ON SHOKUROV'S RATIONAL CONNECTEDNESS CONJECTURE

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Abstract. We prove a conjecture of V. V. Shokurov which in particular implies that the fibers of a resolution of a variety with divisorial log terminal singularities are rationally chain connected.

1. Introduction

In recent years it has become increasingly clear that the geometry of higher dimensional varieties is closely related to the geometry of rational curves on these varieties. From the point of view of the minimal model program, one expects that rational curves on varieties with mild singularities (e.g. log terminal singularities) share many of the basic properties of rational curves on smooth varieties. Surprisingly, very little seems to be known in this direction. The purpose of this paper is to give an affirmative answer to several natural questions that arise in this context. For example we show that:

If (X, Δ) is a divisorially log terminal pair and $f : Y \longrightarrow X$ is a birational morphism, then for any $x \in X$, $f^{-1}(x)$ is rationally chain connected and in particular covered by rational curves.

Two immediate consequences of this are:

(1) *If (X, Δ) is a divisorially log terminal pair, and $g : X \dashrightarrow Z$ is a rational map to a proper variety which is not everywhere defined, then Z contains a rational curve.*
(2) *If (X, Δ) is a divisorially log terminal pair, then X is rationally chain connected if and only if it is rationally connected.*

We now turn to a more detailed discussion of the results of this paper.

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Definition 1.1. *Let X be a reduced, separated scheme of finite type over an algebraically closed field (so that every irreducible component of X is a variety) and let V be any subset. We will say that a curve C is a* ***chain modulo*** *V if C union a subset of V is connected. We will say that X* ***is rationally chain connected modulo*** *V, if any two points x and y, which belong to the same connected component of X, belong to a chain of rational curves modulo V.*

Note that if V is empty and X is irreducible, then X is rationally chain connected in the usual sense. Note also that the disjoint union of two copies of $\mathbb{P}^1$ is rationally chain connected but not connected. Recall that the locus where a pair (X, Δ) is not kawamata log terminal is called the locus of log canonical singularities. Here is the main result of this paper:

Theorem 1.2. *Let (X, Δ) be a log pair and let $f : X \longrightarrow S$ be a proper morphism such that $-K_X$ is relatively big and $\mathcal{O}_X(-m(K_X + \Delta))$ is relatively generated, for some $m > 0$. Let $g : Y \longrightarrow X$ be any birational morphism and let $\pi : Y \longrightarrow S$ be the composite morphism.*

Then every fibre of π is rationally chain connected modulo the inverse image of the locus of log canonical singularities.

We note that we can weaken the hypothesis if we assume that $K_X+\Delta$ is kawamata log terminal, see (7.1). This result has some interesting consequences. We start with a very general result about the fundamental group of a log pair:

Corollary 1.3. *Let (X, Δ) be a proper log pair such that $-(K_X + \Delta)$ is semiample and $-(K_X + \Delta)$ is big.*

Then the fundamental group of X is a quotient of the fundamental group of the locus of log canonical singularities of the pair (X, Δ).

Some other interesting consequences of (1.2) arise when we eliminate the locus of log canonical singularities from the statement:

Corollary 1.4. *Let (X, Δ) be a kawamata log terminal pair. Let $f : X \longrightarrow S$ be a proper morphism such that $-(K_X + \Delta)$ is relatively nef and $-K_X$ is relatively big.*

Then every fibre of f is rationally chain connected.

With a little more work, we can also prove:

Corollary 1.5. *Let (X, Δ) be a log canonical pair. Let $f : X \longrightarrow S$ be a projective morphism such that $-(K_X + \Delta)$ is relatively ample.*

Then every fibre of f is rationally chain connected.

(1.5) was conjectured by Shokurov in [14], and was proved there, assuming the MMP. In the same paper, he also conjectured the following, which he proved under the same assumptions:

Corollary 1.6. *Let (X, Δ) be a divisorially log terminal pair.*

If $g\colon Y \longrightarrow X$ is any birational morphism then the fibres of g are rationally chain connected.

It was pointed out in [14] that one then gets the following:

Corollary 1.7. *Let $f : X \dashrightarrow Y$ be a rational morphism of normal proper varieties such that (X, Δ) is a divisorially log terminal pair for some effective divisor Δ. Then, for each closed point $x \in X$, the indeterminacy locus of x is covered by rational curves.*

Recall that if $W \subset X \times Y$ is the closure of the graph of f and p and q are the projections from W to X, and Y, then the indeterminacy locus of x is defined as $q(p^{-1}(x))$.

Another very interesting consequence of (1.6) is:

Corollary 1.8. *Let (X, Δ) be a divisorially log terminal pair. Then X is rationally chain connected iff it is rationally connected.*

The same methods which are used to prove (1.2), in conjunction with (1.8), yield the following:

Corollary 1.9. *Let (X, Δ) be a kawamata log terminal pair. Let $f\colon X \longrightarrow S$ be a proper morphism with connected fibres such that $-(K_X + \Delta)$ is relatively nef and $-K_X$ is relatively big. Let $g\colon Y \longrightarrow X$ be any birational morphism, and let $\pi\colon Y \longrightarrow S$ be the composition. Let T be any irreducible subset of S and let W be the inverse image of T inside Y.*

Then there is an irreducible closed subset E of W, which dominates T and which has connected and rationally connected fibres.

One interesting feature of rationally connected varieties is that a family of rationally connected varieties over a curve always admits a section. As a consequence of (1.9), we are able to show that a similar result holds for families of Fano varieties, a result which was also conjectured by Shokurov:

Corollary 1.10. *Let (X, Δ) be a kawamata log terminal pair. Let $f\colon X \longrightarrow S$ be a proper morphism with connected fibres such that $-(K_X + \Delta)$ is relatively nef and $-K_X$ is relatively big. Let $g\colon Y \longrightarrow X$ be any birational morphism, and let $\pi\colon Y \longrightarrow S$ be the composition.*

Then π has a section over any curve.

Another reason why rationally connected and rationally chain connected varieties are interesting, is because their intersection theory is particularly simple. In particular this means that the intersection theory of a divisorially log terminal pair and a resolution are closely related:

Corollary 1.11. *Let (X, Δ) be a kawamata log terminal pair, over an algebraically closed field. Let $f\colon X \longrightarrow S$ be a proper morphism with connected fibres such that $-(K_X + \Delta)$ is relatively nef and $-K_X$ is relatively big. Let $g\colon Y \longrightarrow X$ be any birational morphism, and let $\pi\colon Y \longrightarrow S$ be the composition.*

Then the natural map

$$\pi_*\colon\ \mathrm{CH}^0(Y) \longrightarrow \mathrm{CH}^0(S),$$

is an isomorphism.

The following, although not a direct consequence of (1.2), is proved using very similar methods:

Corollary 1.12. *Let (X, Δ) be a kawamata log terminal proper pair and let $X \dashrightarrow Z$ be the Iitaka fibration associated to $-(K_X + \Delta)$.*

Then Z is rationally connected.

As a consequence we obtain:

Corollary 1.13. *Let (X, Δ) be a kawamata log terminal proper pair and suppose that that $-(K_X + \Delta)$ is big and nef.*

Then X is rationally connected.

We remark that (1.13) has also been recently proved by Zhang, using a similar argument, see [16]. We were working on this paper, when his result appeared on the archive. Note that (1.10) follows from (1.13), in the case when the morphism f is flat, using the fact that moduli space of stable maps is proper over Z, see [15].

As Zhang points out in [16], (1.13) implies the following:

Corollary 1.14. *Let (X, Δ) be a kawamata log terminal proper pair and suppose that that $-(K_X + \Delta)$ is big and nef.*

Then X is simply connected.

It should be pointed out though that it is more natural to prove the stronger result that the smooth locus of X has finite fundamental group. The only known case of this much stronger result, is for surfaces, see [7]. It is proved in [11] that at least the algebraic fundamental group of the smooth locus is finite.

Remark 1.15. *Note that* (1.4) *and* (1.14)*, which are stated for kawamata log terminal pairs extend to the case when the locus of log canonical singularities is a point (equivalently, by connectedness, when V has dimension zero). These results generalise easily to the case when the coefficients of Δ are real, and we indicate how to do this at the end of the paper.*

As previously pointed out, Shokurov conjectured many of these results, and proved them assuming the MMP, [13] and [14], where the boundary Δ has real coefficients and the characteristic is arbitrary. Of course the first important results in these directions was proved by Kollár, Miyaoka and Mori [9], who proved that every smooth Fano variety is rationally connected, and even bounded the degree of the covering rational curves. In fact Shokurov conjectures, in (1.2), that one can even bound the degree of each component of the connecting chain, see [12] for a special case. Finally, as already pointed out, Zhang proved the absolute case of (1.2), when S is a point and the locus of log canonical singularities is empty. In fact we are informed by Shokurov that Zhang [16] was very close to a proof of the absolute case, when S is a point, but the singularities are arbitrary.

We now give a quick sketch of the proof of (1.2). Let us suppose that we want to prove that F is rationally connected. By the main result of [3], it suffices to prove that whenever we have a test rational map $t\colon F \dashrightarrow Z$, then either Z is uniruled or it is a point, see (4.2). Suppose Z is not uniruled. By the main result of [1], K_Z is pseudo-effective.

Suppose for a moment, that we can produce a divisor Θ with three key properties:

(a) $K_F + \Theta$ has Kodaira dimension zero,
(b) there is an ample $\mathbb{Q}$-divisor H on Z such that $t^*H \leq \Theta$, and
(c) $K_F + \Theta$ is log terminal on the general fibre.

By log additivity, it follows that the Kodaira dimension of $K_F + \Theta$ is at least the dimension of Z, so that Z is a point by (a), see (4.1). It remains to indicate how to produce the divisor Θ. In our case, F is a smooth divisor and a component of the fibre of a resolution of a Fano fibration $f\colon X \longrightarrow S$. The existence of a divisor Δ on the total space of the Fano fibration satisfying (a), (b) and (c) is quite straightforward, and it is equally straightforward to lift this to a resolution $g\colon Y \longrightarrow X$ of the total space. Conditions (b) and (c) then descend to F. The tricky part is ensuring that condition (a) continues to hold.

The first step is to realise F as a component of the locus of log canonical singularities, with respect to some divisor. At this point we apply the main technical result of [4], which says, roughly speaking, that we can lift appropriate sections from F to Y.

2. Notation and conventions

We work over the field of complex numbers $\mathbb{C}$. A $\mathbb{Q}$-Cartier divisor D on a normal variety X is *nef* if $D \cdot C \geq 0$ for any curve $C \subset X$. We say that two $\mathbb{Q}$-divisors D_1, D_2 are $\mathbb{Q}$-linearly equivalent ($D_1 \sim_{\mathbb{Q}} D_2$) if there exists an integer $m > 0$ such that mD_i are linearly equivalent. Given a morphism of normal varieties $g\colon Y \longrightarrow X$, we say that two $\mathbb{Q}$-divisors D_1 and D_2 are $\mathbb{Q}$-g-linearly equivalent ($D_1 \sim_{g,\mathbb{Q}} D_2$ if there is a positive integer m and a Cartier divisor B on X such that $mD_1 \sim mD_2 + g^*B$. We say that a $\mathbb{Q}$-Weil divisor D is big if we may find an ample divisor A and an effective divisor B, such that $D \sim_{\mathbb{Q}} A + B$. A *log pair* (X, Δ) is a normal variety X and an effective $\mathbb{Q}$-Weil divisor Δ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. A projective morphism $g\colon Y \longrightarrow X$ is a *log resolution* of the pair (X, Δ) if Y is smooth and $g^{-1}(\Delta) \cup \{\text{ exceptional set of } g\,\}$ is a divisor with normal crossings support. We write $g^*(K_X + \Delta) = K_Y + \Gamma$ and $\Gamma = \sum a_i \Gamma_i$ where Γ_i are distinct reduced irreducible divisors. The log discrepancy of Γ_i is $1 - a_i$. The *locus of log canonical singularities of the pair* (X, Δ), denoted $\mathrm{LCS}(X, \Delta)$, is equal to the image of those components of Γ of coefficient at least one (equivalently log discrepancy at most zero). The pair (X, Δ) is *kawamata log terminal* if for every (equivalently for one) log resolution $g\colon Y \longrightarrow X$ as above, the coefficients of Γ are strictly less than one, that is $a_i < 1$ for all i. Equivalently, the pair (X, Δ) is kawamata log terminal if the locus of log canonical singularities is empty. We say that the pair (X, Δ) is *divisorially log terminal* if the coefficients a_i of Δ lie between zero and one, $0 \leq a_i \leq 1$, and there is a log resolution such that the coefficients of the g-exceptional divisors are all less than one.

Given a pair (X, Δ) and a birational morphism $g\colon Y \longrightarrow X$, we may write

$$K_Y + \Gamma = g^*(K_X + \Delta) + E,$$

where Γ and E are effective, with no common components, and E is g-exceptional. Note that this decomposition is unique. Sometimes we will replace equality of $\mathbb{Q}$-divisors, by $\mathbb{Q}$-linearly equivalence,

$$K_Y + \Gamma \sim_{\mathbb{Q}} g^*(K_X + \Delta) + E,$$

where Γ and E have the same properties. In this case the decomposition is far from unique.

Let $\phi\colon X \dashrightarrow Y$ be a rational map. We say that a subset V of X *dominates* Y, if the inverse image of V, in the graph of ϕ, dominates Y.

Let $f\colon X \longrightarrow S$ be a morphism. We say that a divisor E is exceptional for f, if $f(E)$ has codimension at least two in S.

3. Some Examples

In this section we collect together some examples, whose purpose is to show that the results stated in §1 are in some sense best possible, and to motivate their proofs.

We remark that in theorem (1.13), one can not remove the hypothesis that $-(K_X+\Delta)$ is nef or that (X, Δ) is kawamata log terminal as shown by the following well known example. Let $f\colon S \longrightarrow C$ be any $\mathbb{P}^1$-bundle over an elliptic curve and let E be a section of minimal self-intersection. Then f is the maximal rationally connected fibration of S. Suppose first that $E^2 < 0$. Then

$$-(K_S + tE) \qquad \text{is} \qquad \begin{cases} \text{big} & \text{for any } t < 2 \\ \text{ample} & \text{for any } 1 < t < 2 \\ \text{nef} & \text{for any } 1 \leq t \leq 2 \\ \text{lc} & \text{for any } t \leq 1 \\ \text{klt} & \text{for any } t < 1 \end{cases}$$

Clearly S is not rationally chain connected. If $t = 1$ then $K_S + E$ is log canonical and $-(K_S + E)$ is nef and big. If $t < 1$ then $K_S + tE$ is kawamata log terminal and $-(K_S + tE)$ is big but not nef. If $1 < t < 2$ then $-(K_S + tE)$ is ample but not log canonical. Notice that by contracting the negative section E, we get a rationally chain connected surface T. Therefore, rational chain connectedness is not a birational property of S; of course if $f\colon X' \longrightarrow X$ is a birational morphism of varieties and X' is rationally chain connected, then X is also rationally chain connected.

One might well ask the following:

Question 3.1. *Let (X, Δ) be a klt pair, Δ effective and $-(K_X + \Delta)$ nef. If $X \dashrightarrow W$ is the MRCC fibration then does it follow that*

$$\dim X \geq \dim W + \kappa(-(K_X + \Delta)) \ ?$$

Now suppose that we let E_2 be the unique non-split extension of $\mathcal{O}_C$ by $\mathcal{O}_C$ and set S to be the projectivisation of E_2. Then E has self-intersection zero. $-K_S = 2E$, so $-K_S$ is nef of numerical dimension one. But in fact no multiple of E moves. Indeed if it did, then there would be an étale cover $\pi\colon C' \longrightarrow C$ such that π^*E_2 splits. It is easy to see that in characteristic zero this never happens. Thus the Kodaira dimension of $-K_S$ is zero and the sum of the Kodaira dimension plus the dimension of C is less than the dimension of S so that the answer to (3.1) is no, in general.

Finally, we give an easy example, to illustrate the fact that one cannot expect every component of the exceptional locus of a resolution of a divisorially log terminal singularity to be rationally connected. In fact it is rarely the case that each individual component is rationally chain connected.

Let X be a smooth threefold. Pick a point $x \in X$ and let $Y_1 \longrightarrow X$ blow up this point. Now let $Y \longrightarrow Y_1$ blow up a cubic curve in the exceptional divisor, which is a copy of $\mathbb{P}^2$. Consider the birational morphism $\pi\colon Y \longrightarrow X$. Denote the first exceptional divisor, a copy of $\mathbb{P}^2$, by E and the other by F. It is easy to check that $-(K_Y + 4/5E + 1/5F)$ is relatively ample. Note that $E \cup F$ is rationally chain connected, but that F is not rationally chain connected. Indeed F is once again a $\mathbb{P}^1$-bundle over an elliptic curve.

4. Uniruled, rationally connected and rationally chain connected varieties

In this section we give sufficient conditions to ensure that a variety is either uniruled, or rationally connected, or rationally chain connected. First a criterion to ensure that a variety is uniruled:

Proposition 4.1. *Let (X, Δ) be a projective log pair and let $h\colon X \longrightarrow F$ and $t\colon F \dashrightarrow Z$ be a morphism and a rational map, where F and Z are projective, with the following properties:*

1. *the locus of log canonical singularities of $K_X + \Delta$ does not dominate Z, where $X \dashrightarrow Z$ is the composition of t and h,*
2. *$K_X + \Delta$ has Kodaira dimension at least zero on the general fibre of $X \dashrightarrow Z$, (that is, if $g\colon Y \longrightarrow X$ resolves the indeterminacy of $X \dashrightarrow Z$ then $g^*(K_X + \Delta)$ has Kodaira dimension at least zero on the general fibre of the induced morphism $Y \longrightarrow Z$),*
3. *$K_X + \Delta$ has Kodaira dimension at most zero, and*
4. *there is an ample divisor A on F such that $h^*A \leq \Delta$.*

Then either Z is a point or it is uniruled.

Proof. Suppose that Z is not uniruled. Blowing up Z, we may assume that Z is smooth. Let $g\colon Y \longrightarrow X$ be a log resolution of X, such that the induced rational map $Y \dashrightarrow Z$ is in fact a morphism $\psi\colon Y \longrightarrow Z$. We may write

$$K_Y + \Theta = g^*(K_X + \Delta) + E,$$

where Θ and E are effective, with no common components and E is exceptional. Now set $\Gamma = \Theta + \epsilon E'$, where ϵ is a sufficiently small rational number, and E' is the support of the exceptional locus. With this choice of Γ, the Kodaira dimension of $K_Y + \Gamma$ is at least zero on the general fibre of ψ, and the support of Γ contains the full exceptional locus.

As Z is not uniruled, it follows by the main result of [1] that K_Z is pseudo-effective. By (4), and our choice of Γ, Γ contains the pullback of an ample divisor from F. Possibly replacing Γ by a linearly equivalent divisor, we may find an ample divisor G on Z such that $\psi^* G \leq \Gamma$. Since $K_Y + \Gamma$ is log terminal on the general fibre of ψ, it follows by log additivity of the Kodaira dimension, see Corollary 2.11 of [4], that the Kodaira dimension of $K_Y + \Gamma$ is at least the dimension of Z. As the Kodaira dimension of $K_Y + \Gamma$ is at most the Kodaira dimension of $K_X + \Delta$, it follows that Z is a point. □

It is easy to use (4.1) to prove that a variety is rationally chain connected:

Lemma 4.2. *Let F be a normal variety.*

(1) *F is rationally connected iff for every non-constant dominant rational map $t\colon F \dashrightarrow Z$, Z is uniruled.*
(2) *F is rationally chain connected modulo V iff for every non-constant dominant rational map $t\colon F \dashrightarrow Z$, either Z is uniruled or V dominates Z.*

Proof. It is clear that the image of every rationally connected variety is rationally connected. In particular the image of every rationally connected variety is either uniruled or a point. If V does not dominate Z, then pick two general points x and y of F. Then they are connected by a chain of rational curves modulo V, and the image of one of these curves must be a rational curve through the general point of Z.

To prove the reverse direction, first observe that F is uniruled, by applying the basic criterion to the identity map $F \longrightarrow F$. Let F' be a smooth model of F, and let $F' \dashrightarrow Z$ be the maximal rationally connected fibration of F'. Blowing up, we may assume that this map is a morphism. Since F is birational to F', this induces a rational map $t\colon F \dashrightarrow Z$. By assumption either Z is a point, or Z is uniruled, or V dominates Z.

As F is uniruled, $F' \longrightarrow Z$ is a non-trivial fibration. This morphism has the defining property that if a rational curve C meets a very general fibre G, then C is contained in G. By the main result of [3], we can lift any rational curve which passes through the general point of Z to F.

Thus Z is not uniruled. If Z is a point, then F is rationally connected. The only other possibility is that V dominates Z. In this case F is certainly rationally chain connected modulo V. □

Corollary 4.3. *Let (X, Δ) be a log pair and let $h\colon X \longrightarrow F$ be a morphism. Suppose that for every rational map $t\colon F \dashrightarrow Z$, either the locus of log canonical singularities of $K_X + \Delta$ dominates Z, or (2-4) of* (4.1) *hold.*

Then F is rationally chain connected modulo the image R of the locus where $K_X + \Delta$ is not kawamata log terminal.

Proof. We are going to apply the criterion of (4.2). Suppose we are given a rational map $t\colon F \dashrightarrow Z$. If R dominates Z there is nothing to prove. Otherwise we just need to prove that Z is either uniruled or a point and for this we just need to observe that conditions (1-4) of (4.1) hold. □

5. A fine analysis of the fibres of a Fano fibration

In this section we state a detailed theorem about the fibres of a Fano fibration, which despite being technical in nature, we expect will be of independent interest. We fix some notation which will hold throughout the section. Let (X, Δ) be a log pair and let $f\colon X \longrightarrow S$ be a morphism such that $K_X + \Delta \sim_{\mathbb{Q},f} 0$ and Δ is f-big.

Let $s \in S$ be any closed point and let G be any effective $\mathbb{Q}$-Cartier divisor on S. Let $g\colon Y \longrightarrow X$ be any birational morphism, such that the fibre of the composite morphism $\pi\colon Y \longrightarrow S$ over s union the exceptional locus of g union the strict transform of $f^*G + \Delta$ is a divisor with normal crossings. Let $F_1, F_2, \dots, F_k$ be the components of the fibre $\pi^{-1}(s)$, which have log discrepancy greater than zero, with respect to $K_X + \Delta$, and let F be their union.

Theorem 5.1. *With the notation above, we may pick G and divisors Γ and E on Y, with the following properties:*

1. *The equation*
$$K_Y + \Gamma \sim g^*(K_X + \Delta) + E,$$
holds, where Γ and E are effective, with no common components and E is g-exceptional. Moreover, we may write $\Gamma = A + B$, where A is π-ample and B is effective.
2. *Given $t \in [0, 1]$, set $\Delta_t = \Delta + tf^*G$, $\Gamma_t = \Gamma + t\pi^*G$ (where Γ is defined below) and let V_t be the closure of*
$$\mathrm{LCS}(Y, \Gamma_t) - \mathrm{LCS}(Y, \Gamma).$$
We let Γ'_t denote the fractional part of Γ_t. Then $F = V_1$.

(3) *Possibly relabelling and rescaling, we may assume that there are rational numbers,* $0 = t_0, t_1, \dots, t_k = 1$*, such that* $V_i = F_1 \cup F_2 \cup \dots \cup F_i$*, where, here and elsewhere, we adopt the shorthand subscript* i *in lieu of* t_i*. For* ϵ *sufficiently small,* $V_{t_i - \epsilon} = V_{t_{i-1}}$*. Denote by* Θ_i *the restriction of* Γ'_i *to* F_i*.*

(4) *Let* H *be any* π*-ample divisor. Then there is a constant* M *such that*

$$h^0(F_i, \mathcal{O}_X(m(K_{F_i} + \Theta_i) + H|_{F_i})) \leq M$$

for all sufficiently divisible positive integers m*, and any* $1 \leq i \leq k$*.*

(5) F_i *is rationally chain connected modulo* $W_i = F_i \cap \mathrm{LCS}(Y, \Gamma_{i-1})$*.*

(6) *If* $K_Y + \Gamma$ *is kawamata log terminal then* F_1 *is rationally connected.*

(7) *If* $K_X + \Delta$ *is kawamata log terminal then we may pick* Γ *so that* $K_Y + \Gamma$ *is kawamata log terminal.*

We prove each part of (5.1) in a series of lemmas:

Lemma 5.2. *(1) of* (5.1) *holds.*

Proof. We may write

$$K_Y + \Gamma_1 = g^*(K_X + \Delta) + E_1,$$

where Γ_1 and E_1 are effective, with no common components and E_1 is g-exceptional. Now let E' be the sum of all the exceptional divisors, taken with coefficient one. Set $\Gamma_2 = \Gamma_1 + \delta E'$, $E_2 = E_1 + \delta E'$, for some positive rational number δ. If we choose δ small enough, then we do not change the locus of log canonical singularities. As Γ_2 and the strict transform of Δ union the exceptional locus have the same support, it follows that Γ_2 is π-big. It follows that we may write $\Gamma_2 \sim_{\mathbb{Q},\pi} A + B$, where A is π-ample and B is effective. Let

$$K_Y + \Gamma = g^*(K_X + \Delta) + E,$$

be the decomposition obtained by cancelling like terms on both sides of

$$K_Y + ((1-\epsilon)\Gamma_2 + \epsilon B) + \epsilon A = g^*(K_X + \Delta) + E_2.$$

Thus Γ and E are effective with no common components, E is exceptional and of course $\epsilon A \leq \Gamma$ is relatively ample. □

Lemma 5.3. *(2) of* (5.1) *holds.*

Proof. If we pick G sufficiently singular at s, then we may assume that $F \subset V_1$. Possibly replacing G by a linearly equivalent $\mathbb{Q}$-divisor, we may then assume that $F = V_1$. □

Lemma 5.4. *(3) of* (5.1) *holds.*

Proof. By (2), for every $0 \leq t \leq 1$, V_t is a union of components of F. Let t_i be the smallest value of t, such that F_i is a component of $\mathrm{LCS}(Y, \Gamma_t)$. Possibly perturbing G, we may assume that $t_i \neq t_j$, if $i \neq j$ and so possibly re-ordering and rescaling, we may assume that $0 < t_1 < t_2 < \cdots < t_k = 1$. $\square$

One of the key ideas is to use the main result of [4], which we recall:

Theorem 5.5. *Let $F \subset Y$ be a smooth divisor in a smooth variety, and let $\pi\colon Y \longrightarrow S$ be a projective morphism. Let H be a sufficiently π-very ample divisor and set $A = (\dim F + 1)H$. Assume that*

1. *Γ is a $\mathbb{Q}$-divisor with simple normal crossings support such that Γ contains F with coefficient one and (Y, Γ) is log canonical;*
2. *$C \geq 0$ is a $\mathbb{Q}$-divisor not containing F;*
3. *$K_F + \Delta$ is π-pseudo-effective, where $\Delta = (\Gamma - F)|_F$ and*
4. *there is a positive integer m and a divisor $G \in |m(K_Y+\Gamma+C)|$, which does not contain any log canonical centre of $(Y, \ulcorner\Gamma\urcorner)$.*

For any sufficiently divisible positive integer m, the image of the natural homomorphism

$$\pi_*\mathcal{O}_Y(m(K_Y + \Gamma + C) + H + A) \longrightarrow \pi_*\mathcal{O}_F(m(K_F + \Delta + C) + H + A),$$

contains the image of the sheaf $\pi_*\mathcal{O}_F(m(K_F + \Delta) + H)$ considered as a subsheaf of $\pi_*\mathcal{O}_F(m(K_F + \Delta + C) + H + A)$ by the inclusion induced by any divisor in $mC + |A|$ not containing F.

Lemma 5.6. *(4) of* (5.1) *holds.*

Proof. This is the most technical part of (5.1).

We are certainly free to replace H by a multiple. Pick a divisor G on X such that $g^*G \geq (\dim X + 2)H$. Then

$$\begin{aligned} g_*\mathcal{O}_Y(m(K_Y + \Gamma_i) + (\dim X + 2)H) &\subset g_*\mathcal{O}_Y(g^*G + mE) \\ &= \mathcal{O}_X(G), \end{aligned}$$

for m sufficiently divisible. Therefore $\pi_*\mathcal{O}_Y(m(K_Y + \Gamma_i) + (\dim X + 2)H) \subset f_*\mathcal{O}_X(G)$. On the other hand, by (5.5), since Θ_i does not contain F_i, we may lift any section of

$$\pi_*(\mathcal{O}_X(m(K_{F_i} + \Theta_i) + H)) = H^0(X, \mathcal{O}_X(m(K_{F_i} + \Theta_i) + H)),$$

to $\pi_*(\mathcal{O}_Y(m(K_Y + \Gamma_i) + (\dim X + 2)H))$. $\square$

Lemma 5.7. *(5) of* (5.1) *holds.*

Proof. We are going to apply (4.3). Let $\psi\colon F_i \dashrightarrow Z$ be any rational map, such that the locus where $K_{F_i} + (\Gamma_i - F_i)|_{F_i}$ is not kawamata log terminal does not dominate Z. We just need to check that conditions (2-4) of (4.1) hold. Now

$$K_Y + \Gamma_i = E_i,$$

where E_i is effective and exceptional, and F_i is not contained in the support of E_i. It follows that the Kodaira dimension of $K_Y + \Gamma_i$ restricted to the general fibre of ψ is at least zero. We may assume that W_i does not dominate Z, so that $\Gamma_i = \Gamma'_i$, on the general fibre of ψ. It follows that $K_{F_i} + \Theta_i$ has Kodaira dimension at least zero on the general fibre of ψ. Thus (2) holds. (3) is an immediate consequence of (4) of (5.1). (4) holds by (1) of (5.1). □

Lemma 5.8. *(6) of* (5.1) *holds.*

Proof. Since $W_0 = \mathrm{LCS}(Y, \Gamma)$ is by assumption empty, it follows that F_1 is rationally chain connected. As F_1 is smooth, it is in fact rationally connected. □

Lemma 5.9. *(7) of* (5.1) *holds.*

Proof. Clear from the construction of Γ given in (5.2). □

6. Proof of (1.2)

Lemma 6.1. *If $-(K_X + \Delta)$ is semiample and $-K_X$ is big, then we may find a divisor $\Delta' \geq \Delta$, such that some multiple of $K_X + \Delta'$ is linearly equivalent to zero, where Δ' contains an ample divisor, and where the locus of log canonical singularities of $K_X + \Delta'$ is contained in the locus of log canonical singularities of $K_X + \Delta$.*

Proof. If we pick a general element $D \in |-m(K_X + \Delta)|$, where m is sufficiently divisible, then $K_X + \Delta + D/m$ has the same locus of log canonical singularities as $K_X + \Delta$. Replacing Δ by $\Delta + D/m$, we may assume therefore that Δ is big and that some multiple of $K_X + \Delta$ is linearly equivalent to zero. As Δ is big, we may write $\Delta \sim_{\mathbb{Q}} A + B$, where A is ample and B is effective. Let $\Delta' = (1-\epsilon)\Delta + \epsilon A + \epsilon B$. Then for ϵ sufficiently small, the locus of log canonical singularities of $K_X + \Delta'$ is contained in the locus of log canonical singularities of $K_X + \Delta$. □

Proof of (1.2). Let $s \in S$. This result is local over $s \in S$. Passing to an open neighbourhood of $s \in S$, we may as well assume that $-(K_X + \Delta)$ is semiample. By (6.1) we may assume therefore that $\Delta = A + B$, where A is ample and B is effective, and that some multiple of $K_X + \Delta$ is linearly equivalent to zero.

Note that as g has connected fibres, it follows that if $Y' \longrightarrow Y$ is any birational map, then we are free to replace Y by Y'. Thus we may assume that the hypothesis and notation of §5 hold.

By induction on i, it suffices to prove that F_i is rationally chain connected modulo W_i, which is (5) of (5.1). $\square$

7. Proof of Corollaries

Proof of (1.3). Since X is covered by rational curves which intersect an irreducible component of V, the main theorem of [2] implies that the image of the fundamental group of V is of finite index in the fundamental group of X.

Let $\pi\colon Y \longrightarrow X$ be any étale morphism of degree r, where Y is connected, let W be the inverse image of V, and set $\Gamma = \pi^*\Delta$. Then $K_Y + \Gamma = \pi^*(K_X + \Delta)$, so that W is the locus of log canonical singularities of $K_Y + \Gamma$ and $-(K_Y + \Gamma)$ is big and nef. Note that W is connected, by the connectedness theorem.

The proof now divides into two cases. If V is non-empty, then the result follows immediately from the fact that π is arbitrary and W is always connected.

Now suppose that V is empty. In this case, we adopt the convention that the fundamental group of the empty set is the trivial group. By what we have already proved the fundamental group is finite and so it suffices to prove that the algebraic fundamental group is trivial. Now

$$\chi(\mathcal{O}_Y) = r\chi(\mathcal{O}_X).$$

On the other hand,

$$h^i(Y, \mathcal{O}_Y) = h^i(X, \mathcal{O}_X) = 0 \qquad \text{for } i > 0,$$

by Kawamata-Viehweg vanishing, so that

$$\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_X) = 1,$$

whence $r = 1$. But then the fundamental group of X is trivial and the fundamental group of V certainly surjects onto the fundamental group of X. $\square$

Lemma 7.1. *Let (X, Δ) be a kawamata log terminal pair and let $f\colon X \longrightarrow S$ be a projective morphism. Suppose that $-(K_X + \Delta)$ is relatively nef and $-K_X$ is relatively big.*

Then $-(K_X + \Delta)$ is relatively semiample.

Proof. It suffices to exhibit a divisor Θ, such that $K_X + \Theta$ is kawamata log terminal and $-(K_X + \Theta)$ is ample, since then, by the base point free theorem, every nef divisor is relatively semiample.

By assumption we may write

$$-K_X \sim_{\mathbb{Q},f} A + B,$$

where A is relatively ample and B is effective. Let

$$\Theta = (1-\epsilon)\Delta + \epsilon B.$$

Then

$$-(K_X + \Theta) \sim_{\mathbb{Q},f} -(1-\epsilon)(K_X + \Delta) + \epsilon A$$

is relatively ample, as it is the sum of a relatively nef divisor and a relatively ample divisor. In particular $K_X + \Theta$ is $\mathbb{Q}$-Cartier. But then

$$K_X + \Theta = K_X + (1-\epsilon)\Delta + \epsilon B,$$

is certainly kawamata log terminal, for ϵ small enough. □

Proof of (1.4). By (7.1) $\mathcal{O}_X(-m(K_X + \Delta))$ is relatively generated, for some $m > 0$, and so the result follows from (1.2), since the locus of log canonical singularities is empty. □

Proof of (1.5). By (1.2), we already know that the fibres of f are rationally chain connected modulo the locus V of log canonical singularities. It remains to prove that the fibres of V over S are rationally chain connected. By induction on the dimension, it suffices to prove that if W is a log canonical centre, then W is rationally chain connected modulo the union R of the log canonical centres properly contained in W. As in the proof of (1.2), by (6.1), we may assume that Δ contains an ample divisor and that some multiple of $K_X + \Delta$ is linearly equivalent to zero. Let $g\colon Y \longrightarrow X$ be a log resolution of the pair (X, Δ). We may write

$$K_Y + \Gamma = g^*(K_X + \Delta) + E,$$

where Γ and E are effective, with no common components and E is g-exceptional. Let F a log canonical centre of $K_Y + \Gamma$ which dominates W, minimal with this property. Then F is the intersection of divisors of log discrepancy zero, with centre W. In particular we may find a divisor Θ such that

$$(K_Y + \Gamma)|_F = K_F + \Theta.$$

We are going to apply (4.3) to $h\colon F \longrightarrow W$. Let $t\colon W \dashrightarrow Z$ be a rational map. Blowing up further, we may assume that the induced map $\psi\colon F \longrightarrow Z$ is a morphism. Note that the image in W of the locus of log canonical singularities of $K_F + \Theta$ is equal to R, by minimality of

F. So we may assume that that the locus of log canonical singularities of $K_F + \Theta$ does not dominate Z. We need to check that conditions (2-4) hold. (2) holds, exactly as in the proof of (1.2). (3) holds by (4) of (5.1) and (4) holds by assumption. $\square$

Proof of (1.6). By (2.43) of [10], we may assume that the pair (X, Δ) is kawamata log terminal, and in this case we may apply (1.2) with f the identity. $\square$

Proof of (1.7). Immediate from (1.6). $\square$

Proof of (1.8). If X is rationally connected then it is certainly rationally chain connected, so suppose that X is rationally chain connected. Let $g\colon Y \longrightarrow X$ be a resolution of singularities. By (1.6) g has rationally chain connected fibres. It follows that Y is rationally chain connected. But then Y is rationally connected, as it is smooth, and so X is rationally connected as well. $\square$

Proof of (1.9). We work locally in a neighbourhood of the generic point of T. (7.1) implies that $-(K_X + \Delta)$ is relatively semiample. As in the proof of (1.2) and (6.1), we may therefore assume $-(K_X+\Delta)$ is ample and that Δ contains an ample divisor A,

Suppose that $T \neq S$. Pick any ample divisor H in S which contains T. Let t be the largest rational number such that $K_X+\Delta+tf^*H$ is log canonical. Possibly perturbing Δ by an ample divisor in X, we may assume that there is a unique log canonical centre V of $K_X+\Delta+tf^*H$. By connectedness V has connected fibres over T. Since V is the unique log canonical centre, it is certainly a minimal log canonical centre. Thus V is normal. It follows that V has irreducible fibres. By Kawamata's subadjunction formula, see Theorem 1 of [6], there is a boundary Θ on V, such that

$$(K_X + \Delta)|_V = K_V + \Theta,$$

where $K_V + \Theta$ is kawamata log terminal. Possibly replacing A by a linearly equivalent divisor, we may assume that Θ contains an ample divisor. Thus we are free to restrict to V and replace X by V and S by the image of V.

Thus by induction on the dimension, we may assume that $T = S$. In this case the general fibre over T is kawamata log terminal and so by (1.2) and (1.8) the general fibre is rationally connected. $\square$

Proof of (1.10). Pick a resolution $g\colon Y \longrightarrow X$ and let $\pi\colon Y \longrightarrow S$ be the composition. Pick a curve $C \subset S$, and let E be a subvariety of Y with rationally connected fibres over C, whose existence is guaranteed by (1.9). Now apply the main result of [3]. $\square$

Proof of (1.11). Even though this would seem to be a fairly standard result, we could not find a reference, and so we include a proof for completeness.

First note that if π is any surjective morphism, then π_* is always surjective, as $\mathrm{CH}^0(S)$ is generated by the points of S. Now suppose that α is a cycle in $\mathrm{CH}^0(Y)$, whose image β in $\mathrm{CH}^0(S)$ is zero. It remains to prove that α is equivalent to zero. First pick a curve Σ in Y which is the intersection of general hyperplanes, containing the support of α. Then we may replace α by an equivalent cycle in Σ, in such a way that the support of α belongs to the general fibre.

As β is equivalent to zero, by definition of $\mathrm{CH}^0(S)$, it follows that there is a curve C in S such that the support of β belongs to C, and such that β is linearly equivalent to zero in C. By (1.10) we may find a curve $D \subset Y$ which maps birationally down to C. Let α' be the pullback of β to D. As α is equivalent to zero in C, α' is certainly equivalent to zero in D.

On the the other hand, by our choice of α, the fibres of π over the support of β are rationally connected. It follows that α and α' are equivalent, so that α is indeed zero in $\mathrm{CH}^0(Y)$. $\square$

Proof of (1.12). If $K_X + \Delta$ is kawamata log terminal, then the locus of log canonical singularities is empty and the result is therefore immediate from (4.3). $\square$

Proof of (1.13). By assumption the Iitaka fibration of $-(K_X + \Delta)$ is birational, and as rational connectedness is a birational invariant, the result follows from (1.12). $\square$

Proof of (1.14). The pair (X, Δ) is kawamata log terminal iff the locus of log canonical singularities is empty, and so this result is an immediate consequence of (1.3). $\square$

As noted in the introduction, these results extend readily to the case of real coefficients (see [5] for the definitions involving real coefficients). We briefly indicate how to do this. Since most results are direct consequences of (1.2), the key point is to replace the condition that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, by the condition that $K_X + \Delta$ is $\mathbb{R}$-Cartier. In particular it no longer makes sense to assume that $\mathcal{O}_X(-m(K_X + \Delta))$ is relatively generated, and we replace this by the weaker condition that $-(K_X + \Delta)$ is relatively $\mathbb{R}$-semiample. As in the proof of (1.2), this means that locally over S, we may assume that $-(K_X + \Delta)$ is ample, where we also use the fact that Δ is big. But then by (2.11) of [8], applied to $D = K_X + \Delta$, we may assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier (note that the proof of (2.11) of [8] implies that we may assume that the rational coefficients of D are equal to the corresponding coefficients D', in the notation of (2.11) of [8]) and we can apply (1.2) directly.

As (7.1) easily generalises to the case of real coefficients, using the base point free theorem for real divisors, see for example (7.1) of [5], it follows that (1.4)-(1.14) generalise to the case of real coefficients.

References

[1] S. Boucksom, J-P. Demailly, M. Paun, and T. Peternell, *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension*, arXiv:math.AG/0405285.

[2] F. Campana, *On twistor spaces of the class C*, J. Diff. Geom. **33** (1991), 541–549.

[3] T. Graber, J. Harris, and J. Starr, *Families of rationally connected varieties*, J. Amer. Math. Soc. **16** (2003), no. 1, 57–67 (electronic).

[4] C. Hacon and J. M^{c}Kernan, *Boundedness of pluricanonical maps of varieties of general type*, math.AG/0504327.

[5] ———, *On the existence of flips*, math.AG/0507597.

[6] Y. Kawamata, *On Fujita's freeness conjecture for 3-folds and 4-folds*, Math. Ann. **308** (1997), no. 3, 491–505.

[7] S. Keel and J. M^{c}Kernan, *Rational curves on quasi-projective surfaces*, Mem. Amer. Math. Soc. **140** (1999), no. 669, viii+153.

[8] J. Kollár et al., *Flips and abundance for algebraic threefolds*, Société Mathématique de France, Paris, 1992, Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, Astérisque No. 211 (1992).

[9] J. Kollár, Y. Miyaoka, and S. Mori, *Rational curves on Fano varieties*, J. Diff. Geom. **3** (1992), 765–779.

[10] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge tracts in mathematics, vol. 134, Cambridge University Press, 1998.

[11] J. M^{c}Kernan, *Boundedness of log terminal Fano pairs of bounded index*, arXiv:math.AG/0205214.

[12] V. Nikulin, *The diagram method for 3-folds and its application to the Kähler cone and Picard number of Calabi-Yau 3-folds. I*, Higher-dimensional complex varieties (Trento, 1994), de Gruyter, Berlin, 1996, With an appendix by Vyacheslav V. Shokurov, pp. 261–328.

[13] V. V. Shokurov, *Letters of a bi-rationalist. I. A projectivity criterion*, Birational algebraic geometry (Baltimore, MD, 1996), Contemp. Math., vol. 207, Amer. Math. Soc., Providence, RI, 1997, pp. 143–152.

[14] ———, *On rational connectedness*, Math. Notes **68** (2000), no. 5-6, 652–660.

[15] J. Starr, *A pencil of Enriques surfaces of index one with no section*, `http://www-math.mit.edu/~jstarr/papers/explicit4.pdf`.

[16] Q. Zhang, *Rational connectedness of log Q-Fano varieties*, arXiv:math.AG/0408301.

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