operatorname{vol}(X, K_X + B) \mid (X, B) \in \mathfrak{B}_0\}$$

*satisfies the DCC.*

*Proof.* We may assume that $1 \in I$. Since $\mathfrak{B}_0$ is log birationally bounded, there exists a projective morphism $Z \longrightarrow T$ where $T$ is of finite type and a log pair $(Z, D)$ such that for any $(X, B) \in \mathfrak{B}_0$, there is a birational morphism $f\colon X \dashrightarrow Z_t$ such that the support of $D_t$ contains the support of $f_*B$ and the $Z_t \dashrightarrow X$ exceptional divisors. Let $\nu\colon X' \longrightarrow X$ be a resolution such that $X' \longrightarrow Z_t$ is a morphism. If $B' = \mathbf{M}_{B,X'}$, then $\mathrm{vol}(X', K_{X'} + B') = \mathrm{vol}(X, K_X + B)$ (cf. Lemma 1.5.1). Replacing $(X, B)$ by $(X', B')$ we may assume that $f$ is a morphism. By a standard argument, after replacing $T$ by a finite cover and $X$ by the corresponding fibre product, we may assume that $T$ is smooth (and possibly reducible), $(Z, D)$ is log smooth over $T$, the strata of $(Z, D)$ are geometrically irreducible over $T$.

For any birational morphism $f\colon X \longrightarrow Z_t$ as above, consider the finite set $\mathcal{E}$ of divisors $E$ on $X$ such that $a_E(Z_t, f_*B) < 0$. Since $(Z_t, D_t)$ is log smooth, there exists a finite sequence of blow ups along strata of $\mathbf{M}_{D_t}$ say $X' \longrightarrow Z_t$ such that the divisors in $\mathcal{E}$ are not $X \dashrightarrow X'$ exceptional. Let $p\colon W \longrightarrow X$ and $q\colon W \longrightarrow X'$ be a common resolution, then by Lemma 1.5.1

$$\mathrm{vol}(X, K_X + B) = \mathrm{vol}(W, K_W + \mathbf{M}_{B,W}) = \mathrm{vol}(X', K_{X'} + B')$$

where $B' = \mathbf{M}_{B,X'}$. Hence, replacing $(X, B)$ by $(X', B')$ we may assume that each $f\colon X \longrightarrow Z_t$ is induced by a finite sequence of blow ups along strata of $(Z_t, D_t)$. Notice that since $(Z, D)$ is log smooth over $T$, there is a sequence of blow ups along strata of $(Z, D)$ say $Z' \longrightarrow Z$ such that $Z'_t \cong X'$. Let $\Phi$ be the divisor supported on the strict transform of $D$ and the $Z' \longrightarrow Z$ exceptional divisors such that $\Phi_t = B'$. Fix a closed point $0 \in T$, then by Theorem 1.9.2,

$$\mathrm{vol}(X, K_X + B) = \mathrm{vol}(Z'_0, K_{Z'_0} + \Phi_0).$$

By Theorem 2.1.2, the set of these volumes satisfies the DCC. □

**Theorem 2.1.4.** [15, 3.5.2] *Fix $n \in \mathbb{N}$, $M > 0$, and a set $I \subset [0, 1]$ which satisfies the DCC. Suppose that $\mathfrak{B}_0$ is a set of log canonical pairs $(X, B)$ such that*

*(1) $X$ is projective of dimension $n$,*
*(2) $\mathrm{coeff}(B) \subset I$, and*
*(3) there exists an integer $k > 0$ such that $\phi_{k(K_X+B)}$ is birational and $\mathrm{vol}(X, k(K_X + B)) \leq M$.*

*Then the set*

$$\{\, \mathrm{vol}(X, K_X + B) \mid (X, B) \in \mathfrak{B}_0 \,\}$$

*satisfies the DCC.*

*Proof.* Proposition 1.10.4 implies that $\mathfrak{B}_0$ is log birationally bounded and so the result follows from Corollary 2.1.3. □

2.2. **Adjunction.** In this section, we discuss various versions of adjunction. The main new result is Theorem 2.2.5, which is an adjunction for pairs with hyperstandard coefficients. This is a key result for doing induction on the dimension. We remark that the proof of Theorem

2.2.5 is different from the one in [15, 4.2], as the argument presented here does not need the ACC for log canonical thresholds. On the other hand the argument only works for hyperstandard coefficients.

**Theorem 2.2.1** (Shokurov log adjunction)**.** *Let $(X, S+B)$ be a log canonical pair, $B = \sum b_i B_i$ and $S$ a prime divisor with normalisation $\nu\colon S^\nu \longrightarrow S$. Then*

$$(K_X + S + B)|_{S^\nu} = K_{S^\nu} + \mathrm{Diff}_{S^\nu}(B) = K_{S^\nu} + \mathrm{Diff}_{S^\nu}(0) + B|_{S^\nu},$$

*where the coefficients of $\mathrm{Diff}_{S^\nu}(B)$ are of the form*

$$\frac{r - 1 + \sum n_i b_i}{r} \qquad \text{for some} \qquad n_i \in \mathbb{N}$$

*and $r$ is the index of the corresponding codimension 2 point $P \in X$. In particular if $\mathrm{coeff}(B) \subset I$, then $\mathrm{coeff}(\mathrm{Diff}_{S^\nu}(B)) \subset D(I)$.*

We have the following easy consequence.

**Lemma 2.2.2.** *Let $(X, S+B)$ be a log canonical pair, $B = \sum b_i B_i$ an effective $\mathbb{R}$-Cartier divisor and $S$ a prime divisor with normalisation $\nu\colon S^\nu \longrightarrow S$. Then for any $0 \leq \lambda \leq 1$ we have*

$$\mathrm{Diff}_{S^\nu}(\lambda B) \geq \lambda\, \mathrm{Diff}_{S^\nu}(B).$$

*Proof.* The coefficients of $\mathrm{Diff}_{S^\nu}(\lambda B)$ are of the form

$$\frac{r - 1 + \lambda \sum n_i b_i}{r} \geq \lambda \left( \frac{r - 1 + \sum n_i b_i}{r} \right). \qquad \square$$

**Theorem 2.2.3** (Kawamata Subadjunction)**.** *Let $(X, \Delta)$ be a pair such that $X$ is quasi-projective and normal and $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Assume $V$ is a subvariety such that $(X, \Delta)$ is log canonical at the generic point $\eta$ of $V$ and $V$ is the only non-klt centre of $(X, \Delta)$ at $\eta$. Then there is a $\mathbb{Q}$-divisor $B \geq 0$ and a $\mathbb{Q}$-divisor class $J$ on the normalisation $V^n$ of $V$, such that*

$$(K_X + \Delta)|_{V^n} \sim_{\mathbb{Q}} K_{V^n} + B + J.$$

*If $X$ is projective, then $J$ is pseudo-effective.*

*Furthermore, if there is a generically finite morphism $\pi\colon Y \longrightarrow X$ such that if we write $f^*(K_X + \Delta) = K_Y + \Delta_Y$, $\Delta_Y$ is effective and $W \longrightarrow \pi^{-1}(V)$ is a finite morphism on to a non-klt centre of $(X, \Delta)$ and we denote by $p\colon W^n \longrightarrow V^n$ the natural map between the normalisations, then if we apply Kawamata subadjunction to $W^n$ and $(Y, \Delta_Y)$ and write $(K_Y + \Delta_Y)|_{W^n} \sim_{\mathbb{Q}} K_{W^n} + B_W + J_W$, then we have $J_W = p^*J$.*

*Proof.* These statements follow from [22, 8.4-8.6], especially [8.4.9] for the properties of $J$. In particular, the last statement is an immediate consequence of the results there: if we choose sufficiently high models $\alpha\colon V' \longrightarrow V^n$ and $\beta\colon W' \longrightarrow W^n$, we also assume there is a (generically finite) morphism $q\colon W' \longrightarrow V'$, we know that there is a divisor class $J'$ on $V'$ such that we have $J = \alpha_* J'$ and $J_W = \beta_* q^* J'$. But since $p\colon W^n \longrightarrow V^n$ is a finite morphism, this immediately implies that $p^*J = J_W$. $\square$

**Lemma 2.2.4.** *Fix $q \in \mathbb{N}$ and let $I_0 = \{\frac{1}{q}, ..., \frac{q-1}{q}\}$. Let $(X, \Delta)$ be $\mathbb{Q}$-factorial pair such that $\operatorname{coeff}(\Delta) \subset I = D(I_0)$. Then, for any point $x \in X$, there is a finite morphism $\pi\colon Y \longrightarrow U$ for some neighbourhood $x \in U \subset X$ with Galois group $G$, such that $\pi^*(K_U + \Delta|_U) = K_Y + \Delta_Y$, and $Y$ is Gorenstein canonical, the coefficients of $\Delta_Y$ are in $I_0$ and the components of $\Delta_Y$ are Cartier.*

*Proof.* We begin by constructing a finite cover $p_1\colon Y_1 \longrightarrow U$ of normal varieties, which is étale in codimension 1, such that $p_1^* K_X$ and $p_1^*\Delta_i$ are Cartier for all components $\Delta_i$ of $\Delta$. To this end, take $D$ to be either $K_X$ or a component of $\Delta$. Then $D$ is a Weil divisor. Let $n$ be the smallest integer such that $nD$ is Cartier in a neighbourhood $U$ of $x \in X$. Pick an isomorphism $\mathcal{O}_U(nD|_U) \cong \mathcal{O}_U$ and let $\pi\colon U' \longrightarrow U$ be the normalisation of the corresponding cyclic cover. Then $D' = \pi^{-1}(D|_U)$ is Cartier. Since $D$ is Cartier in codimension 1 (as $X$ is normal and hence $R_1$), it follows that $U' \longrightarrow U$ is étale in codimension 1. We let $p_1\colon Y_1 \longrightarrow U$ be the normalisation of the fibre product of these cyclic covers. Note that as $p_1$ is étale in codimension 1, writing $K_{Y_1} + \Delta_1 = (K_X + \Delta)|_{Y_1}$, we have $\operatorname{coeff}(\Delta_1) \subset I$.

If $\Delta^i$ is a component of the support of $\Delta_1$ then the coefficient of $\Delta^i$ in $\Delta_1$ is of the form $\frac{m_i - 1 + a_i}{m_i}$ where $a_i = r_i/q$ from some positive integers $m_i, r_i$. By Kawamata's trick, we may take a branched cover $\mu\colon Y \longrightarrow Y_1$, which is branched of degree $m_i$ along each $\Delta_i$. Let $P_1$ be the generic point of $\Delta^i$ and $P$ be the generic point of an irreducible component of $\mu^{-1}(P)$. If $K_Y + \Delta_Y = \mu^*(K_{Y_1} + \Delta_1)$ then by an easy local computation one sees that $\operatorname{mult}_P(\Delta_Y) = r_i/q$ as required. $\square$

**Theorem 2.2.5.** *Fix $n$ and $q \in \mathbb{N}$. Let $I = D(I_0)$ where $I_0 = \{\frac{1}{q}, ..., \frac{q-1}{q}\}$ for some $q \in \mathbb{N}$. Let $(X, B)$ be an $n$-dimensional projective Kawamata log terminal log pair with $\operatorname{coeff}(B) \subset I$.*

*Assume that there is a flat projective family $h\colon \mathcal{V} \longrightarrow S$ over a smooth base $S$ with a generically finite morphism $\beta\colon \mathcal{V} \longrightarrow X$ such that for a general point $v \in \mathcal{V}$, there exists a $\mathbb{Q}$-Cartier divisor $B' \geq 0$, such that $(X, B + B')$ is log canonical in a neighbourhood of $\beta(v)$ and* *if $s = h(v)$ then the fibre $\mathcal{V}_s = \mathcal{V} \underset{S}{\times} \{s\}$ is mapped isomorphically to the unique non-klt centre $V$ of $(X, B + B')$ containing $\beta(v)$.*

*If $\nu\colon W \longrightarrow V$ is the normalisation then there is a divisor $\Theta$ on $W$ such that* $\operatorname{coeff}(\Theta) \in I$ *and*

$$(K_X + B + B')|_W - (K_W + \Theta)$$

*is pseudo-effective. Moreover, there is a log resolution $\psi\colon W' \longrightarrow W$ of $(W, \Theta)$ such that*

$$K_{W'} + \Omega \geq (K_X + B)|_{W'}$$

*where $\Omega = \psi^{-1}_* \Theta + \operatorname{Exc}(\psi)$.*

*Proof.* Let $\mathcal{W}$ be the normalisation of $\mathcal{V}$. Replacing $S$ by a dense open subset we may assume that for any $s \in S$, the fibre $\mathcal{W}_s := \mathcal{W} \underset{S}{\times} \{s\}$ is the normalisation of $\mathcal{V}_s = V$. We denote $\mathcal{W}_s$ by $W$.

For any point $P$ on $W$, if we consider $\nu(P) \in V$ as a point in $X$ then there exists a Zariski open subset $U_i \subset X$ containing $\nu(P)$ with a finite morphism $\pi_i\colon Y_i \longrightarrow U_i$ as in Lemma 2.2.4. Let $G = G_i$ be the corresponding Galois group and we denote by

$$\pi_i^*(K_{U_i} + B|_{U_i}) = K_{Y_i} + \Delta_{Y_i}.$$

Consider $\mathcal{V}_i := \mathcal{V} \times_X Y_i$ and its normalisation $\mathcal{W}_i \longrightarrow \mathcal{V}_i$. Let

$$K_{\mathcal{W}_i} + \Psi_i' = (K_X + B)|_{\mathcal{W}_i} = (K_{Y_i} + \Delta_{Y_i})|_{\mathcal{W}_i} \qquad \text{and} \qquad \Psi_i = \Psi_i' \vee 0.$$

Let $F = (\mathcal{W}_i)_s$ and $E = (\mathcal{V}_i)_s$ be the fibres of $\mathcal{W}_i$ and $\mathcal{V}_i$ over a general point $s \in S$. Then $E \longrightarrow \pi_i^{-1}(V \cap U_i)$ is an isomorphism and $\pi_i^{-1}(V \cap U_i)$ is a union of non-klt centres of $(Y_i, \Delta_{Y_i} + \pi_i^* B'|_{U_i})$ with an induced $G$-action and $F$ is the normalisation of $E$.

We denote $\Psi_i|_F = \Phi_i$. We note that since $F$ is a general fibre of $\mathcal{W}_i$ over $S$, we have $K_{\mathcal{W}_i}|_F = K_F$. There is a natural isomorphism

$$W_{U_i} := W \times_X U_i \cong F/G,$$

so we have a morphism $p_i\colon F \longrightarrow W$. We define $\Theta_i$ on $W_{U_i}$ via the equality

$$p_i^*(K_{W_{U_i}} + \Theta_i) = K_F + \Phi_i.$$

Note in fact that $K_F + \Psi_i'|_F = (K_X + B)|_F$ is pulled back from $W_{U_i}$, each component of the support of $\Psi - \Psi'$ is Cartier (by Lemma 2.2.4), $(\Psi - \Psi')|_F$ is $G$-invariant and so it is a pull back of a $\mathbb{Q}$-Cartier divisor on $F/G$. Since $\Phi_i \geq 0$, it follows that $\Theta_i \geq 0$.

We choose finitely many open subsets $\{U_i\}_{i \in I}$ covering $X$, and we define the divisor $\Theta$ on $W$ by defining the coefficient of any prime divisor $P$ in $\Theta$ to be

$$\operatorname{mult}_P(\Theta) = \max\{ \operatorname{mult}_P(\Theta_i) \mid P \in W_{U_i} \}$$

We now check that $(W, \Theta)$ is the pair we are looking for.

We first check that the coefficients of $\Theta$ are in $I = D(I_0)$. It suffices to show that the coefficients of $\Psi_i$ are in $I_0$, as this implies that the coefficients of $\Phi_i$ are in $I_0$, and we can then conclude from the usual Hurwitz formula that the coefficients of $\Theta$ are in $I = D(I_0)$. By our construction $\mathcal{W}_i \longrightarrow Y_i$ is generically finite, and

$$K_{\mathcal{W}_i} + \Psi_i' = (K_{Y_i} + \Delta_{Y_i})|_{\mathcal{W}_i}.$$

Since $Y_i$ is Gorenstein, the components of $\Delta_{Y_i}$ are Cartier and all their coefficients are in $I_0$, it follows that all coefficients of $\Psi_i'$ are of the form $m + \sum m_j i_j$ where $m$, $m_j \in \mathbb{N}$ and $i_j \in I_0$ and so they belong to $\frac{1}{q}\mathbb{N}$. Since $K_X + B$ is Kawamata log terminal, $K_{\mathcal{W}_i} + \Psi_i'$ is sub Kawamata log terminal and so $\operatorname{coeff}(\Psi_i) \subset \frac{1}{q}\mathbb{N} \cap [0, 1) = I_0$.

Next we check that

$$(K_X + B + B')|_W - (K_W + \Theta)$$

is pseudo-effective. By Kawamata subadjunction, we may write

$$(K_X + B + B')|_W = K_W + (B + B')_W + J_W$$

where $(B + B')_W \geq 0$ is an $\mathbb{R}$-divisor and $J_W$ is a pseudo-effective $\mathbb{R}$-divisor defined up to $\mathbb{R}$-linear equivalence. Define $(B')_W$ by the equation $(B + B')_W = B|_W + (B')_W \geq 0$. Since $s$ is general, we may assume that $E$ does not belong to $\operatorname{Supp}(\Delta_{Y_i})$. Then $F$ is a union of log canonical centres of $K_{\mathcal{W}_i} + B'|_{\mathcal{W}_i}$. Applying Kawamata subadjunction, we may write

$$\big(K_{Y_i} + \pi_i^*(B'|_{U_i})\big)|_F = K_F + B_F' + J_F,$$

where $B_F' \geq 0$ and since $F \longrightarrow W_{U_i}$ is finite, we have $J_F = p_i^*(J_W)$.

If we let $K_{\mathcal{W}_i} + \Gamma_i = (K_{Y_i})|_{\mathcal{W}_i}$ then we know that the discrepancy of any divisor with respect to $(\mathcal{W}_i, \Gamma_i)$ is positive as $K_{Y_i}$ is canonical. In particular, $\Gamma_i \leq 0$. Thus

$$\Phi_i = ((K_{Y_i} + \Delta_{Y_i})|_{\mathcal{W}_i} - K_{\mathcal{W}_i})|_F = (\Gamma_i + \Delta_{Y_i}|_{\mathcal{W}_i})|_F \vee 0 \leq \Delta_{Y_i}|_F.$$

By Kawamata subadjunction, we know that

$$(K_X + B + B')|_W - (K_W + \Theta + J_W) = (B + B')_W - \Theta$$

is a well defined $\mathbb{Q}$-divisor on $W$, which we claim to be effective. For this purpose, we only need to check the multiplicities at each codimension 1 point $P$ on $W$. It suffices to verify this after pulling back via $Y_i \longrightarrow X$, where $U_i$ contains $P$ and $\operatorname{mult}_P(\Theta) = \operatorname{mult}_P(\Theta_i)$.

We have

$$\begin{aligned}
&p_i^*\big((K_X+B+B')|_W-(K_W+\Theta+J_W)\big)\\
=&\big(K_{Y_i}+\Delta_{Y_i}+\pi_i^*(B'|_{U_i})\big)|_F-(K_F+\Phi_i+J_F)\\
=&B'_F+(\Delta_{Y_i}|_F-\Phi_i)\geq 0.
\end{aligned}$$

where we used the fact that $p_i^*J_W=J_F$ in the first equality.

Finally we check the last statement. Let

$$W'_{U_i}:=W_{U_i}\underset{W}{\times}W' \quad \psi_i\colon W'_{U_i}\longrightarrow W_{U_i} \quad \text{and} \quad \Omega_i=\psi_{i,*}^{-1}\Theta_i+\mathrm{Exc}(\psi_i).$$

Note that $\Omega|_{W'_{U_i}}\geq\Omega_i$. After possibly shrinking $S$, we can assume that there is a resolution $\mathcal{W}'\longrightarrow\mathcal{W}$ which induces a log resolution of each fibre and in particular, $\mathcal{W}'_s\longrightarrow\mathcal{W}_s$ induces the log resolution $W'\longrightarrow W$ of $(W,\Theta)$. We may also assume that there is a $G$-equivariant resolution $\mathcal{W}'_i\longrightarrow\mathcal{W}_i$ over $S$ which gives a $G$-invariant log resolution $\psi_F\colon F'\longrightarrow F$, such that there is a proper morphism $F'/G\longrightarrow W'_{U_i}$. Let

$$\Omega_{F'}=\psi_{F*}^{-1}(\Theta_i)+\mathrm{Exc}(\psi_F).$$

It follows that

$$K_{F'}+\Omega_{F'}-(K_{\mathcal{W}_i}+\Psi'_i)|_{F'}\geq 0,$$

as $(K_{\mathcal{W}_i}+\Psi')|_F$ is Kawamata log terminal. Let $\Omega_{F'/G}$ be the $\mathbb{Q}$-divisor defined by $(K_{F'/G}+\Omega_{F'/G})|_{F'}=K_{F'}+\Omega_{F'}$. Then

$$K_{F'/G}+\Omega_{F'/G}-(K_X+B)|_{F'/G}\geq 0.$$

The claim follows by pushing forward to $W'_{U_i}$. □

2.3. **DCC of volumes and birational boundedness.** In this section we prove a result on the ACC for volumes and on log birational boundedness of pairs $(X,B)$ with hyperstandard coefficients. The general case, Theorem 3.0.1 is covered in the next section.

**Theorem 2.3.1.** *Fix $n\in\mathbb{N}$ and a finite set $I_0\subset[0,1]\cap\mathbb{Q}$. Let $J=D(I_0)\subset[0,1]$ and $\mathfrak{D}$ be the set of projective log canonical pairs $(X,B)$ such that $\dim X=n$ and $\mathrm{coeff}(B)\subset J$.*

*Then there is a constant $\delta>0$ and a positive integer $m$ such that*

1. *the set*
$$\{\,\mathrm{vol}(X,K_X+B)\mid(X,B)\in\mathfrak{D}\,\}$$
*also satisfies the DCC,*
2. *if $\mathrm{vol}(X,K_X+B)>0$ then $\mathrm{vol}(X,K_X+B)\geq\delta$, and*
3. *if $K_X+B$ is big then $\phi_{m(K_X+B)}$ is birational.*

*Proof.* We may assume that $I_0 = \{\, \frac{j}{q} \,|\, 1 \leq j \leq q \,\}$ for some $q \in \mathbb{N}$. We proceed by induction on the dimension. Replacing $X$ by a log resolution and $B$ by its strict transform plus the exceptional divisor, we may assume that $(X, B)$ is log smooth. Replacing $B$ by $\{B\} + (1 - \frac{1}{r})\lfloor B \rfloor$ for some $r \gg 0$, we may assume that $(X, B)$ is Kawamata log terminal. Replacing $(X, B)$ by the log canonical model [3], we may assume that $K_X + B$ is ample.

By induction, there is a positive integer $l \in \mathbb{N}$ such that if $(U, \Psi)$ is a projective log canonical pair of dimension $\leq n - 1$, $\operatorname{coeff}(\Psi) \subset J$, and $K_U + \Psi$ is big then $\phi_{l(K_U+\Psi)}$ is birational. Fix $k \in \mathbb{N}$ such that

$$\operatorname{vol}(X, k(K_X + B)) > (2n)^n.$$

**Claim 2.3.2.** *There is an integer $m_0 > 0$ such that $\phi_{m_0 k(K_X+B)}$ is birational.*

*Proof.* By Lemma 1.6.2, there is a family $V \longrightarrow T$ of subvarieties of $X$ such that for any two general points $x$, $y \in X$ there exists $t \in T$ and $0 \leq D_t \sim_{\mathbb{R}} k(K_X + B)$ such that $(X, B + D_t)$ is not Kawamata log terminal at both $x$ and $y$ and there is a unique non Kawamata log terminal place whose centre $V_t$ contains $x$ (so that in particular $(X, B + D_t)$ is log canonical at $x$). Let $\nu \colon V^\nu \longrightarrow V = V_t$ be the normalisation. By Theorem 2.2.5 there exists a $\mathbb{Q}$-divisor $\Theta$ on $V^\nu$ such that

(1) $(K_X + B + D_t)|_{V^\nu} - (K_{V^\nu} + \Theta)$ is pseudo-effective,
(2) $\operatorname{coeff}(\Theta) \subset D(I_0)$, and
(3) there exists $\psi \colon U \longrightarrow V^\nu$ a log resolution of $(V^\nu, \Theta)$ such that if $\Psi = \psi^{-1}_*\Theta + \operatorname{Exc}(\psi)$ then $(K_U + \Psi) \geq (K_X + B)|_U$. In particular $(K_U + \Psi)$ is big.

By induction, $\phi_{l(K_U+\Psi)}$ is birational and so $\phi_{l(K_{V^\nu}+\Theta)}$ is birational as well.

Let $x$, $y \in X$ be general points. We may assume that $x \in V$ is a general point. If $v = \dim V$ and $H_i \in |l(K_{V^\nu} + \Theta)|$ are general divisors passing through $x$ and we set

$$H = \frac{v}{v+1}(H_1 + \ldots + H_{v+1}),$$

then $x$ is an isolated component of the non Kawamata log terminal locus of $(V^\nu, \Theta + H)$. If in addition $y \in V$, then, arguing as above, and at the expense of replacing $H$ by something $\mathbb{Q}$-linearly equivalent to $3H$ (or indeed any multiple of $H$ greater than 2), we may arrange that $(V^\nu, \Theta + H)$ is not log canonical at $y$. Since

$$(K_X + B + D_t)|_{V^\nu} - (K_{V^\nu} + \Theta)$$

is pseudo-effective, we may pick

$$\tilde{H} \sim_{\mathbb{R}} (k+1)vl(K_X + B)$$

such that $\tilde{H}|_V = H$ in a neighbourhood of $x \in V$. Let $\lambda = \operatorname{lct}_x(V^\nu, \Theta; H)$, then $\lambda \leq 1$.

By inversion of adjunction

(1) $x \in X$ is a non Kawamata log terminal centre of $(X, B + D_t + \lambda\tilde{H})$,
(2) $(X, B + D_t + \lambda\tilde{H})$ is log canonical at $x \in X$, and
(3) $(X, B + D_t + \lambda\tilde{H})$ is not Kawamata log terminal at $y \in X$ and it is not log canonical at $y$ if $y \in V$.

Since $K_X + B$ is big, we may write $K_X + B \sim_{\mathbb{R}} A + E$ where $E \geq 0$ and $A$ is ample. As $x$ and $y$ are general we may assume that $x$ and $y$ don't belong to $E$. If $y$ does not belong to $V$ then $x$ and $y$ belong to different connected components of the non kawamata log terminal locus. In this case we can increase $\lambda$ a little bit and use $A$ to make $(X, B + D_t + \lambda\tilde{H})$ not log canonical at $y$. Using $A$ to tie-break (cf. Proposition 1.6.1) we may assume that $x$ is an isolated non kawamata log terminal centre.

By Lemma 1.10.1, $\phi_{K_X + \lceil t(K_X+B)\rceil}$ is birational for any $t \geq (k+1)v$. We claim there is an inequality

$$(m+1)(K_X + B) \geq K_X + \lceil m(K_X + B)\rceil$$

for any integer $m > 0$ which is divisible by $q$. Grant this for the time being. It follows that that $\phi_{(m+1)(K_X+B)}$ is birational for any integer $m$ divisible by $q$ such that $m > (k+1)v$. Claim 2.3.2 now follows.

To see the inequality note that if $k \in \mathbb{N}$, then $\lceil k/r\rceil \leq k/r + (r-1)/r$ and so since $m/q \in \mathbb{N}$, we have

$$\begin{aligned}\lceil m(\frac{r-1+\frac{a}{q}}{r})\rceil &\leq m(\frac{r-1+\frac{a}{q}}{r}) + \frac{r-1}{r} \\ &\leq (m+1)(\frac{r-1+\frac{a}{q}}{r}).\end{aligned}$$

Since the coefficients of $B$ are of the form $\frac{r-1+\frac{a}{q}}{r}$, we have shown that $(m+1)B \geq \lceil mB\rceil$. □

If $\operatorname{vol}(X, K_X + B) \geq 1$ then let $k = 2(n+1)$. The result follows in this case. Therefore, we may assume that $\operatorname{vol}(X, K_X + B) < 1$ and we pick $k \in \mathbb{N}$ such that

$$(2n)^n \leq \operatorname{vol}(X, k(K_X + B)) < (4n)^n.$$

But then $\mathrm{vol}(X, m_0k(K_X + B)) \leq (4m_0n)^n$. By Theorem 2.1.4 the set of these volumes satisfies the DCC and so there exists a constant $\delta > 0$ such that $\mathrm{vol}(K_X + B) \geq \delta$. In particular

$$k = \max(\lfloor \frac{2n}{\delta} \rfloor + 1, 2(n+1)). \qquad \square$$

## 3. Birational boundedness: the general case

The purpose of this section is to prove [15, 1.4]:

**Theorem 3.0.1.** *Fix $n \in \mathbb{N}$ and a set $I \subset [0,1]$ which satisfies the DCC. Let $\mathfrak{D}$ be the set of projective log canonical pairs $(X, B)$ such that $\dim X = n$ and $\mathrm{coeff}(B) \subset I$.*

*Then there is a constant $\delta > 0$ and a positive integer $m$ such that*

*(1) the set*
$$\{\, \mathrm{vol}(X, K_X + B) \ \mid (X, B) \in \mathfrak{D} \,\}$$
*also satisfies the DCC,*
*(2) if $\mathrm{vol}(X, K_X + B) > 0$ then $\mathrm{vol}(X, K_X + B) \geq \delta$, and*
*(3) if $K_X + B$ is big then $\phi_{m(K_X+B)}$ is birational.*

The proof is by induction on the dimension. We will prove the following four statements ([15]).

**Theorem 3.0.2** (Boundedness of the anticanonical volume)**.** *Fix $n \in \mathbb{N}$ and a set $I \subset [0,1)$ which satisfies the DCC. Let $\mathfrak{D}$ be the set of Kawamata log terminal pairs $(X, B)$ such that $X$ is projective, $\dim X = n$, $K_X + B \equiv 0$, and $\mathrm{coeff}(B) \subset I$.*

*Then there exists a constant $M > 0$ depending only on $n$ and $I$ such that $\mathrm{vol}(X, -K_X) < M$ for any pair $(X, B) \in \mathfrak{D}$.*

**Theorem 3.0.3** (Effectively birational)**.** *Fix $n \in \mathbb{N}$ and a set $I \subset [0,1]$ which satisfies the DCC. Let $\mathfrak{B}$ be the set of log canonical pairs $(X, B)$ such that $X$ is projective, $\dim X = n$, $K_X + B$ is big, and $\mathrm{coeff}(B) \subset I$.*

*Then there exists a positive integer $m = m(n, I)$ such that $\phi_{m(K_X+B)}$ is birational for any $(X, B) \in \mathfrak{B}$.*

**Theorem 3.0.4** (The ACC for numerically trivial pairs)**.** *Fix $n \in \mathbb{N}$ and a DCC set $I \subset [0,1]$.*

*Then there is a finite subset $I_0 \subset I$ such that if*

*(1) $(X, B)$ is an $n$-dimensional projective log canonical pair,*
*(2) $\mathrm{coeff}(B) \subset I$, and*
*(3) $K_X + B \equiv 0$,*

*then the coefficients of $B$ belong to $I_0$.*

**Theorem 3.0.5** (The ACC for the LCT)**.** *Fix $n \in \mathbb{N}$ and a set $I \subset [0,1]$ which satisfies the DCC. Then there exists a constant $\delta > 0$ such that if*

*(1) $(X, B)$ is an $n$-dimensional log pair with* $\operatorname{coeff}(B) \in I$*,*
*(2) $(X, \Phi)$ is Kawamata log terminal for some $\Phi \geq 0$ and*
*(3) $B' \geq (1-\delta)B$ where $(X, B')$ is a log canonical pair,*

*then $(X, B)$ is log canonical.*

*Proof of Theorems 3.0.2, 3.0.3, 3.0.4, and 3.0.5.* The proof is by induction on the dimension. The case $n = 1$ is obvious. The proof subdivided into the following 4 steps.

(1) Theorems 3.0.4 and 3.0.5 in dimension $n-1$ imply Theorem 3.0.2 in dimension $n$ (cf. Theorem 3.1.1),
(2) Theorem 3.0.3 in dimension $n-1$ and Theorem 3.0.2 in dimension $n$ imply Theorem 3.0.3 in dimension $n$ (cf. Theorem 3.2.4),
(3) Theorem 3.0.4 in dimension $n-1$ and Theorem 3.0.3 in dimension $n$ imply Theorem 3.0.4 in dimension $n$ (cf. Theorem 3.3.1), and
(4) Theorems 3.0.2, 3.0.3, 3.0.4 and 3.0.5 in dimension $n-1$ imply Theorem 3.0.5 in dimension $n$ (cf. Theorem 3.4.1). □

*Proof of Theorem 3.0.1.* (3) follows from Theorem 3.0.3. To prove (1), we may fix $M > 0$ and consider pairs $(X, B)$ such that $0 < \operatorname{vol}(X, K_X + B) \leq M$. By Proposition 1.10.4 the pairs $(X, B)$ are log birationally bounded. (1) now follows from Corollary 2.1.3 and (2) is an easy consequence of (1). □

## 3.1. **Boundedness of the anticanonical volume.**

**Theorem 3.1.1.** *Theorems 3.0.4 and 3.0.5 in dimension $n-1$ imply Theorem 3.0.2 in dimension $n$.*

*Proof.* Suppose that $(X, B)$ is a pair in $\mathfrak{D}$ with $\operatorname{vol}(X, -K_X) > 0$. By Theorem 1.7.2, there exists a small proper birational morphism $\nu \colon X' \longrightarrow X$ such that $X'$ is $\mathbb{Q}$-factorial. Let

$$K_{X'} + B' = \nu^*(K_X + B) \equiv 0.$$

Then $(X', B') \in \mathfrak{D}$ and $\operatorname{vol}(X', -K_{X'}) = \operatorname{vol}(X, -K_X)$. Replacing $X$ by $X'$, we may therefore assume that $X$ is $\mathbb{Q}$-factorial. If $x \in X$ is a general point, then by a standard argument (cf. [29, 10.4.12]), there exists $G \sim_{\mathbb{R}} -K_X$ with

$$\operatorname{mult}_x(G) > \frac{1}{2}\big(\operatorname{vol}(X, -K_X)\big)^{1/n}.$$

It follows that

$$\sup\{\, t \geq 0 \mid (X, tG) \text{ is log canonical} \,\} < \frac{2n}{\big(\operatorname{vol}(X, -K_X)\big)^{1/n}}$$

(cf. [29, 9.3.2]). Therefore, we may assume that

$$(X, \Phi := (1-\delta)B + \delta G)$$

is log canonical but not Kawamata log terminal for some

$$\delta < 2n/(\operatorname{vol}(X, -K_X))^{1/n}.$$

Note that

$$K_X + \Phi \sim_{\mathbb{R}} (1-\delta)(K_X + B) \equiv 0.$$

By tie breaking (cf. Proposition 1.6.1), we may assume that $(X, \Phi)$ has a unique non Kawamata log terminal centre say $Z$ with a unique non Kawamata log terminal place say $E$. Let $\nu \colon X' \longrightarrow X$ be the corresponding divisorial extraction so that $\rho(X'/X) = 1$ (cf. Theorem 1.7.2(3)) and the exceptional locus is given by the prime divisor $E \subset X'$. Let $\Phi' = \nu_*^{-1}\Phi$ and $B' = \nu_*^{-1}B$ and write

$$K_{X'} + B' + aE = \nu^*(K_X + B), \qquad K_{X'} + \Phi' + E = \nu^*(K_X + \Phi)$$

where $a < 1$. In particular $K_{X'} + \Phi' + E$ is purely log terminal. We now run the $K_{X'} + \Phi' \equiv -E$ minimal model program $\psi \colon X' \dashrightarrow X''$ (cf. Proposition 1.7.3) until we obtain a Mori fibre space $\pi \colon X'' \longrightarrow W$. Let $E'' = \psi_* E$, $\Phi'' = \psi_*\Phi$, and $B'' = \psi_* B'$, so that $E''$ is $\pi$-ample. Note that since $K_{X'} + \Phi' + E \equiv 0$, it follows that $K_{X''} + \Phi'' + E''$ is purely log terminal. After restricting to a general fibre, we may assume that $E''$ is ample and we write

$$(K_{X''} + B'' + E'')|_{E''} = K_{E''} + B_{E''}, \qquad (K_{X''} + \Phi'' + E'')|_{E''} = K_{E''} + \Phi_{E''}.$$

Note that

(1) $\operatorname{coeff}(B_{E''}) \subset D(I)$ (by Theorem 2.2.1, since $\operatorname{coeff}(B'') \subset I$),
(2) $K_{E''} + \Phi_{E''}$ is Kawamata log terminal (since $K_{X''} + \Phi'' + E''$ is purely log terminal), and
(3) $\Phi_{E''} \geq (1-\delta)B_{E''}$ (by Lemma 2.2.2, since $\Phi'' \geq (1-\delta)B''$).

If $\operatorname{vol}(X, -K_X) \gg 0$, then $\delta \ll 1$, and so by Theorem 3.0.5 in dimension $\leq n-1$, we have that $K_{E''} + B_{E''}$ is log canonical.

Since $\Phi'' \geq (1-\delta)B''$ and $K_{X''} + \Phi'' + E'' \equiv 0$, we have

$$K_{X''} + (1-\eta)B'' + E'' \equiv_W 0 \qquad \text{for some} \qquad 0 < \eta < \delta,$$

and so

$$K_{E''} + \operatorname{Diff}_{E''}\big((1-\eta)B''\big) \equiv 0.$$

By Lemma 2.2.2,

$$\operatorname{Diff}_{E''}\big((1-\eta)B''\big) \geq (1-\eta)B_{E''}.$$

We claim that 0 is not an accumulation point for the possible values of $\eta$. If this where not the case then there would be a decreasing sequence $\eta_k > 0$ with $\lim \eta_k = 0$. But then, it is easy to see that the coefficients of $\operatorname{Diff}_{E''}((1-\eta_k)B'')$ belong to a DCC set and so we obtain a contradiction by Theorem 3.0.4 in dimension $n-1$. Since

$$\eta < \delta \leq 2n/\operatorname{vol}(X, -K_X),$$

it follows that $\operatorname{vol}(X, -K_X)$ is bounded from above. $\square$

3.2. **Birational boundedness.**

**Theorem 3.2.1.** *Assume that Theorem 3.0.3 holds in dimension $n-1$ and Theorem 3.0.2 holds in dimension $n$. Then there is a constant $\beta < 1$ such that if $(X, B)$ is an $n$-dimensional projective log canonical pair where $K_X + B$ is big and* $\operatorname{coeff}(B) \subset I$*, then the pseudo-effective threshold satisfies*

$$\lambda := \inf\{\, t \in \mathbb{R} \,|\, K_X + tB \textit{ is big}\,\} \leq \beta.$$

*Proof.* Suppose that we have a sequence of pairs $(X_i, B_i)$ with increasing pseudo-effective thresholds $\lambda_i < \lambda_{i+1}$ such that $\lim \lambda_i = 1$. In particular we may assume that $1 > \lambda_i \geq 1/2$.

**Claim 3.2.2.** *We may assume that there is a sequence of $\mathbb{Q}$-factorial projective Kawamata log terminal pairs $(Y_i, \Gamma_i)$ such that* $\operatorname{coeff}(\Gamma_i) \subset I$*, $-K_{Y_i}$ is ample, $K_{Y_i} + \lambda_i \Gamma_i \equiv 0$ and* $\dim Y_i \leq n$*.*

*Proof.* We may assume that $1 \in I$. As a first step, we will show that we may assume that $(X, B) = (X_i, B_i)$ is log smooth. Let $\nu \colon X' \longrightarrow X$ be a log resolution of $(X, B)$ and write

$$K_{X'} + B' = \nu^*(K_X + B) + E$$

where $B' = \nu^{-1}_* B + \operatorname{Exc}(\nu)$. Note that $K_{X'} + B'$ is big and if $K_{X'} + tB'$ is big, then so is $K_X + tB = \nu_*(K_{X'} + tB')$. Thus

$$1 > \lambda' := \inf\{\, t \in \mathbb{R} \,|\, K_{X'} + tB' \text{ is big}\,\} \geq \lambda$$

and we may replace $(X, B)$ by $(X', B')$. Therefore we may assume that $(X, B)$ is log smooth.

Since $K_X + B$ is big, we may pick an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} K_X + B$ and so for $0 < \epsilon \ll 1$ we have

$$(1+\epsilon)(K_X + \lambda B) \sim_{\mathbb{Q}} K_X + \mu B + \epsilon D$$

where $0 < \mu := \lambda(1+\epsilon) - \epsilon < \lambda$ and $K_X + \mu B + \epsilon D$ is Kawamata log terminal. Since $\mu B + \epsilon D$ is big, by [3] we may run the $K_X + \mu B + \epsilon D$ minimal model program say $f\colon X \dashrightarrow X'$. Since this is also a $K_X + \lambda B$ minimal model program, we may assume that $K_{X'} + \lambda B'$ is nef and Kawamata log terminal where $B' = f_* B$ and $D' = f_* D$.

We may now run a $K_{X'} + \mu B'$ minimal model program. By Proposition 1.7.3 after finitely many $K_{X'} + \mu B' + \epsilon D'$ flops $g\colon X' \dashrightarrow X''$ we obtain a $K_{X''} + \mu B'' + \epsilon D''$-trivial contraction of fibre type $X'' \longrightarrow Z$ (where $B'' = g_* B'$ and $D'' = g_* D'$) such that $\epsilon D''$ is ample over $Z$. Therefore $-K_{X''}$ is ample over $Z$ (since $B'' \geq 0$ and $\rho(X''/Z) = 1$). It follows that $K_{X''} + \lambda B''$ is Kawamata log terminal and $K_{X''} + \lambda B'' \equiv_Z 0$. Letting $(Y, \Gamma) = (F, B''|_F)$ where $F$ is a general fibre of $X'' \longrightarrow Z$, the claim follows. □

Let $\nu_i\colon Y_i' \longrightarrow Y_i$ be a log resolution of $(Y_i, \Gamma_i)$, $D_i = (\Gamma_i)_{\mathrm{red}}$ and $\Gamma_i'$ (resp. $D_i'$) be the strict transform of $\Gamma_i$ (resp. $D_i$) plus the $\nu_i$-exceptional divisors. Since $(Y_i, \lambda_i \Gamma_i)$ is klt, then for any $0 < \delta \ll 1$,

$$K_{Y_i'} + \Gamma_i' \geq \nu_i^*(K_{Y_i} + (\lambda_i + \delta)\Gamma_i) \equiv \delta \nu_i^* \Gamma_i$$

and so both $K_{Y_i'} + \Gamma_i'$ and $K_{Y_i'} + D_i'$ are big. By Theorem 3.0.2 in dimension $n$ there exists a constant $C$ such that $\mathrm{vol}(Y_i, \lambda_i \Gamma_i) < C$. Since $I$ satisfies the DCC, there exists a smallest non-zero element $\alpha \in I$.

**Claim 3.2.3.** *The pairs $(Y_i', D_i')$ are log birationally bounded.*

*Proof.* Since $K_{Y_i'} + D_i'$ is big, then so is $K_{Y_i'} + \frac{r-1}{r} D_i'$ for any $r \gg 0$. If $\Theta_i := \frac{r-1}{r} D_i'$ then

$$\begin{aligned}
\mathrm{vol}(Y_i', K_{Y_i'} + \Theta_i) &\leq \mathrm{vol}(Y_i', K_{Y_i'} + D_i') \\
&\leq \mathrm{vol}(Y_i, K_{Y_i} + D_i) \\
&= \mathrm{vol}(Y_i, D_i - \lambda_i \Gamma_i) \\
&\leq \mathrm{vol}(Y_i, D_i) \\
&\leq \mathrm{vol}(Y_i, \frac{1}{\alpha}\Gamma_i) \\
&\leq \frac{C}{(\lambda_i \alpha)^{n'}} \\
&\leq C \left(\frac{2}{\alpha}\right)^{n'}
\end{aligned}$$

where $n' = \dim Y_i$. Since

$$\mathrm{coeff}(\Theta_i) \subset \{\, 1 - \frac{1}{r} \,|\, r \in \mathbb{N} \,\}$$

then by Theorem 2.3.1, it follows that there exists a constant $m > 0$ such that $\phi_{m(K_{Y_i'}+\Theta_i)}$ is birational. By Proposition 1.10.4, the pairs $(Y_i', \Theta_i)$ are log birationally bounded and hence so are the pairs $(Y_i', D_i')$. $\square$

By Corollary 2.1.3, it follows that there exists a constant $\delta > 0$ such that

$$\operatorname{vol}(Y_i', K_{Y_i'} + \Gamma_i') \geq \delta.$$

But then we have

$$\begin{aligned}
\delta &\leq \operatorname{vol}(Y_i', K_{Y_i'} + \Gamma_i') \\
&\leq \operatorname{vol}(Y_i, K_{Y_i} + \Gamma_i) \\
&= \operatorname{vol}(Y_i, \frac{1-\lambda_i}{\lambda_i}\lambda_i\Gamma_i) \\
&= (\frac{1-\lambda_i}{\lambda_i})^{n'} \operatorname{vol}(Y_i, \lambda_i\Gamma_i) \\
&\leq (\frac{1-\lambda_i}{\lambda_i})^{n'} C.
\end{aligned}$$

Thus, if

$$\beta = \frac{1}{1 + (\frac{\delta}{C})^{1/n'}} < 1,$$

then $\lambda_i \leq \beta$. $\square$

**Theorem 3.2.4.** *Theorem 3.0.3 in dimension $n-1$ and Theorem 3.0.2 in dimension $n$ imply Theorem 3.0.3 in dimension $n$.*

*Proof.* By Theorem 3.2.1 there exists a constant $\gamma < 1$, such that $K_X + \gamma\Delta$ is big. Fix a positive integer $q$, such that $(1-\gamma)\delta > \frac{1}{q}$, where $\delta = \min(I \cap (0, 1])$ and let $I_0 = \{\frac{1}{q}, \frac{2}{q}, \dots, \frac{q-1}{q}, 1\}$. It is easy to see that there exists a $\mathbb{Q}$-divisor $\Delta_0$ such that

$$\gamma\Delta \leq \Delta_0 \leq \Delta \qquad \text{and} \qquad \operatorname{coeff}(\Delta_0) \subset I_0.$$

By Theorem 2.3.1, there exists a constant $m \in \mathbb{N}$ such that $\phi_{m(K_X+\Delta_0)}$ is birational. Since $\Delta_0 \leq \Delta$, $\phi_{m(K_X+\Delta)}$ is also birational. $\square$

## 3.3. **ACC for numerically trivial pairs.**

**Theorem 3.3.1.** *Theorem 3.0.4 in dimension $n-1$ and Theorem 3.0.3 in dimension $n$ implies Theorem 3.0.4 in dimension $n$.*

*Proof.* Let $J_0$ be the finite subset given by applying Theorem 3.0.4 in dimension $\leq n-1$ for $J := D(I)$ and $I_1 \subset I$ the finite subset defined in Lemma 1.8.1.

Let $(X, B)$ be an $n$-dimensional projective log canonical pair such that $K_X + B \equiv 0$ and $\operatorname{coeff}(B) \subset I$. By Theorem 1.7.2 we may assume that $(X, B)$ is dlt and in particular $(X, B)$ is klt if and only if $\lfloor B \rfloor = 0$. Let $B = \sum b_i B_i$ where $b_i \in I$. If $B_i$ intersects a component $S$ of $\lfloor B \rfloor$, then let $K_S + \Theta = (K_X + B)|_S$. Note that the coefficients of $\Theta$ belong to the DCC set $J = D(I)$ (cf. Theorem 2.2.1). Since $(S, \Theta)$ is log canonical and $K_S + \Theta \equiv 0$, by Theorem 3.0.4 in dimension $\leq n - 1$, it follows that $\operatorname{coeff}(\Theta) \subset J_0$. If $P$ is an irreducible component of $\operatorname{Supp}(B_i)|_S$, then

$$\operatorname{mult}_P(\Theta) = \frac{m - 1 + f + kb_i}{m} \qquad \text{for some } f \in J \text{ and } m,\, k \in \mathbb{N}.$$

By Lemma 1.8.1, $b_i$ belongs to the finite subset $I_1 \subset I$.

We may therefore assume that if $b_i \notin I_1$, then $B_i \cap \lfloor B \rfloor = 0$. Pick one such component $B_i$ and run the $K_X + B - b_i B_i$ minimal model program with scaling of an ample divisor. Since

$$K_X + B - b_i B_i \equiv -b_i B_i,$$

every step of this minimal model program is $B_i$ positive and hence does not contract $B_i$. Since $K_X + B - b_i B_i \equiv -b_i B_i$ is not pseudo effective, after finitely many steps we obtain a Mori fibre space

$$X \dashrightarrow X' \longrightarrow Z.$$

If at any point we contract a component $S$ of $\lfloor B \rfloor$, then the strict transforms of $B_i$ and $S$ must intersect and so $b_i \in I_1$ contradicting our assumptions. In particular no components of $\lfloor B \rfloor$ are contracted. If $\dim Z > 0$, then replacing $X$ by a general fibre of $X' \longrightarrow Z$, we see that $b_i$ belongs to $J_0$ (by Theorem 3.0.4 in dimension $\leq n - 1$). Therefore, we may assume that $\dim Z = 0$ and so $\rho(X') = 1$ so that every component of the strict transform of $\lfloor B \rfloor$ intersects the strict transform of $B_i$. Arguing as above, if $\lfloor B \rfloor \neq 0$, it follows that $b_i \in I_1$ which is a contradiction. Therefore we may assume that $\lfloor B \rfloor = 0$, that is, $(X, B)$ is Kawamata log terminal. Replacing $(X, B)$ by $(X', B')$ we may also assume that that $\rho(X) = 1$.

Let $m = m(n, I) > 0$ be the constant whose existence is guaranteed by Theorem 3.0.3 in dimension $n$, so that if $(X, C)$ is a projective $n$-dimensional log canonical pair such that $K_X + C$ is big and $\operatorname{coeff}(C) \subset I$, then $\phi_{m(K_X+C)}$ is birational. It suffices to show that $I \cap [(l-1)/m, l/m)$ contains at most one element (for any integer $1 \leq l \leq m$). Suppose to the contrary that $I \cap [(l-1)/m, l/m)$ contains two elements say $i_1 < i_2$. We may assume that there is a Kawamata log terminal pair $(X, B)$ as above such that $B = \sum b_j B_j$ where

$b_1 = i_1$. Let $\nu\colon X' \longrightarrow X$ be a log resolution and consider the pair $(X', B' := \nu_*^{-1}(B + (i_2 - i_1)B_1) + \operatorname{Exc}(\nu))$. Since

$$K_{X'} + B' = \nu^*(K_X + B) + (i_2 - i_1)\nu_*^{-1}B_1 + F$$

where $F \geq 0$ and its support contains $\operatorname{Exc}(\nu)$, it follows that $K_{X'}+B'$ is big and the coefficients of $B'$ are in $I$. So by Theorem 3.0.3, $\phi_{m(K_{X'}+B')}$ is birational. In particular

$$\lfloor m(K_X + \nu_*B')\rfloor = m(K_X + \lfloor m\nu_*B'\rfloor/m)$$

is big. Since

$$(l-1)/m \leq i_1 < i_2 < l/m,$$

it follows that $\lfloor mi_2\rfloor = l - 1$ and so $B \geq \lfloor m\nu_*B'\rfloor/m$. But since $K_X + B \equiv 0$ this contradicts the bigness of $K_X + \lfloor m\nu_*B'\rfloor/m$. $\square$

### 3.4. **ACC for the log canonical threshold.**

**Theorem 3.4.1.** *Theorems 3.0.2, 3.0.3, and 3.0.5 in dimension $n-1$ imply Theorem 3.0.5 in dimension $n$.*

*Proof.* Since $(X, \Phi)$ is klt, Theorem 1.7.2 implies that there exists a small birational morphism $\nu\colon X' \longrightarrow X$ such that $X'$ is $\mathbb{Q}$-factorial. Since $K_{X'} + \nu_*^{-1}B = \nu^*(K_X + B)$, it follows that $(X', \nu_*^{-1}B)$ is log canonical if and only if $(X, B)$ is log canonical. Replacing $X$, $\Phi$, and $B$ by $X'$, $\nu_*^{-1}\Phi$ and $\nu_*^{-1}B$, we may assume that $X$ is $\mathbb{Q}$-factorial.

Let $\lambda$ be the log canonical threshold of $(X, B)$ so that $(X, \lambda B)$ is log canonical but not Kawamata log terminal. As we are assuming Theorem 3.0.2 and Theorem 3.0.3 in dimension $n-1$, we know that Theorem 3.2.1 holds in dimension $n-1$. Let $\beta < 1$ be the constant defined by Theorem 3.2.1 in dimension $n-1$ (where we take $D(I)$ to be the coefficient set). Let $\mu < 1$ be the constant defined by Theorem 3.0.5 in dimension $n-1$. It suffices to show that if $\mu < \lambda < 1$, then $\lambda \leq \beta$.

If $\lambda B$ has a component of coefficient 1, then as $\operatorname{coeff}(B) \subset I \subset [0,1]$, it follows that $\lambda = 1$ and hence $(X, B)$ is log canonical. We may therefore assume that all non Kawamata log terminal centres of $(X, \lambda B)$ have codimension $\geq 2$. Since $(X, \Phi)$ is klt, by tie breaking (cf. Proposition 1.6.1), there exists a non Kawamata log terminal place $E$ of $(X, \lambda B)$ and a log canonical pair $(X, \Psi)$ such that $E$ is the unique non-Kawamata log terminal place of $(X, \Psi)$. By Theorem 1.7.2, there exists a projective birational morphism $\nu\colon X' \longrightarrow X$ such that $\rho(X'/X) = 1$, $\operatorname{Exc}(\nu) = E$ is an irreducible divisor and

$$K_{X'} + \lambda\nu_*^{-1}B + E = \nu^*(K_X + \lambda B)$$

is log canonical so that

$$K_E + \mathrm{Diff}_E(\lambda\nu_*^{-1}B) = (K_{X'} + \lambda\nu_*^{-1}B + E)|_E$$

is log canonical. Note that

$$K_{X'} + \nu_*^{-1}\Psi + E = \nu^*(K_X + \Psi)$$

is plt and hence $K_E + \Psi_E = (K_{X'} + \nu_*^{-1}\Psi + E)|_E$ is klt. Since $\lambda \leq 1$, then by Lemma 2.2.2,

$$\mathrm{Diff}_E(\lambda\nu_*^{-1}B) \geq \lambda\,\mathrm{Diff}_E(\nu_*^{-1}B).$$

As $\lambda > \mu$, Theorem 3.0.5 in dimension $n-1$ implies that $K_E + \mathrm{Diff}_E(\nu_*^{-1}B)$ is log canonical. Let $H$ be a general sufficiently ample divisor on $X$, since

$$K_E + \mathrm{Diff}_E(\nu_*^{-1}B) \sim_{\mathbb{Q},X} \mathrm{Diff}_E(\nu_*^{-1}B) - \mathrm{Diff}_E(\lambda\nu_*^{-1}B)$$

is ample over $X$, then

$$K_E + \mathrm{Diff}_E(\nu_*^{-1}B) + \nu^*H|_E$$

is ample and so by Theorem 3.2.1 in dimension $\leq n-1$, $K_E + t\,\mathrm{Diff}_E(B) + \nu^*H|_E$ is big for any $t > \beta$. In particular since

$$K_E + \lambda\,\mathrm{Diff}_E(\nu_*^{-1}B) \leq K_E + \mathrm{Diff}_E(\lambda\nu_*^{-1}B) \equiv_X 0$$

it follows that $\lambda \leq \beta$. □

We have the following corollary.

**Theorem 3.4.2** (The ACC for the log canonical threshold)**.** *Fix $n \in \mathbb{N}$ and sets $I \subset [0,1]$, $J \subset (0,+\infty)$ which satisfy the DCC. Let $\mathcal{L}$ be the set of log canonical thresholds of pairs $(X,B)$ with respect to an $\mathbb{R}$-Cartier divisor $D$ such that*

*(1) $(X,B)$ is an $n$-dimensional log canonical pair,*
*(2) $\mathrm{coeff}(B) \subset I$, and*
*(3) $\mathrm{coeff}(D) \subset J$.*

*Then $\mathcal{L}$ satisfies the ACC.*

*Proof.* Replacing $X$ by a $\mathbb{Q}$-factorial modification (cf. Theorem 1.7.2), we may assume that $X$ is $\mathbb{Q}$-factorial. Suppose that there is a sequence of triples $(X_i, B_i, D_i)$ as above such that $\lambda_i = \mathrm{lct}(X_i, B_i; D_i)$ is non-decreasing. If $\lambda = \lim \lambda_i$ and $K = I + \lambda J$ then $K$ satisfies the DCC and $(X_i, B_i + \lambda_i D_i)$ is log canonical but not Kawamata log terminal for all $i = 1, 2, \ldots$. We claim that all but finitely many of the coefficients of $B_i + \lambda D_i$ belong to $[0,1]$. If this were not the case, then consider a subsequence such that $\mathrm{mult}_{P_i}(B_i + \lambda D_i) > 1$. We may assume that

$\lambda_i \geq \lambda/2 > 0$. Since $\operatorname{mult}_{P_i}(\lambda_i D_i) \leq 1$, it follows that $\operatorname{mult}_{P_i}(D_i) \leq 1/\lambda_i \leq 2/\lambda$. But then

$$\begin{aligned} 1 &\leq \operatorname{mult}_{P_i}(B_i + \lambda D_i) \\ &= \operatorname{mult}_{P_i}(B_i + \lambda_i D_i) + (\lambda - \lambda_i) \operatorname{mult}_{P_i}(D_i) \\ &\leq 1 + \frac{2(\lambda - \lambda_i)}{\lambda}. \end{aligned}$$

Since $\operatorname{mult}_{P_i}(B_i + \lambda D_i)$ belongs to the DCC set $K$ and since $\lim \frac{2(\lambda-\lambda_i)}{\lambda} = 0$, this is a contradiction. Therefore we may assume that the coefficients of $B_i + \lambda D_i$ belong to the DCC set $K \cap [0, 1]$.

Note that for any $\delta > 0$,

$$(1 - \delta)(B_i + \lambda D_i) \leq (B_i + \lambda_i D_i)$$

for all $i \gg 0$ and hence $(X_i, (1 - \delta)(B_i + \lambda D_i))$ is log canonical. By Theorem 3.0.5, $(X_i, B_i + \lambda D_i)$ is also log canonical. But then $\lambda = \lambda_i$ and hence the sequence $\lambda_i$ is eventually constant as required. □

## 4. Boundedness

**Proposition 4.1.** *Fix $w \in \mathbb{R}_{>0}$, $n \in \mathbb{N}$ and a set $I \subset [0, 1]$ which satisfies the DCC. Let $(Z, D)$ be a projective log smooth $n$-dimensional pair where $D$ is reduced and $\mathbf{M}_D$ the b-divisor corresponding to the strict transform of $D$ plus the exceptional divisors. Then there exists $f\colon Z' \longrightarrow Z$, a finite sequence of blow ups along strata of $\mathbf{M}_D$, such that if*

*(1) $(X, B)$ is a projective log smooth $n$-dimensional pair*
*(2) $g\colon X \longrightarrow Z$ is a finite sequence of blow ups along strata of $\mathbf{M}_D$,*
*(3) $\operatorname{coeff}(B) \subset I$,*
*(4) $g_*B \leq D$, and*
*(5) $\operatorname{vol}(X, K_X + B) = w$,*

*then $\operatorname{vol}(Z, K_{Z'} + \mathbf{M}_{B,Z'}) = w$ where $\mathbf{M}_{B,Z'}$ is the strict transform of $B$ plus the $Z' \dashrightarrow X$ exceptional divisors.*

*Proof.* We may assume that $1 \in I$. Let

$$V = \{ \operatorname{vol}(Y, K_Y + \Gamma) \mid (Y, \Gamma) \in \mathfrak{D} \}$$

where $\mathfrak{D}$ is the set of all $n$-dimensional projective log smooth pairs such that $K_Y + \Gamma$ is big, $\operatorname{coeff}(\Gamma) \subset I$, $g\colon Y \longrightarrow Z$ is a birational morphism and $g_*\Gamma \leq D$. By Theorem 2.1.2, $V$ satisfies the DCC. Therefore, there is a constant $\delta > 0$ such that if $\operatorname{vol}(Y, K_Y + \Gamma) \leq w + \delta$, then $\operatorname{vol}(Y, K_Y + \Gamma) = w$. Notice also that by Theorem 3.2.1 there exists an integer $r > 0$ such that if $(Y, \Gamma) \in \mathfrak{D}$, then $K_Y + \frac{r-1}{r}\Gamma$ is big. We now fix $\epsilon > 0$ such that

$$(1-\epsilon)^n > \frac{w}{w+\delta}, \qquad \text{and let} \qquad a = 1 - \frac{\epsilon}{r}.$$

Since

$$K_Y + a\Gamma = (1-\epsilon)(K_Y + \Gamma) + \epsilon(K_Y + \frac{r-1}{r}\Gamma)$$

it follows that

$$\begin{aligned}\operatorname{vol}(Y, K_Y + a\Gamma) &\geq \operatorname{vol}((1-\epsilon)(Y, K_Y + \Gamma)) \\ &= (1-\epsilon)^n \operatorname{vol}(Y, K_Y + \Gamma) \\ &> \frac{w}{w+\delta} \operatorname{vol}(Y, K_Y + \Gamma).\end{aligned}$$

Since $(Z, aD)$ is Kawamata log terminal, there is a sequence of blow ups $f\colon Z' \longrightarrow Z$ of the strata with the following property: if $K_{Z'} + \Psi = f^*(K_Z + aD) + E$ where $\Psi \wedge E = 0$, then $(Z', \Psi)$ is terminal. Let $\mathfrak{F}$ be the set of pairs $(X, B)$ satisfying properties (1-5) above such that $\phi\colon X \dashrightarrow Z'$ is a morphism. If $(X, B) \in \mathfrak{F}$ and $B_{Z'} = \phi_* B$, then $f_*(aB_{Z'}) \leq aD$ so that if

$$K_{Z'} + \Phi = f^*(K_Z + f_*(aB_{Z'})) + F$$

where $\Phi \wedge F = 0$, then $(Z, \Phi)$ is terminal. We then have

$$\begin{aligned}\operatorname{vol}(Z', K_{Z'} + aB_{Z'}) &= \operatorname{vol}(Z', K_{Z'} + aB_{Z'} \wedge \Phi) \\ &= \operatorname{vol}(X, K_X + \phi^{-1}_*(aB_{Z'} \wedge \Phi)) \\ &\leq \operatorname{vol}(X, K_X + B),\end{aligned}$$

where the first line follows from Lemma 1.5.1(3), the second since $(Z', aB_{Z'} \wedge \Phi)$ is terminal and the third since $\phi^{-1}_*(aB_{Z'} \wedge \Phi) \leq B$. But then

$$w \leq \operatorname{vol}(Z', K_{Z'} + B_{Z'}) \leq \frac{w+\delta}{w} \operatorname{vol}(Z', K_{Z'} + aB_{Z'}) \leq w + \delta.$$

By what we observed above, we then have $\operatorname{vol}(Z', K_{Z'} + B_{Z'}) = w$ as required.

To conclude the proof, it suffices to observe that if $(X, B)$ is a pair satisfying properties (1-5) above, then after blowing up $X$ along finitely many strata of $\mathbf{M}_D$ and replacing $B$ by its strict transform plus the exceptional divisors, we may assume that $X \dashrightarrow Z'$ is a morphism and hence that $(X, B) \in \mathfrak{F}$. $\square$

**Proposition 4.2.** *Fix $n \in \mathbb{N}$, $d > 0$ and a set $I \subset [0,1] \cap \mathbb{Q}$ which satisfies the DCC. Let $\mathfrak{F}_{\mathrm{lc}}(n, d, I)$ be the set of pairs $(X, B)$ which are* *the disjoint union of log canonical models $(X_i, B_i)$ where* $\dim X_i = n$, $\mathrm{coeff}(B_i) \subset I$ *and* $(K_X + B)^n = d$*. Then* $\mathfrak{F}_{\mathrm{lc}}(n, d, I)$ *is bounded.*

*Proof.* Since $d = \sum d_i$ where $d_i = (K_{X_i} + B_i)^n$ and by Theorem 3.0.1 the $d_i$ belong to a DCC set, it follows easily that there are only finitely many possibilities for the $d_i$. We may therefore assume that $X$ is irreducible. It suffices to show that there is an integer $N > 0$ such that if $(X, B)$ is an $n$-dimensional log canonical model with $\mathrm{coeff}(B) \subset I$ and $(K_X + B)^n = d$, then $N(K_X + B)$ is very ample. Suppose that this is not the case and let $(X_i, B_i)$ be a sequence of $n$-dimensional log canonical models with $\mathrm{coeff}(B_i) \subset I$ and $(K_{X_i} + B_i)^n = d$ such that $i!(K_{X_i} + B_i)$ is not very ample for all $i > 0$.

By Theorem 3.0.1 and Proposition 1.10.4, the set of such pairs $(X_i, B_i)$ is log birationally bounded. Therefore there is a projective morphism $\pi\colon Z \longrightarrow T$ and a log pair $(Z, D)$ which is log smooth over a variety $T$, such that for any pair $(X_i, B_i)$ as above, there is a closed point $t_i \in T_i$ and a birational map $f_i\colon Z_{t_i} \dashrightarrow X_i$ such that the support of $D_{t_i}$ contains the strict transform of $B_i$ plus the $f_i$ exceptional divisors. Passing to a subsequence, we may assume that the $t_i$ belong to a fixed irreducible component of $T$. We may therefore assume that $T$ is irreducible and the components of $D$ are geometrically irreducible over $T$.

Applying Proposition 4.1 to $(Z_{t_1}, D_{t_1})$, we obtain a model $Z'_{t_1} \longrightarrow Z_{t_1}$ and $Z' \longrightarrow Z$ the morphism obtained by blowing up the corresponding strata of $\mathbf{M}_D$.

Denote by $\Phi_{t_i} = ({f'}_i^{-1})_* B_i + \mathrm{Exc}(f'_i) \leq D'_{t_i}$, where $f'_i\colon Z'_{t_i} \dashrightarrow X_i$ is the induced birational map. Passing to a subsequence, we may also assume that for any irreducible component $P$ of the support of $D' := \mathbf{M}_{D,Z'}$, the coefficients of $\Phi_{t_i}$ along $P_{t_i}$ are non-decreasing. Let $\Phi^i$ be the divisor with support contained in $D'$ such that $\Phi^i|_{Z'_{t_i}} = \Phi_{t_i}$.

We claim that for any pair $(X_i, B_i)$ as above

$$\mathrm{vol}(Z'_{t_i}, K_{Z'_{t_i}} + \Phi_{t_i}) = d.$$

To see this, by the proof of Corollary of 2.1.3, we can construct $Z'' \longrightarrow Z'$ by a sequence of blow ups along strata of $\mathbf{M}_D$ such that $Z''_{t_i} \dashrightarrow X_i$ is a rational map and we have

$$\mathrm{vol}(Z''_{t_i}, K_{Z''_{t_i}} + \Psi_{t_i}) = d$$

where $\Psi_{t_i}$ is the strict transform of $B_i$ plus the $Z''_{t_i}/X_i$ exceptional divisors. If $\Psi$ is the divisor supported on $\mathrm{Supp}(\mathbf{M}_{D,Z''})$ such that

$\Psi|_{Z''_{t_i}} = \Psi_{t_i}$, then

$$\begin{aligned} d &= \operatorname{vol}(Z''_{t_i}, K_{Z''_{t_i}} + \Psi_{t_i}) \\ &= \operatorname{vol}(Z''_{t_1}, K_{Z''_{t_1}} + \Psi|_{Z''_{t_1}}) \\ &= \operatorname{vol}(Z'_{t_1}, K_{Z'_{t_1}} + \Phi^i|_{Z'_{t_1}}) \\ &= \operatorname{vol}(Z'_{t_i}, K_{Z'_{t_i}} + \Phi_{t_i}), \end{aligned}$$

where the second and fourth equalities follow from Theorem 1.9.2 and the third one follows from Proposition 4.1.

By Theorem 1.9.1, we may assume $(Z', \Phi^1)$ has a relative log canonical model $\psi\colon Z' \dashrightarrow W$ over $T$, which fibere by fibre $\psi_{t_i}\colon Z'_{t_i} \dashrightarrow W_{t_i}$ gives the log canonical model for $(Z'_{t_i}, \Phi^1_{t_i})$ for all $i \geq 1$. Notice that by Theorem 1.9.2,

$$d = \operatorname{vol}(Z'_{t_k}, K_{Z'_{t_k}} + \Phi^k_{t_k}) = \operatorname{vol}(Z'_{t_1}, K_{Z'_{t_1}} + \Phi^k_{t_1})$$

for all $k > 0$. Since we have assumed that

$$\Phi^1 \leq \Phi^2 \leq \Phi^3 \leq \dots,$$

it follows by Lemma 4.3 that $\psi_{t_i}\colon Z'_{t_i} \dashrightarrow W_{t_i}$ is also a log canonical model of $(Z'_{t_i}, \Phi^k_{t_i})$ for all $k \geq 1$, $\psi_{t_i *}\Phi^k_{t_i} = \psi_{t_i *}\Phi^1_{t_i}$ and there is an isomorphism $\alpha_i\colon W_{t_i} \cong X_i$.

There is an integer $N > 0$ such that $N(K_W + \psi_*\Phi^1)$ is very ample over $T$ and so $N(K_{W_{t_i}} + \psi_{t_i *}\Phi^1_{t_i})$ is very ample for all $i > 0$. Since

$$\psi_{t_i *}\Phi^1_{t_i} = \psi_{t_i *}\Phi^i_{t_i} = \alpha_i^*(B_i),$$

it follows that $N(K_{X_i} + B_i)$ is very ample for all $i > 0$ which is the required contradiction. □

**Lemma 4.3.** *Let $(X, B)$ be a log canonical pair such that $K_X + B$ is big and $f\colon X \dashrightarrow W$ the log canonical model of $(X, B)$. If $B' \geq B$, $(X, B')$ is log canonical and* $\operatorname{vol}(X, K_X + B) = \operatorname{vol}(X, K_X + B')$, *then $f$ is also the log canonical model of $(X, B')$.*

*Proof.* Replacing $X$ by an appropriate resolution, we may assume that $f\colon X \longrightarrow W$ is a morphism. Let $A = f_*(K_X + B)$, then $A$ is ample and $F := K_X + B - f^*A$ is effective and $f$-exceptional. We have for any $t \geq 0$

$$\begin{aligned} \operatorname{vol}(X, K_X + B) &= \operatorname{vol}(X, K_X + B + t(B' - B)) \\ &\geq \operatorname{vol}(X, f^*A + t(B' - B)) \\ &\geq \operatorname{vol}(X, f^*A) \\ &= \operatorname{vol}(X, K_X + B). \end{aligned}$$

But then

$$\mathrm{vol}(X, f^*A + t(B' - B)) = \mathrm{vol}(X, f^*A) \qquad \forall t \in [0,1]$$

is a constant function. If $E$ is a component of $B' - B$ then by [30] we have

$$\begin{aligned} 0 &= \frac{d}{dt}\,\mathrm{vol}(X, f^*A + tE)|_{t=0} \\ &= n \cdot \mathrm{vol}_{X|E}(H) \\ &\geq n \cdot E \cdot f^*A^{n-1} \\ &= n \cdot \deg f_*E. \end{aligned}$$

Therefore $E$ is $f$-exceptional and so

$$\begin{aligned} H^0(X, \mathcal{O}_X(m(K_X + B'))) &= H^0(X, \mathcal{O}_X(mf^*A + m(E + F))) \\ &= H^0(X, \mathcal{O}_X(mf^*A)) \\ &= H^0(X, \mathcal{O}_X(m(K_X + B))) \end{aligned}$$

and thus $f$ is the log canonical model of $(X, B')$. □

*Proof of 1.2.1.* Let $(X, B) \in \mathfrak{F}_{\mathrm{slc}}(n, I, d)$ and $X^\nu \longrightarrow X$ be its normalisation. By Proposition 4.2, if we write

$$X^\nu = \coprod X_i \qquad \text{and} \qquad (K_X + B)|_{X_i} = K_{X_i} + B_i,$$

then the pairs $(X_i, B_i)$ are bounded. In particular, there exists a finite set of rational numbers $I_0 \subset I$ such that

$$\mathrm{coeff}(B_i) \subset I_0 \qquad \text{and} \qquad (K_{X_i} + B_i)^n = d_i \in \mathfrak{D}.$$

By [24] and [26, 5.3], the slc models $(X, B)$ are in one to one correspondence with pairs $(X^\nu, B^\nu)$ and involutions $\tau\colon S^\nu \longrightarrow S^\nu$ of the normalisation of a divisor $S \subset \lfloor B^\nu \rfloor$ (the divisor $S$ corresponds to the double locus of $X^\nu \longrightarrow X$) such that $\tau$ sends the different $\mathrm{Diff}_{S^\nu}(B^\nu)$ to itself. Since $\tau$ is an involution that fixes the ample $\mathbb{Q}$-divisor $(K_{X^\nu} + B^\nu)|_S$, it follows that $\tau$ belongs to an algebraic group. Since fixing the different $\mathrm{Diff}_{S^\nu}(B^\nu)$ is a closed condition the set of possible involutions $\tau$ corresponds to a closed subset of this algebraic group and so $\tau$ is bounded. Therefore the quadruples $(X, B, S, \tau)$ are bounded. □

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