EXISTENCE OF MINIMAL MODELS FOR VARIETIES OF LOG GENERAL TYPE II

CHRISTOPHER D. HACON AND JAMES M$^{\text{C}}$KERNAN

ABSTRACT. Assuming finite generation in dimension $n-1$, we prove that pl-flips exist in dimension n.

CONTENTS

1. INTRODUCTION

This is the second of two papers whose purpose is to establish:

Theorem 1.1. *The canonical ring*

$$R(X, K_X) = \bigoplus_{m \in \mathbb{N}} H^0(X, \mathcal{O}_X(mK_X)),$$

is finitely generated for every smooth projective variety X.

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Note that Siu has announced a proof of finite generation for varieties of general type, using analytic methods, see [13].

Our proof relies on the ideas and techniques of the minimal model program and roughly speaking in this paper we will show that finite generation in dimension $n-1$ implies the existence of flips in dimension n. To be more precise, we first recall some definitions:

Definition 1.2. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties, and let V be a finite dimensional affine subspace of the real vector space $\mathrm{WDiv}_{\mathbb{R}}(X)$ of Weil divisors on X. Fix an $\mathbb{R}$-divisor $A \geq 0$ and define*

$$V_A = \{\, \Delta \,|\, \Delta = A + B, B \in V \,\},$$
$$\mathcal{L}_A(V) = \{\, \Delta = A + B \in V_A \,|\, K_X + \Delta \text{ is log canonical and } B \geq 0 \,\}.$$

Then, assuming the following:

Theorem F. *Let $\pi\colon X \longrightarrow Z$ be a projective morphism to a normal affine variety. Let $(X, \Delta = A + B)$ be a $\mathbb{Q}$-factorial kawamata log terminal pair of dimension n, where $A \geq 0$ is an ample $\mathbb{Q}$-divisor and $B \geq 0$. If $K_X + \Delta$ is pseudo-effective, then*

(1) *The pair (X, Δ) has a log terminal model $\mu\colon X \dashrightarrow Y$. In particular if $K_X + \Delta$ is $\mathbb{Q}$-Cartier then the log canonical ring*

$$R(X, K_X + \Delta) = \bigoplus_{m \in \mathbb{N}} H^0(X, \mathcal{O}_X(\llcorner m(K_X + \Delta) \lrcorner)),$$

is finitely generated.

(2) *Let $V \subset \mathrm{WDiv}_{\mathbb{R}}(X)$ be the vector space spanned by the components of Δ. Then there is a constant $\delta > 0$ such that if G is a prime divisor contained in the stable base locus of $K_X + \Delta$ and $\Xi \in \mathcal{L}_A(V)$ such that $\|\Xi - \Delta\| < \delta$, then G is contained in the stable base locus of $K_X + \Xi$.*

(3) *Let $W \subset V$ be the smallest affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$ containing Δ, which is defined over the rationals. Then there is a constant $\eta > 0$ and a positive integer $r > 0$ such that if $\Xi \in W$ is any divisor and k is any positive integer such that $\|\Xi - \Delta\| < \eta$ and $k(K_X + \Xi)/r$ is Cartier, then every component of $\mathrm{Fix}(k(K_X + \Xi))$ is a component of the stable base locus of $K_X + \Delta$.*

we prove the existence of pl-flips:

Theorem A. *Pl-flips exist in dimension n.*

Theorem F and Theorem A are taken directly from [2]. Theorem A_n refers to Theorem A in the case when the dimension of X is n.

Theorem 1.3. *Theorem F_{n-1} implies Theorem A_n.*

It is proved in [2] that (1.3) completes the proof of (1.1).

The main ideas used in this paper have their origins in the work of Shokurov on the existence of flips [12] together with the use of the extension theorem of [5] which in turn was inspired by the work of Kawamata, Siu and Tsuji (cf. [7], [14] and [17]). For further history about the details of this problem see [3, §2.1].

In this paper, however we do not make use of the concept of "asymptotic saturation" introduced by Shokurov, and in fact we prove a more general result which does not require the relative weak log Fano condition (see also [1]).

Further treatments of the results of this paper may be found in [1] and [6] (which follows Shokurov's approach more explicitly).

We now turn to a more detailed description of the results and techniques used in this paper. Recall the following:

Definition 1.4. *Let (X, Δ) be a purely log terminal pair and $f\colon X \longrightarrow Z$ be a projective morphism of normal varieties. Then f is a **pl-flipping contraction** if Δ is a $\mathbb{Q}$-divisor and*

1. *f is small, of relative Picard number one,*
2. *$-(K_X + \Delta)$ is f-ample,*
3. *X is $\mathbb{Q}$-factorial,*
4. *$S = \llcorner \Delta \lrcorner$ is irreducible and $-S$ is f-ample.*

*The **flip** of a pl-flipping contraction $f\colon X \longrightarrow Z$ is a small projective morphism $g\colon Y \longrightarrow Z$ of relative Picard number one, such that $K_Y + \Gamma$ is g-ample, where Γ is the strict transform of Δ.*

The flip g is unique, if it exists at all, and it is given by

$$Y = \operatorname{Proj}_Z \mathfrak{R} \qquad \text{where} \qquad \mathfrak{R} = \bigoplus_{m \in \mathbb{N}\,:\, k|m} f_*\mathcal{O}_X(m(K_X + \Delta)),$$

and k is any positive integer such that $k(K_X+\Delta)$ is integral. Therefore, in order to prove the existence of pl-flips, it suffices to show that $\mathfrak{R}$ is a finitely generated $\mathcal{O}_Z$-algebra. Since this problem is local over Z, we may assume that $Z = \operatorname{Spec} A$ is affine and it suffices to prove that

$$R(X, k(K_X + \Delta)) = \bigoplus_{m \in \mathbb{N}\,:\, k|m} H^0(X, \mathcal{O}_X(m(K_X + \Delta))),$$

is a finitely generated A-algebra. It is then natural to consider the restricted algebra

$$R_S(X, k(K_X + \Delta)) = \operatorname{Im}\left(R(X, k(K_X + \Delta)) \longrightarrow R(S, k(K_S + \Omega))\right),$$

whose graded pieces correspond to the images of the restriction homomorphisms

$$H^0(X, \mathcal{O}_X(m(K_X + \Delta))) \longrightarrow H^0(S, \mathcal{O}_S(m(K_S + \Omega))),$$

where $m = kl$ is divisible by k and Ω is defined by the adjunction formula

$$(K_X + \Delta)|_S = K_S + \Omega,$$

and $k(K_X + \Delta)$ is Cartier. Shokurov has shown, cf. (3.2), that the algebra $R(X, k(K_X + \Delta))$ is finitely generated if and only if the restricted algebra is finitely generated.

Now, if the natural inclusion

$$R_S(X, k(K_X + \Delta)) \subset R(S, k(K_S + \Omega)),$$

were an isomorphism, then (1.3) would follow from (1) of Theorem F_{n-1}. In fact the pair (S, Ω) is kawamata log terminal, $\dim S = \dim X - 1 = n - 1$ and since $f|_S$ is birational, Ω is automatically big so that, by a standard argument, (1) of Theorem F_{n-1} applies and $R(S, k(K_S + \Omega))$ is finitely generated. (3.2) also implies $\mathfrak{R}$ is finitely generated.

Unluckily this is too much to hope for. However it does suggest that one should concentrate on the problem of lifting sections and the main focus of this paper is to prove the extension result (6.3). In fact (1.3) is a straightforward consequence of (6.3).

To fix ideas, let us start with an example where we cannot lift sections. Let X be the blow up of $\mathbb{P}^2$ at a point o, with exceptional divisor E. Let S be the strict transform of a line through o, let L_1, L_2 and L_3 be the strict transforms of general lines in $\mathbb{P}^2$, let $p = E \cap S$ and let $p_i = L_i \cap S$. Then the pair

$$(X, \Delta = S + (2/3)(E + L_1 + L_2 + L_3)),$$

is purely log terminal but the homomorphism

$$H^0(\mathcal{O}_X(3l(K_X + \Delta))) \longrightarrow H^0(\mathcal{O}_S(3l(K_S + \Omega))) \simeq H^0(\mathcal{O}_{\mathbb{P}^1}(2l)),$$

is never surjective, where $\Omega = (\Delta - S)|_S = 2/3(p + p_1 + p_2 + p_3)$ and l is a positive integer. The problem is that the stable base locus of $K_X + \Delta$ contains E and yet $|3(K_S + \Omega)|$ is base point free. Notice, however, that

$$|3l(K_X + \Delta)|_S = |3l(K_S + \Theta)| + 3l(\Omega - \Theta),$$

where $\Theta = (2/3)(p_1 + p_2 + p_3)$ is obtained from Ω by throwing away p. In other words, Θ is obtained from Ω by removing some part of each component contained in the stable base locus of $K_X + \Delta$.

Returning to the general setting, one may then hope that the restricted algebra $R_S(S, l(K_X + \Delta))$ is given by an algebra of the form

$R(S, l(K_S + \Theta))$ for some kawamata log terminal pair (S, Θ) where $0 \leq \Theta \leq \Omega$ is a $\mathbb{Q}$-divisor obtained from Ω by subtracting components of Ω contained in the stable base locus of $K_X + \Delta$. We will now explain how this may be achieved. The tricky thing is to determine exactly how much of the stable base locus to throw away.

It is not hard to reduce to the following situation: $\pi\colon X \longrightarrow Z$ is a projective morphism to a normal affine variety Z, where $(X, \Delta = S + A + B)$ is a purely log terminal pair of dimension n, $S = \llcorner \Delta \lrcorner$ is irreducible, X and S are smooth, $A \geq 0$ is an ample $\mathbb{Q}$-divisor, $B \geq 0$, $(S, \Omega = (\Delta - S)|_S)$ is canonical and the stable base locus of $K_X + \Delta$ does not contain S.

Let

$$\Theta_m = \Omega - \Omega \wedge F_m \qquad \text{where} \qquad F_m = \mathrm{Fix}(|m(K_X + \Delta)|_S)/m,$$

and $m(K_X + \Delta)$ is Cartier. $\Omega \wedge F_m$ is the minimum of Ω and F_m, where the minimum is taken component by component. In particular $m(\Omega - \Theta_m)$ is the biggest divisor contained in $\mathrm{Fix}(|m(K_X + \Delta)|_S)$ such that $0 \leq \Theta_m \leq \Omega$. It follows that

$$(\supset) \qquad |m(K_S + \Theta_m)| + m(\Omega - \Theta_m) \supset |m(K_X + \Delta)|_S.$$

A simple consequence of the main lifting result (6.3) of this paper implies that this tautological inclusion $(\supset)$ is actually an equality,

$$(=) \qquad |m(K_S + \Theta_m)| + m(\Omega - \Theta_m) = |m(K_X + \Delta)|_S.$$

A technical, but significant, improvement on the proof of the existence of flips which appears in [6] is that the statement of $(=)$ and of (6.3) involves only linear systems and divisors on X and S, even though the proof of (6.3) involves passing to a higher model. The key point is that since (S, Ω) is canonical, it suffices to keep track only of the fixed divisor on S and not of the whole base locus.

To prove $(=)$ we use the method of multiplier ideal sheaves. In fact the main point is to establish an inclusion of multiplier ideal sheaves, (5.3). A proof of (5.3) appeared originally in [5]. We chose to include a proof of this result for the convenience of the reader and we decided to use notation closer to the well established notation used in [10]. Note however that the multiplier ideal sheaves we use, see (4.2), must take into account the divisor Δ (for example consider the case worked out above) and the fact that (S, Ω) is canonical.

In fact, if one assumes the MMP then one should expect $(=)$ to hold. Indeed, if one runs $f\colon X \dashrightarrow Y$ the $(K_X + \Delta)$-MMP, almost by definition this will not change the linear systems $|m(K_X + \Delta)|$. Since $K_Y + \Gamma = K_X + f_*\Delta$ is nef, one can lift sections on Y from the strict transform T of S, by an easy application of Kawamata-Viehweg vanishing. In general, however, the linear systems $|m(K_T + g_*\Theta)|$ are bigger than the linear systems $|m(K_S+\Theta)|$, since the induced birational map $g\colon S \dashrightarrow T$ might extract some divisors. However any such divisor must have log discrepancy at most one, so this cannot happen, almost by definition, if $K_S + \Theta$ is canonical.

In order to establish that $R_S(X, k(K_X+\Delta))$ is finitely generated, cf. (7.1), and thereby to finish the proof of (1.3), it is necessary and sufficient to show that $\Theta = \lim(\Theta_{m!}/m!)$ is rational (the seemingly strange use of factorials is so that we can use limits rather than limsups). At this point we play off two facts. The first is that since we are assuming that Theorem F holds on S, if $m > 0$ is sufficiently divisible and Φ is an appropriately chosen $\mathbb{Q}$-divisor sufficiently close to Θ, then the base locus of $|m(K_S+\Phi)|$ and the stable base locus of $K_S+\Theta$ are essentially the same (basically because $K_S + \Theta$ and $K_S + \Phi$ share a log terminal model $\mu\colon S \dashrightarrow S'$ and these two sets of divisors are precisely the divisors contracted by μ). The second is that using (4.1), (6.3) is slightly stronger than (=); one is allowed to overshoot Θ_m by an amount ϵ/m, where $\epsilon > 0$ is fixed. (It seems worth pointing out that (4.1) seems to us a little mysterious. In particular, unlike (6.3), we were unable to show that this result follows from the MMP.)

More precisely, since the base locus of $|m(K_S + \Theta_m)|$ contains no components of Θ_m, by (2) of Theorem F it follows that the stable base locus of $K_S + \Theta$ contains no components of Θ. If Θ is not rational, then by Diophantine approximation there is a $\mathbb{Q}$-divisor $0 \leq \Phi \leq \Omega$ very close to Θ and an integer $k > 0$ such that $k\Phi$ is integral and $\operatorname{mult}_G \Phi > \operatorname{mult}_G \Theta$, for some prime divisor G. By (6.3), it actually follows that

$$|k(K_S + \Phi)| + k(\Omega - \Phi) = |k(K_X + \Delta)|_S.$$

The condition $\operatorname{mult}_G \Phi > \operatorname{mult}_G \Theta$ ensures that G is a component of $\operatorname{Fix}(k(K_S + \Phi))$, and hence of the stable base locus of $K_S + \Phi$. But then G is a component of Θ and of the stable base locus of $K_S + \Theta$. This is the required contradiction.

2. Notation and conventions

We work over the field of complex numbers $\mathbb{C}$. Let X be a normal variety. A

$$\left.\begin{array}{r}\textit{(integral) divisor}\\ \mathbb{Q}\textit{-divisor}\\ \mathbb{R}\textit{-divisor}\end{array}\right\}\quad\text{is a}\quad\left\{\begin{array}{l}\mathbb{Z}\text{-linear}\\ \mathbb{Q}\text{-linear}\\ \mathbb{R}\text{-linear,}\end{array}\right.$$

combination of prime divisors. Given an integral Weil divisor D, we let

$$R(X, D) = \bigoplus_{m\in\mathbb{N}} H^0(X, \mathcal{O}_X(mD)).$$

Set

$$\begin{aligned} \mathrm{WDiv}_{\mathbb{Q}}(X) &= \mathrm{WDiv}(X) \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \\ \mathrm{WDiv}_{\mathbb{R}}(X) &= \mathrm{WDiv}(X) \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \end{aligned}$$

where $\mathrm{WDiv}(X)$ is the group of Weil divisors on X. The definitions below for $\mathbb{R}$-divisors reduce to the usual definitions for $\mathbb{Q}$-divisors and integral divisors, see [2]. Note that the group of $\mathbb{R}$-divisors forms a vector space, with a canonical basis given by the prime divisors. If $C = \sum c_i B_i$ and $D = \sum d_i B_i$, where B_i are distinct prime divisors, then we write $D \geq 0$ if $d_i \geq 0$ and we will denote by

$$\begin{aligned} \|C\| &= \max_i c_i & \qquad C \wedge D &= \sum_i \min\{c_i, d_i\} B_i \\ \llcorner C \lrcorner &= \sum_i \llcorner c_i \lrcorner B_i & \qquad \{C\} &= C - \llcorner C \lrcorner. \end{aligned}$$

Two $\mathbb{R}$-divisors C and D are

$$\left.\begin{array}{rl} \text{linearly equivalent,} & C \sim D \\ \mathbb{Q}\text{-linearly equivalent,} & C \sim_{\mathbb{Q}} D \\ \mathbb{R}\text{-linearly equivalent,} & C \sim_{\mathbb{R}} D \end{array}\right\} \quad \text{if } C - D \text{ is a} \quad \left\{\begin{array}{l} \mathbb{Z}\text{-linear} \\ \mathbb{Q}\text{-linear} \\ \mathbb{R}\text{-linear,} \end{array}\right.$$

combination of principal divisors. Note that if $C \sim_{\mathbb{Q}} D$ then $mC \sim mD$ for some positive integer m, but this fails in general for $\mathbb{R}$-linear equivalence. Note also that if two $\mathbb{Q}$-divisors are $\mathbb{R}$-linearly equivalent then they are in fact $\mathbb{Q}$-linearly equivalent, but that two integral divisors might be $\mathbb{Q}$-linearly equivalent without being linearly equivalent. Let

$$\begin{aligned} |D| &= \{\, C \in \mathrm{WDiv}(X) \,|\, C \geq 0\,,\, C \sim D \,\} \\ |D|_{\mathbb{Q}} &= \{\, C \in \mathrm{WDiv}_{\mathbb{Q}}(X) \,|\, C \geq 0\,,\, C \sim_{\mathbb{Q}} D \,\} \\ |D|_{\mathbb{R}} &= \{\, C \in \mathrm{WDiv}_{\mathbb{R}}(X) \,|\, C \geq 0\,,\, C \sim_{\mathbb{R}} D \,\}. \end{aligned}$$

If T is a subvariety of X, not contained in the base locus of $|D|$, then $|D|_T$ denotes the image of the linear system $|D|$ under restriction to T. If D is an integral divisor, $\mathrm{Fix}(D)$ denotes the fixed divisor of D so that $|D| = |D - \mathrm{Fix}(D)| + \mathrm{Fix}(D)$ where the base locus of $|D - \mathrm{Fix}(D)|$ contains no divisors. More generally $\mathrm{Fix}(V)$ denotes the fixed divisor of the linear system V.

The *stable base locus* of D, denoted by $\mathbf{B}(D)$, is the intersection of the support of the elements of $|D|_{\mathbb{R}}$ (if $|D|_{\mathbb{R}}$ is empty then by convention the stable base locus is the whole of X). The *stable fixed divisor* is the divisorial support of the stable base locus. The *augmented stable base locus* of D, denoted by $\mathbf{B}_+(D)$, is given by the stable base locus of $D - \epsilon A$ for some ample divisor A and any rational number $0 < \epsilon \ll 1$. The *diminished stable base locus* is defined by

$$\mathbf{B}_-(D) = \bigcup_{\epsilon > 0} \mathbf{B}(D + \epsilon A).$$

In particular we have

$$\mathbf{B}_-(D) \subset \mathbf{B}(D) \subset \mathbf{B}_+(D).$$

This notation was established in [4] for projective varieties but we will use it in a slightly more general setting, see for example [11].

An $\mathbb{R}$-*Cartier divisor* D is an $\mathbb{R}$-linear combination of Cartier divisors. An $\mathbb{R}$-Cartier divisor D is *nef* if $D \cdot \Sigma \geq 0$ for any curve $\Sigma \subset X$. An $\mathbb{R}$-Cartier divisor D is *ample* if it is $\mathbb{R}$-linearly equivalent to a positive linear combination of ample divisors (in the usual sense). An $\mathbb{R}$-Cartier divisor D is *big* if $D \sim_{\mathbb{R}} A + B$, where A is ample and $B \geq 0$. A $\mathbb{Q}$-Cartier divisor D is a *general ample* $\mathbb{Q}$-*divisor* if there is an integer $m > 0$ such that mD is very ample and $mD \in |mD|$ is very general.

A *log pair* (X, Δ) is a normal variety X and an $\mathbb{R}$-Weil divisor $\Delta \geq 0$ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. We say that a log pair (X, Δ) is *log smooth*, if X is smooth and the support of Δ is a divisor with global normal crossings. A projective birational morphism $g \colon Y \longrightarrow X$ is a *log resolution* of the pair (X, Δ) if X is smooth and the inverse image of Δ union the exceptional locus is a divisor with global normal crossings. Note that in the definition of log resolution we place no requirement that the indeterminacy locus of g is contained in the locus where the pair (X, Δ) is not log smooth. If V is a linear system on X, a *log resolution of* V *and* (X, Δ) is a log resolution of the pair (X, Δ) such that if $|M| + F$ is the decomposition of g^*V into its mobile and fixed parts, then $|M|$ is base point free and F union the exceptional locus union the strict transform of Δ is a divisor with simple normal crossings support. If g is a log resolution, then we may write

$$K_Y + \Gamma = g^*(K_X + \Delta) + E,$$

where $\Gamma \geq 0$ and $E \geq 0$ have no common components, $g_*\Gamma = \Delta$ and E is g-exceptional. Note that this decomposition is unique. The *log discrepancy* of a divisor F over X

$$a(X, \Delta, F) = 1 + \operatorname{mult}_F(E - \Gamma).$$

Note that with this definition, a component F of Δ with coefficient b has log discrepancy $1 - b$. The log discrepancy does not depend on the choice of model Y, so that the log discrepancy is also a function defined on valuations. A *non kawamata log terminal place* is any valuation of log discrepancy at most zero and the centre of a non kawamata log terminal place is called a *non kawamata log terminal centre*. A *non terminal centre* is any centre of log discrepancy at most one. Note that every divisor on X is by definition a non terminal centre, so the only interesting non terminal centres are of codimension at least two.

The pair (X, Δ) is *kawamata log terminal* if there are no non kawamata log terminal centres. We say that the pair (X, Δ) is *purely log terminal* (respectively *canonical* or *terminal*) if the log discrepancy of any exceptional divisor is greater than zero (respectively at least one or greater than one). We say that the pair is *divisorially log terminal* if there is a log resolution $g\colon Y \longrightarrow X$ such that all exceptional divisors $E \subset Y$ have log discrepancy greater than zero.

3. Preliminary results

In this section we recall several results about finitely generated algebras and in particular we will give a proof of Shokurov's result that the pl-flip exists if and only if the restricted algebra is finitely generated.

Definition 3.1. *Let X be a normal variety, S be a prime divisor and B an integral Weil divisor which is $\mathbb{Q}$-Cartier and whose support does not contain S. The* ***restricted algebra*** *$R_S(X, B)$ is the image of the homomorphism $R(X, B) \longrightarrow R(S, B|_S)$.*

We remark that as B is $\mathbb{Q}$-Cartier then $B|_S$ is a well defined $\mathbb{Q}$-Cartier divisor on S.

Theorem 3.2. *Let $f\colon X \longrightarrow Z$ be a pl-flipping contraction with respect to (X, Δ). Pick an integer k such that $k(K_X + \Delta)$ is Cartier.*

Then

1. *The flip of f exists if and only if the flip of f exists locally over Z.*
2. *If $Z = \operatorname{Spec} A$ is affine then the flip $f^+\colon X^+ \longrightarrow Z$ exists if and only if the restricted algebra $R_S(X, k(K_X + \Delta))$ is a finitely generated A-algebra.*

Definition 3.3. *Let A be a ring, and let R be any graded A-algebra. A* ***truncation*** *of R is any A-algebra of the form*

$$R_{(d)} = \bigoplus_{m \in \mathbb{N}} R_{md},$$

for a positive integer d.

We start with the following well known result:

Lemma 3.4. *Let R be a graded algebra which is an integral domain and let d be a positive integer.*

Then R is finitely generated if and only if $R_{(d)}$ is finitely generated.

Proof. Suppose that R is finitely generated. It is easy to write down an action of the cyclic group $\mathbb{Z}_d$ on R so that the invariant ring is $R_{(d)}$. Thus $R_{(d)}$ is finitely generated by the Theorem of E. Noether which states that the ring of invariants of a finitely generated ring under the action of a finite group is finitely generated.

Suppose now that $R_{(d)}$ is finitely generated. Let $f \in R_i$. Then f is a root of the monic polynomial $x^d - f^d \in R_{(d)}[x]$. It follows that R is integral over $R_{(d)}$ and the result follows by another Theorem of E. Noether on finiteness of integral closures. $\square$

Lemma 3.5. *Let S be a normal prime divisor on X and let B an integral Weil divisor which is $\mathbb{Q}$-Cartier and whose support does not contain S.*

- *If $R(X, B)$ is finitely generated then $R_S(X, B)$ is finitely generated.*
- *If $S \sim B$ and $R_S(X, B)$ is finitely generated then $R(X, B)$ is finitely generated.*

Proof. Since there is a surjective homomorphism $\phi\colon R(X, B) \longrightarrow R_S(X, B)$, it is clear that if $R(X, B)$ is finitely generated then $R_S(X, B)$ is finitely generated.

Suppose now that $R_S(X, B)$ is finitely generated and $S \sim B$. Then there is a rational function g_1 such that $(g_1) = S - B$. If we consider the elements of $R(X, B)_m$ as rational functions, then a rational function g belongs to $R(X, B)_m$ if and only if $(g) + mB \geq 0$. But if g is in the kernel of ϕ, then there is a divisor $S' \geq 0$ such that $(g) + mB = S + S'$. It follows that $(g/g_1) + (m-1)B = S'$ so that $g/g_1 = h \in R(X, B)_{m-1}$. But then the kernel of ϕ is the principal ideal generated by g_1. $\square$

Proof of (3.2). It is well known that the flip $f^+\colon X^+ \longrightarrow Z$ exists if and only if the sheaf of graded $\mathcal{O}_Z$-algebras

$$\bigoplus_{m \in \mathbb{N}\,:\,k|m} f_*\mathcal{O}_X(m(K_X + \Delta)),$$

is finitely generated, cf. [8, 6.4]. Since this can be checked locally, this gives (1).

If $Z = \operatorname{Spec} A$ is affine it suffices to check that $R(X, k(K_X + \Delta))$ is a finitely generated A-algebra. Since the relative Picard number is one, there are real numbers a and b such that $a(K_X+\Delta)$ and bS are numerically equivalent over Z. As both $-(K_X+\Delta)$ and $-S$ are ample $\mathbb{Q}$-divisors we may assume that a and b are both positive integers. Moreover, as $a(K_X+\Delta)-bS$ is numerically trivial over Z, it is semi-ample over Z by the base point free theorem. In particular, we may replace numerical equivalence by linear equivalence,

$$a(K_X+\Delta)\sim_Z bS.$$

But then there is a rational function g and a divisor D on Z such that

$$a(K_X+\Delta)=bS+f^*D+(g).$$

As any line bundle on a quasi-projective variety is locally trivial, possibly passing to an open subset of Z, and using (1), we may assume that $D\sim 0$, so that

$$a(K_X+\Delta)\sim bS.$$

By (3.4) it follows that $R(X,k(K_X+\Delta))$ is finitely generated if and only if $R(X,S)$ is finitely generated. Since Z is affine and f is small, S is mobile (that is the fixed divisor is empty) so that $S\sim S'$ where $S'\geq 0$ is a divisor whose support does not contain S. By (3.5), $R(X,S)$ is finitely generated if and only if $R_S(X,S')$ is finitely generated. Since $a(K_X+\Delta)|_S\sim bS'|_S$ the result follows by (3.4). □

4. Multiplier ideal sheaves

The main result of this section is:

Theorem 4.1. *Let $\pi\colon X\longrightarrow Z$ be a projective morphism to a normal affine variety Z, where $(X,\Delta=S+A+B)$ is a log pair, $S=\llcorner\Delta\lrcorner$ is irreducible, (X,S) is log smooth, and both $A\geq 0$ and $B\geq 0$ are $\mathbb{Q}$-divisors. Let k be any positive integer and $0\leq\Phi\leq\Omega=(\Delta-S)|_S$ be any divisor such that both $k(K_S+\Phi)$ and $k(K_X+\Delta)$ are Cartier. Let $C=A/k$.*

If there is an integer $l>1$ and an integral divisor $P\geq 0$ such that lA is Cartier, $C-\frac{(k-1)}{m}P$ is ample, $(X,\Delta+\frac{k-1}{m}P)$ is purely log terminal and

$$l|k(K_S+\Phi)|+m(\Omega-\Phi)+(mC+P)|_S\subset|m(K_X+\Delta+C)+P|_S,$$

where $m=kl$, then

$$|k(K_S+\Phi)|+k(\Omega-\Phi)\subset|k(K_X+\Delta)|_S.$$

See also [14] and [16] for related statements. To prove (4.1), we need a variant of multiplier ideal sheaves:

Definition-Lemma 4.2. *Let (X, Δ) be a log smooth pair where Δ is a reduced divisor and let V be a linear system whose base locus contains no non kawamata log terminal centres of (X, Δ). Let $\mu\colon Y \longrightarrow X$ be a log resolution of V and (X, Δ) and let F be the fixed divisor of the linear system μ^*V. Let $K_Y + \Gamma = \mu^*(K_X + \Delta) + E$ where $\Gamma = \sum P_i$ is the sum of the divisors on Y of log discrepancy zero.*

*Then for any real number $c \geq 0$, define the **multiplier ideal sheaf***

$$\mathcal{J}_{\Delta, c\cdot V} := \mu_* \mathcal{O}_Y(E - \llcorner cF \lrcorner).$$

If $\Delta = 0$ we will write $\mathcal{J}_{c\cdot V}$ and if $D = cG$, where $G > 0$ is a Cartier divisor, we define

$$\mathcal{J}_{\Delta, D} := \mathcal{J}_{\Delta, c\cdot V},$$

where $V = \{G\}$.

Proof. We have to show that the definition of the multiplier ideal sheaf is independent of the choice of log resolution. Let $\mu\colon Y \longrightarrow X$ and $\mu'\colon Y' \longrightarrow X$ be two log resolutions of (X, Δ) and V. We may assume that μ' factors through μ via a morphism $\nu\colon Y' \longrightarrow Y$. Then $F' = \nu^* F$ as $\mu^*V - F$ is free, and

$$\begin{aligned}
E' - cF' &= K_{Y'} + \Gamma' - \mu'^*(K_X + \Delta) - cF' \\
&= K_{Y'} + \Gamma' - \nu^*(K_Y + \Gamma - E + cF) \\
&= \nu^*(E - \llcorner cF \lrcorner) + K_{Y'} + \Gamma' - \nu^*(K_Y + \Gamma + \{cF\}) \\
&= \nu^*(E - \llcorner cF \lrcorner) + G.
\end{aligned}$$

Since $(Y, \Gamma + E + F)$ is log smooth, it follows that $(Y, \Gamma + \{cF\})$ is log canonical and has the same non kawamata log terminal places as (Y, Γ) and hence as (X, Δ). Thus $\ulcorner G \urcorner \geq 0$ and since $\nu_*(K_{Y'} + \Gamma') = K_Y + \Gamma$, $\ulcorner G \urcorner$ is ν-exceptional. Then

$$\begin{aligned}
\mu'_* \mathcal{O}_{Y'}(E' - \llcorner cF' \lrcorner) &= \mu_*(\nu_* \mathcal{O}_{Y'}(E' - \llcorner cF' \lrcorner)) \\
&= \mu_*(\nu_* \mathcal{O}_{Y'}(\nu^*(E - \llcorner cF \lrcorner) + \ulcorner G \urcorner)) \\
&= \mu_* \mathcal{O}_Y(E - \llcorner cF \lrcorner).
\end{aligned}$$

□

We need to develop a little of the theory of multiplier ideal sheaves.

Lemma 4.3. *Let (X, Δ) be a log smooth pair where Δ is reduced, let V be a linear system whose base locus contains no non kawamata log terminal centres of (X, Δ) and let $G \geq 0$ and $D \geq 0$ be $\mathbb{Q}$-Cartier divisors whose supports contain no non kawamata log terminal centres of (X, Δ).*

Then

1. *$\mathcal{J}_{\Delta, D} = \mathcal{O}_X$ if and only if $(X, \Delta + D)$ is divisorially log terminal and $\llcorner D \lrcorner = 0$.*

(2) *If* $0 \leq \Delta' \leq \Delta$ *then* $\mathcal{J}_{\Delta,c\cdot V} \subset \mathcal{J}_{\Delta',c\cdot V}$. *In particular,* $\mathcal{J}_{\Delta,c\cdot V} \subset \mathcal{J}_{c\cdot V} \subset \mathcal{O}_X$.
(3) *If* $\Sigma \geq 0$ *is a Cartier divisor,* $D - \Sigma \leq G$ *and* $\mathcal{J}_{\Delta,G} = \mathcal{O}_X$ *then* $\mathcal{I}_\Sigma \subset \mathcal{J}_{\Delta,D}$.

Proof. (1) follows easily from the definitions.

(2) follows from the fact that $a(P, X, \Delta') \geq a(P, X, \Delta)$ for all divisors P on Y.

To see (3), note that we are free to replace Σ by a smaller Cartier divisor Σ' such that $D - \Sigma' \leq G$. As D contains no non kawamata log terminal centres of (X, Δ) we may therefore assume that Σ contains no non kawamata log terminal centres of (X, Δ) as well. Notice that as Σ is Cartier and $\mathcal{J}_{\Delta,G} = \mathcal{O}_X$, we have

$$\mathcal{J}_{\Delta,G}(-\Sigma) = \mathcal{O}_X(-\Sigma) = \mathcal{I}_\Sigma.$$

But since $D \leq G + \Sigma$ and Σ contains no non kawamata log terminal centres of (X, Δ) we also have

$$\mathcal{J}_{\Delta,G}(-\Sigma) = \mathcal{J}_{\Delta,G+\Sigma} \subset \mathcal{J}_{\Delta,D}. \qquad \square$$

We have the following extension of (9.5.1) of [9] or (2.4.2) of [16]:

Lemma 4.4. *Let* $\pi\colon X \longrightarrow Z$ *be a projective morphism to a normal affine variety* Z. *Let* (X, Δ) *be a log smooth pair where* Δ *is reduced, let* S *be a component of* Δ, *let* $D \geq 0$ *be a* $\mathbb{Q}$*-Cartier divisor whose support does not contain any non kawamata log terminal centres of* (X, Δ) *and let* $\Theta = (\Delta - S)|_S$. *Let* N *be a Cartier divisor.*

(1) *There is a short exact sequence*

$$0 \longrightarrow \mathcal{J}_{\Delta-S,D+S} \longrightarrow \mathcal{J}_{\Delta,D} \longrightarrow \mathcal{J}_{\Theta,D|_S} \longrightarrow 0.$$

(2) *(Nadel Vanishing) If* $N - D$ *is ample then*

$$H^i(X, \mathcal{J}_{\Delta,D}(K_X + \Delta + N)) = 0,$$

for $i > 0$.

(3) *If* $N - D$ *is ample then*

$$H^0(X, \mathcal{J}_{\Delta,D}(K_X + \Delta + N)) \longrightarrow H^0(S, \mathcal{J}_{\Theta,D|_S}(K_X + \Delta + N)),$$

is surjective.

Proof. By the resolution lemma of [15], we may find a log resolution $\mu\colon Y \longrightarrow X$ of $(X, \Delta + D)$ which is an isomorphism over the generic point of each non kawamata log terminal centre of (X, Δ). If T is the strict transform of S then we have a short exact sequence

$$0 \longrightarrow \mathcal{O}_Y(E - \llcorner \mu^* D \lrcorner - T) \longrightarrow \mathcal{O}_Y(E - \llcorner \mu^* D \lrcorner) \longrightarrow \mathcal{O}_T(E - \llcorner \mu^* D \lrcorner) \longrightarrow 0,$$

where E is defined in (4.2). Now $\mu_*\mathcal{O}_Y(E - \llcorner\mu^* D\lrcorner) = \mathcal{J}_{\Delta,D}$. If Γ is the sum of the divisors of log discrepancy zero then

$$E - \mu^* D = (K_Y + \Gamma) - \mu^*(K_X + \Delta + D).$$

But then

$$E - \mu^* D - T = (K_Y + \Gamma - T) - \mu^*(K_X + \Delta - S + (D + S)),$$

so that

$$\mu_*\mathcal{O}_Y(E - \llcorner\mu^* D\lrcorner - T) = \mathcal{J}_{\Delta - S, D+S},$$

and

$$(E - \mu^* D)|_T = K_T + (\Gamma - T)|_T - \mu^*(K_S + \Theta + D|_S),$$

so that

$$\mu_*\mathcal{O}_T(E - \llcorner\mu^* D\lrcorner) = \mathcal{J}_{\Theta, D|_S}.$$

Since $(Y, \Gamma + \mu^* D)$ is log smooth and Γ and $\mu^* D$ have no common components, $(Y, \Gamma + \{\mu^* D\})$ is divisorially log terminal. Choose $F \geq 0$ such that $-F$ is μ-ample whose coefficients are sufficiently small so that $K_Y + \Gamma + \{\mu^* D\} + F$ is divisorially log terminal. As

$$E - \llcorner\mu^* D\lrcorner - T - (K_Y + \Gamma - T + \{\mu^* D\} + F) = -\mu^*(K_X + \Delta + D) - F,$$

is μ-ample, Kawamata-Viehweg vanishing implies that

$$R^1\mu_*\mathcal{O}_Y(E - \llcorner\mu^* D\lrcorner - T) = 0,$$

and this gives (1).

Similarly, Kawamata-Viehweg vanishing implies that

$$R^i\mu_*\mathcal{O}_Y(\mu^*(K_X + \Delta + N) + E - \llcorner\mu^* D\lrcorner) = 0,$$

for $i > 0$. As $N - D$ is ample then, possibly replacing F by a small multiple, we may assume that $\mu^*(N - D) - F$ is ample. As

$$\mu^*(K_X + \Delta + N) + E - \llcorner\mu^* D\lrcorner - (K_Y + \Gamma + \{\mu^* D\} + F) = \mu^*(N - D) - F,$$

is ample, Kawamata-Viehweg vanishing implies that

$$H^i(Y, \mathcal{O}_Y(\mu^*(K_X + \Delta + N) + E - \llcorner\mu^* D\lrcorner)) = 0,$$

for $i > 0$. Since the Leray-Serre spectral sequence degenerates, this gives (2), and (3) follows from (2). $\square$

Proof of (4.1). Since $(X, \Delta + \frac{k-1}{m}P)$ is purely log terminal, $(S, \Omega + \frac{k-1}{m}P|_S)$ is kawamata log terminal and S is not contained in the support of A or P. If $\Sigma \in |k(K_S + \Phi)|$ then we may pick a divisor

$$G \in |m(K_X + \Delta + C) + P| \quad \text{such that} \quad G|_S = l\Sigma + m(\Omega - \Phi + C|_S) + P|_S.$$

Let

$$\Lambda = \frac{k-1}{m}G + B \qquad \text{and} \qquad N = k(K_X + \Delta) - K_X - S.$$

As the support of the $\mathbb{Q}$-divisor $\Lambda \geq 0$ does not contain S and by assumption

$$N - \Lambda \sim_{\mathbb{Q}} C - \frac{k-1}{m} P,$$

is ample, (4.4) implies that sections of $H^0(S, \mathcal{J}_{\Lambda|_S}(k(K_S + \Omega)))$ extend to sections of $H^0(X, \mathcal{O}_X(k(K_X + \Delta)))$. Now

$$\begin{aligned}
&\Lambda|_S - (\Sigma + k(\Omega - \Phi)) \\
&= \frac{k-1}{m}(l\Sigma + m(\Omega - \Phi + C|_S) + P|_S) + B|_S - (\Sigma + k(\Omega - \Phi)) \\
&\leq \Omega + \frac{k-1}{m} P|_S.
\end{aligned}$$

As $(S, \Omega + \frac{k-1}{m}P|_S)$ is kawamata log terminal, $\mathcal{J}_{\Omega + \frac{k-1}{m}P|_S} = \mathcal{O}_S$ and we are done by (3) of (4.3). $\square$

5. Asymptotic multiplier ideal sheaves

Definition 5.1. *Let X be a normal variety and let D be a divisor. An **additive sequence of linear systems associated to** D is a sequence $V_\bullet$, such that $V_m \subset \mathbb{P}(H^0(X, \mathcal{O}_X(mD)))$ and*

$$V_i + V_j \subset V_{i+j}.$$

Definition-Lemma 5.2. *Suppose that (X, Δ) is log smooth, where Δ is reduced and let $V_\bullet$ be an additive sequence of linear systems associated to a divisor D. Assume that there is an integer $k > 0$ such that no non kawamata log terminal centre of (X, Δ) is contained in the base locus of V_k.*

If c is a positive real number and p and q are positive integers divisible by k then

$$\mathcal{J}_{\Delta, \frac{c}{p} \cdot V_p} \subset \mathcal{J}_{\Delta, \frac{c}{q} \cdot V_q} \qquad \forall q \text{ divisible by } p.$$

*In particular the **asymptotic multiplier ideal sheaf** of $V_\bullet$*

$$\mathcal{J}_{\Delta, c \cdot V_\bullet} = \bigcup_{p > 0} \mathcal{J}_{\Delta, \frac{c}{p} \cdot V_p},$$

is given by $\mathcal{J}_{\Delta, c\cdot V_\bullet} = \mathcal{J}_{\Delta, \frac{c}{p} \cdot V_p}$, for p sufficiently large and divisible. If we take $V_m = |mD|$ the complete linear system, then define

$$\mathcal{J}_{\Delta, c \cdot \|D\|} = \mathcal{J}_{\Delta, c \cdot V_\bullet},$$

and if S is a component of Δ and we take $W_m = |mD|_S$, then define

$$\mathcal{J}_{\Theta, c \cdot \|D\|_S} = \mathcal{J}_{\Theta, c \cdot W_\bullet},$$

where $\Theta = (\Delta - S)|_S$.

Proof. If p divides q then pick a common log resolution $\mu\colon Y \longrightarrow X$ of V_p, V_q and (X,Δ) and note that

$$\frac{1}{q}F_q \leq \frac{1}{p}F_p,$$

where F_p is the fixed locus of μ^*V_p and F_q is the fixed locus of μ^*V_q. Therefore $\mathcal{J}_{\Delta,\frac{c}{p}\cdot V_p} \subset \mathcal{J}_{\Delta,\frac{c}{q}\cdot V_q}$. The equality $\mathcal{J}_{\Delta,c\cdot V_\bullet} = \mathcal{J}_{\Delta,\frac{c}{p}\cdot V_p}$, now follows as X is Noetherian. □

We are now ready to state the main result of this section:

Theorem 5.3. *Let $\pi\colon X \longrightarrow Z$ be a projective morphism to a normal affine variety Z. Suppose that $(X,\Delta = S+B)$ is log smooth and purely log terminal of dimension n, where $S = \llcorner \Delta \lrcorner$ is irreducible and let k be a positive integer such that $D = k(K_X+\Delta)$ is integral. Let A be any ample $\mathbb{Q}$-divisor on X. Let q and r be any positive integers such that $Q = qA$ is very ample, rA is Cartier and $(j-1)K_X + \Xi + rA$ is ample for every Cartier divisor $0 \leq \Xi \leq j\ulcorner \Delta \urcorner$ and every integer $1 \leq j \leq k+1$.*

If the stable base locus of D does not contain any non kawamata log terminal centres of $(X, \ulcorner \Delta \urcorner)$, then

$$\mathcal{J}_{\|mD|_S\|} \subset \mathcal{J}_{\Theta,\|mD+P\|_S} \qquad \text{for all} \qquad m \in \mathbb{N},$$

where $\Theta = \ulcorner B \urcorner|_S$, $p = qn + r$ and $P = pA$. Moreover, all sections of the linear system determined by $\mathcal{J}_{\|mD|_S\|}(mD+P)$ lift, that is we have

$$\pi_*\mathcal{J}_{\|mD|_S\|}(mD+P) \subset \operatorname{Im}\left(\pi_*\mathcal{O}_X(mD+P) \to \pi_*\mathcal{O}_S(mD+P)\right),$$

for all $m \in \mathbb{N}$.

We will need some results about the sheaves $\mathcal{J}_{\Delta,c\cdot V_\bullet}$, most of which are easy generalisations of the corresponding facts for the usual asymptotic multiplier ideal sheaves.

Lemma 5.4. *Let $\pi\colon X \longrightarrow Z$ be a projective morphism to a normal affine variety Z and let D be a $\mathbb{Q}$-Cartier divisor. Suppose that (X,Δ) is log smooth, Δ is reduced and the stable base locus of D contains no non kawamata log terminal centre of (X,Δ). Then*

(1) *for any real numbers $0 < c_1 \leq c_2$ there is a natural inclusion*

$$\mathcal{J}_{\Delta,c_2\cdot\|D\|} \subset \mathcal{J}_{\Delta,c_1\cdot\|D\|},$$

and

(2) *if D is Cartier and S is a component of Δ, then the image of the map*

$$\pi_*\mathcal{O}_X(D) \longrightarrow \pi_*\mathcal{O}_S(D),$$

is contained in $\pi_*\mathcal{J}_{\Theta,\|D\|_S}(D)$ where $\Theta = (\Delta - S)|_S$.

Proof. (1) is immediate from the definitions.

Suppose that D is Cartier. Pick an integer p such that

$$\mathcal{J}_{\Theta,\|D\|_S} = \mathcal{J}_{\Theta,\frac{1}{p}\cdot|pD|_S},$$

and a log resolution $\mu\colon Y \longrightarrow X$ of $|D|$, $|pD|$ and (X,Δ). Let T be the strict transform of S, let F_1 be the fixed locus of $\mu^*|D|$ and let F_p be the fixed locus of $\mu^*|pD|$. We have

$$(\pi\circ\mu)_*\mathcal{O}_Y(\mu^*D - F_1) = \pi_*\mathcal{O}_X(D) = (\pi\circ\mu)_*\mathcal{O}_Y(E+\mu^*D).$$

The first equality follows by definition of F_1 and the second follows as $E \geq 0$ is exceptional. As there are inequalities

$$\mu^*D - F_1 \leq \mu^*D - \llcorner F_p/p \lrcorner \leq E + \mu^*D - \llcorner F_p/p \lrcorner \leq E + \mu^*D,$$

the image of $\pi_*\mathcal{O}_X(D)$ is equal to the image of

$$(\pi\circ\mu)_*\mathcal{O}_Y(E+\mu^*D - \llcorner F_p/p \lrcorner).$$

Thus the image of $\pi_*\mathcal{O}_X(D)$ is contained in

$$(\pi\circ\mu)_*\mathcal{O}_T(E+\mu^*D - \llcorner F_p/p \lrcorner) = \pi_*\mathcal{J}_{\Theta,\|D\|_S}(D). \qquad \square$$

Lemma 5.5. *Let $\pi\colon X \longrightarrow Z$ be a projective morphism to a normal affine variety Z and let D be a Cartier divisor. Suppose that (X,Δ) is log smooth and Δ is reduced. Let S be a component of Δ and $\Theta = (\Delta - S)|_S$.*

If $\mathbf{B}_+(D)$ contains no non kawamata log terminal centres of (X,Δ) then the image of the map

$$\pi_*\mathcal{O}_X(K_X+\Delta+D) \longrightarrow \pi_*\mathcal{O}_S(K_S+\Theta+D),$$

contains

$$\pi_*\mathcal{J}_{\Theta,\|D\|_S}(K_S+\Theta+D).$$

Proof. Pick an integer $p > 1$ such that

$$\mathcal{J}_{\Theta,\|D\|_S} = \mathcal{J}_{\Theta,\frac{1}{p}\cdot|pD|_S},$$

and there is a divisor $A+B \in |pD|$ where $A \geq 0$ is a general very ample divisor and $B \geq 0$ contains no non kawamata log terminal centres of (X,Δ). By the resolution lemma of [15], we may find a log resolution $\mu\colon Y \longrightarrow X$ of $|pD|$ and of (X,Δ) which is an isomorphism over every non kawamata log terminal centre of (X,Δ). Let F_p be the fixed divisor of $\mu^*|pD|$, $M_p = p\mu^*D - F_p$ and let Γ and T be the strict transforms of Δ and S. We have a short exact sequence

$$0 \longrightarrow \mathcal{O}_Y(G-T) \longrightarrow \mathcal{O}_Y(G) \longrightarrow \mathcal{O}_T(G) \longrightarrow 0,$$

where $G = K_Y + \Gamma + \mu^*D - \llcorner F_p/p \lrcorner$. As μ^*A is base point free and $\mu^*(A+B) \in \mu^*|pD|$, the divisor $C = \mu^*B - F_p \geq 0$. Note that

$M_p - C \sim \mu^* A$. As no component of C is a component of Γ, we may pick $0 < \delta \leq 1/p$ and an exceptional $\mathbb{Q}$-divisor $F \geq 0$ such that $(Y, \Gamma - T + \{F_p/p\} + \delta(C+F))$ is divisorially log terminal and $\mu^* A - F$ is ample. As $|M_p|$ is free, M_p/p is nef and so

$$
\begin{aligned}
G - T - (K_Y + \Gamma - T + \{\frac{1}{p} F_p\} + \delta(C+F)) &= \frac{1}{p} M_p - \delta(C+F) \\
&\sim_{\mathbb{Q}} (\frac{1}{p} - \delta) M_p + \delta(\mu^* A - F),
\end{aligned}
$$

is ample. In particular Kawamata-Viehweg vanishing implies that $R^1 \phi_* \mathcal{O}_Y(G - T) = 0$ where $\phi = \pi \circ \mu$. Therefore the homomorphism

$$
\pi_* \mathcal{O}_X(K_X + \Delta + D) \supset \phi_* \mathcal{O}_Y(G) \longrightarrow \phi_* \mathcal{O}_T(G) = \pi_* \mathcal{J}_{\Theta, \|D\|_S}(K_S + \Theta + D),
$$

is surjective. $\square$

Theorem 5.6. *Let $\pi \colon X \longrightarrow Z$ be a projective morphism, where Z is affine and X is a smooth variety of dimension n.*

If D is a Cartier divisor whose stable base locus is a proper subset of X, A is an ample Cartier divisor and H is a very ample divisor then $\mathcal{J}_{\|D\|}(D + K_X + A + nH)$ is globally generated.

Proof. Pick an integer $p > 0$ such that if $pB \in |pD|$ is a general element, then

$$
\mathcal{J}_{\|D\|} = \mathcal{J}_{\frac{1}{p} \cdot |pD|} = \mathcal{J}_B.
$$

Then by (2) of (4.4), $H^i(X, \mathcal{J}_{\|D\|}(D + K_X + A + mH)) = 0$ for all $i > 0$ and $m \geq 0$ and we may apply (5.7). $\square$

The following result is well known to experts. We include a proof for the benefit of the reader:

Lemma 5.7. *Let $\pi \colon X \longrightarrow Z$ be a projective morphism where X is smooth of dimension n, Z is affine and let H be a very ample divisor.*

If $\mathcal{F}$ is any coherent sheaf such that $H^i(X, \mathcal{F}(mH)) = 0$, for $i > 0$ and for all $m \geq -n$ then $\mathcal{F}$ is globally generated.

Proof. Pick $x \in X$. Let $\mathcal{T} \subset \mathcal{F}$ be the torsion subsheaf supported at x, and let $\mathcal{G} = \mathcal{F}/\mathcal{T}$. Then $H^i(X, \mathcal{G}(mH)) = 0$ for $i > 0$ and for all $m \geq -n$ and $\mathcal{F}$ is globally generated if and only if $\mathcal{G}$ is globally generated. Replacing $\mathcal{F}$ by $\mathcal{G}$ we may therefore assume that $\mathcal{T} = 0$.

Pick a general element $Y \in |H|$ containing x. As $\mathcal{T} = 0$ there is an exact sequence

$$
0 \longrightarrow \mathcal{F}(-Y) \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0,
$$

where $\mathcal{G} = \mathcal{F} \otimes \mathcal{O}_Y$. As $H^i(Y, \mathcal{G}(mH)) = 0$, for $i > 0$ and for all $m \geq -(n-1)$, $\mathcal{G}$ is globally generated by induction on the dimension. As $H^1(X, \mathcal{F}(-Y)) = 0$ it follows that $\mathcal{F}$ is globally generated. $\square$

Proof of (5.3). We follow the argument of [5] which in turn is based on the ideas of [7], [14] and [17].

We proceed by induction on m. The statement is clear for $m = 0$, and so it suffices to show that

$$\mathcal{J}_{\|(m+1)D|_S\|} \subset \mathcal{J}_{\Theta, \|(m+1)D+P\|_S},$$

assuming that

$$\mathcal{J}_{\|tD|_S\|} \subset \mathcal{J}_{\Theta, \|tD+P\|_S} \qquad \text{for all} \qquad t \leq m.$$

If $\Delta = \sum \delta_i \Delta_i$, where each Δ_i is a prime divisor, then for any $1 \leq s \leq k$, put

$$\Delta^s = \sum_{i\,:\,\delta_i > (k-s)/k} \Delta_i.$$

We have

- each Δ^s is integral,
- $S = \Delta^1 \leq \Delta^2 \leq \cdots \leq \Delta^k = \ulcorner \Delta \urcorner$, and
- $\Delta = \dfrac{1}{k} \displaystyle\sum_{s=1}^{k} \Delta^s$,

and these properties uniquely determine the divisors Δ^s. We let $\Delta^{k+1} = \ulcorner \Delta \urcorner$. We recursively define integral divisors $D_{\leq s}$ by the rule

$$D_{\leq s} = \begin{cases} 0 & \text{if } s = 0 \\ K_X + \Delta^s + D_{\leq s-1} & 1 \leq s \leq k. \end{cases}$$

Note that $D_{\leq k} = D$. By (1) of (5.4) there is an inclusion

$$\mathcal{J}_{\|(m+1)D|_S\|} \subset \mathcal{J}_{\|mD|_S\|},$$

and so it suffices to prove that there are inclusions

$$(\star) \qquad \mathcal{J}_{\|mD|_S\|} \subset \mathcal{J}_{\Theta^{s+1}, \|mD+D_{\leq s}+P\|_S},$$

for $0 \leq s \leq k$, where $\Theta^i = (\Delta^i - S)|_S$ for $1 \leq i \leq k+1$. Thus $\Theta^k = \Theta^{k+1} = \Theta$ and $\Theta^1 = 0$.

We proceed by induction on s. Now

$$\mathcal{J}_{\|mD|_S\|} \subset \mathcal{J}_{\Theta, \|mD+P\|_S} \subset \mathcal{J}_{\Theta^1, \|mD+P\|_S}.$$

The first inclusion holds by assumption and since $\Theta^1 \leq \Theta$, (2) of (4.3) implies the second inclusion. Thus $(\star)$ holds when $s = 0$.

Now suppose that $(\star)$ holds for $s \leq t-1$. Note that

$$\begin{aligned} mD + D_{\leq t} + P &= K_X + \Delta^t + (D_{\leq t-1} + P) + mD \\ &= mD + K_X + (\Delta^t + D_{\leq t-1} + rA) + nQ, \end{aligned} \tag{$\dagger$}$$

where, by assumption, both $D_{\leq t-1} + P$ and $\Delta^t + D_{\leq t-1} + rA$ are ample for any $1 \leq t \leq k+1$. In particular $\mathbf{B}_+(mD + D_{\leq t-1} + P)$ contains no log non terminal centres of $(X, \ulcorner \Delta \urcorner)$. Then

$$\begin{aligned} \pi_* \mathcal{J}_{\|mD|_S\|}(mD + D_{\leq t} + P) &\subset \pi_* \mathcal{J}_{\Theta^t, \|mD + D_{\leq t-1} + P\|_S}(mD + D_{\leq t} + P) \\ &\subset \operatorname{Im}\left(\pi_* \mathcal{O}_X(mD + D_{\leq t} + P) \longrightarrow \pi_* \mathcal{O}_S(mD + D_{\leq t} + P)\right) \\ &\subset \pi_* \mathcal{J}_{\Theta^{t+1}, \|mD + D_{\leq t} + P\|_S}(mD + D_{\leq t} + P). \end{aligned}$$

The first inclusion holds as we are assuming $(\star)$ for $s = t-1$, the second inclusion holds by $(\dagger)$ and (5.5) and the last inclusion follows from (2) of (5.4). But $(\dagger)$ and (5.6) imply that

$$\mathcal{J}_{\|mD|_S\|}(mD + D_{\leq t} + P),$$

is generated by global sections and so

$$\mathcal{J}_{\|mD|_S\|} \subset \mathcal{J}_{\Theta^{t+1}, \|mD + D_{\leq t} + P\|_S}.$$

The inclusion

$$\pi_* \mathcal{J}_{\|mD|_S\|}(mD + P) \subset \operatorname{Im}\left(\pi_* \mathcal{O}_X(mD + P) \to \pi_* \mathcal{O}_S(mD + P)\right),$$

is part of the inclusions proved above when $s = k$. □

6. Lifting sections

Lemma 6.1. *Let $D \geq 0$ be a Cartier divisor on a normal variety X, and let $Z \subset X$ be an irreducible subvariety.*

Then

$$\liminf \frac{\operatorname{mult}_Z(|mD|)}{m} = \lim \frac{\operatorname{mult}_Z(|m!D|)}{m!}.$$

Proof. Note that if a divides b then

$$\frac{\operatorname{mult}_Z(|aD|)}{a} \geq \frac{\operatorname{mult}_Z(|bD|)}{b},$$

whence the result. □

The following is essentially proved in [4]; we include a proof for the benefit of the reader:

Lemma 6.2. *Let $D \subset X$ be a divisor on a smooth variety and Z a closed subvariety.*

If $\lim \operatorname{mult}_Z(|m!D|)/m! = 0$ then Z is not contained in $\mathbf{B}_-(D)$.

Proof. Let A be any ample divisor. Pick $l > 0$ such that $lA - K_X$ is ample. If $m > l$ is sufficiently divisible then $\mathcal{J}_{\|mD\|}(m(D+A))$ is globally generated by (5.6). But if $p > 0$ is sufficiently large and divisible and $D_{mp} \in |mpD|$ is general, then $\operatorname{mult}_Z D_{mp} = \operatorname{mult}_Z |mpD| < p$ and

$$\mathcal{J}_{\|mD\|} = \mathcal{J}_{(1/p)D_{mp}}.$$

But since $\operatorname{mult}_Z D_{mp}/p < 1$ it follows that $(X, D_{mp}/p)$ is kawamata log terminal, in a neighbourhood of the generic point of Z. Thus Z is not contained in the co-support of $\mathcal{J}_{\|mD\|}$ and so Z is not contained in the base locus of $m(D+A)$. $\square$

Theorem 6.3. *Let $\pi\colon X \longrightarrow Z$ be a projective morphism to a normal affine variety Z, where $(X, \Delta = S + A + B)$ is a purely log terminal pair, $S = \llcorner \Delta \lrcorner$ is irreducible, (X, S) is log smooth, $A \geq 0$ is a general ample $\mathbb{Q}$-divisor, $B \geq 0$ is a $\mathbb{Q}$-divisor and $(S, \Omega + A|_S)$ is canonical, where $\Omega = (\Delta - S)|_S$. Assume that the stable base locus of $K_X + \Delta$ does not contain S. Let $F = \lim F_{l!}$, where, for any positive and sufficiently divisible integer m, we let*

$$F_m = \operatorname{Fix}(|m(K_X + \Delta)|_S)/m.$$

If $\epsilon > 0$ is any rational number such that $\epsilon(K_X + \Delta) + A$ is ample and if Φ is any $\mathbb{Q}$-divisor on S and $k > 0$ is any integer such that

(1) *both $k\Delta$ and $k\Phi$ are Cartier, and*
(2) *$\Omega \wedge \lambda F \leq \Phi \leq \Omega$, where $\lambda = 1 - \epsilon/k$,*

then

$$|k(K_S + \Omega - \Phi)| + k\Phi \subset |k(K_X + \Delta)|_S.$$

Proof. By assumption $A = H/m$, where H is very ample and a very general element of $|H|$ and $m \geq 2$ is an integer. If $C = A/k$, then

$$A + (k-1)C = \frac{2k-1}{km} H,$$

and so

$$(X, \Delta + (k-1)C = S + \frac{2k-1}{km} H + B)$$

is purely log terminal, as

$$\frac{2k-1}{km} < 1.$$

On the other hand,

$$(S, \Omega + C|_S),$$

is canonical as we are even assuming that $(S, \Omega + A|_S)$ is canonical. Pick $\eta > \epsilon/k$ rational so that $\eta(K_X + \Delta) + C$ is ample and let $\mu =$

$1-\eta < \lambda = 1-\epsilon/k$. If $l > 0$ is any sufficiently divisible integer so that $O = l(\eta(K_X + \Delta) + C)$ is very ample, then

$$
\begin{aligned}
G_l &= \operatorname{Fix}(|l(K_X + \Delta + C)|_S)/l \\
&= \operatorname{Fix}(|l\mu(K_X + \Delta) + O|_S)/l \\
&\leq \operatorname{Fix}(|l\mu(K_X + \Delta)|_S)/l \\
&= \mu F_{\mu l}.
\end{aligned}
$$

Thus

$$\lim G_{l!} \leq \mu \lim F_{l!} = \mu F.$$

On the other hand (6.2) implies that there is a positive integer l such that every prime divisor on S which does not belong to the support of F does not belong to the base locus of $|l(K_X + \Delta + C)|$. Thus we may pick a positive integer l such that

- k divides l,
- lC is Cartier, and
- $G_l \leq \lambda F$.

Let $f\colon Y \longrightarrow X$ be a log resolution of the linear system $|l(K_X + \Delta + C)|$ and of $(X, \Delta + C)$. We may write

$$K_Y + \Gamma = f^*(K_X + \Delta + C) + E,$$

where $\Gamma \geq 0$ and $E \geq 0$ have no common components, $f_*\Gamma = \Delta + C$ and $f_*E = 0$. Then

$$H_l = \operatorname{Fix}(l(K_Y + \Gamma))/l = \operatorname{Fix}(lf^*(K_X + \Delta + C))/l + E.$$

If $\Xi = \Gamma - \Gamma \wedge H_l$ then $l(K_Y + \Xi)$ is Cartier and $\operatorname{Fix}(l(K_Y + \Xi))$ and Ξ share no common components. Since the mobile part of $|l(K_Y + \Xi)|$ is free and the support of $\operatorname{Fix}(l(K_Y + \Xi)) + \Xi$ has normal crossings it follows that the stable base locus of $K_Y + \Xi$ contains no non kawamata log terminal centres of $(Y, \ulcorner \Xi \urcorner)$ (which are nothing but the strata of $\ulcorner \Xi \urcorner$).

Let $H \geq 0$ be any ample divisor on Y. Pick positive integers m and q such that l divides m and $Q = qH$ is very ample. Let T be the strict transform of S, let $\Gamma_T = (\Gamma - T)|_T$ and let $\Xi_T = (\Xi - T)|_T$. If

$$\tau \in H^0(T, \mathcal{O}_T(m(K_T + \Xi_T))) = H^0(T, \mathcal{J}_{\|m(K_T+\Xi_T)\|}(m(K_T + \Xi_T))),$$

and $\sigma \in H^0(T, \mathcal{O}_T(Q))$ then

$$\sigma \cdot \tau \in H^0(T, \mathcal{J}_{\|m(K_T+\Xi_T)\|}(m(K_T + \Xi_T) + Q)).$$

On the other hand, if q is sufficiently large and divisible then by (5.3) $H^0(T, \mathcal{J}_{\|m(K_T+\Xi_T)\|}(m(K_T + \Xi_T) + Q))$ is contained in the image of

$$H^0(Y, \mathcal{O}_Y(m(K_Y + \Xi) + Q)) \longrightarrow H^0(T, \mathcal{O}_T(m(K_T + \Xi_T) + Q)).$$

Hence there is a fixed q such that whenever l divides m, we have

$$|m(K_T + \Xi_T)| + m(\Gamma_T - \Xi_T) + |Q|_T| \subset |m(K_Y + \Gamma) + Q|_T.$$

If $g = f|_T \colon T \longrightarrow S$ then $g_*\Gamma_T = \Omega + C|_S$ and since $g_*\Xi_T \leq \Omega + C|_S$ and $(S, \Omega + C|_S)$ is canonical, we have $|m(K_S + g_*\Xi_T)| = g_*|m(K_T + \Xi_T)|$. Therefore, applying g_*, we obtain

$$|m(K_S + g_*\Xi_T)| + m(\Omega + C|_S - g_*\Xi_T) + P|_S \subset |m(K_X + \Delta + C) + P|_S,$$

where $P = f_*Q$.

Since for every prime divisor L on S we have

$$\operatorname{mult}_L G_l = \operatorname{mult}_{L'} \operatorname{Fix}(|l(K_Y + \Gamma)|_T)/l = \operatorname{mult}_{L'} H_l|_T,$$

where L' is the strict transform of L on T, it follows that

$$g_*\Xi_T - C|_S = \Omega - \Omega \wedge G_l \geq \Omega - \Omega \wedge \lambda F \geq \Omega - \Phi \geq 0.$$

Therefore

$$|m(K_S + \Omega - \Phi)| + m\Phi + (mC + P)|_S \subset |m(K_X + \Delta + C) + P|_S,$$

for any m divisible by l. In particular if we pick m so that $C - \frac{k-1}{m}P$ is ample and $(X, \Delta + \frac{k-1}{m}P)$ is purely log terminal then the result follows by (4.1). $\square$

7. Rationality of the restricted algebra

In this section we will prove:

Theorem 7.1. *Assume Theorem F_{n-1}.*

Let $\pi \colon X \longrightarrow Z$ be a projective morphism to a normal affine variety Z, where $(X, \Delta = S + A + B)$ is a purely log terminal pair of dimension n, $S = \llcorner \Delta \lrcorner$ is irreducible, (X, S) is log smooth, $A \geq 0$ is a general ample $\mathbb{Q}$-divisor, $B \geq 0$ is a $\mathbb{Q}$-divisor and $(S, \Omega + A|_S)$ is canonical, where $\Omega = (\Delta - S)|_S$. Assume that the stable base locus of $K_X + \Delta$ does not contain S. Let $F = \lim F_{l!}$ where, for any positive and sufficiently divisible integer m, we let

$$F_m = \operatorname{Fix}(|m(K_X + \Delta)|_S)/m.$$

Then $\Theta = \Omega - \Omega \wedge F$ is rational. Moreover if both $k\Delta$ and $k\Theta$ are Cartier then

$$|k(K_S + \Theta)| + k(\Omega - \Theta) = |k(K_X + \Delta)|_S,$$

and

$$R_S(X, k(K_X + \Delta)) \simeq R(S, k(K_S + \Theta)).$$

Proof. Suppose that Θ is not rational. Let $V \subset \mathrm{WDiv}_{\mathbb{R}}(S)$ be the vector space spanned by the components of Θ. Then there is a constant $\delta > 0$ such that if $\Phi \in V$ and $\|\Phi - \Theta\| < \delta$ then $\Phi \geq 0$ has the same support as Θ and moreover, by (2) of Theorem F_{n-1}, if G is a prime divisor contained in the stable base locus of $K_S + \Theta$ then it is also contained in the stable base locus of $K_S + \Phi$.

If $l(K_X + \Delta)$ is Cartier and $\Theta_l = \Omega - \Omega \wedge F_l$ then

$$|l(K_X + \Delta)|_S \subset |l(K_S + \Theta_l)| + l(\Omega \wedge F_l).$$

Hence $\mathrm{Fix}(l(K_S + \Theta_l))$ does not contain any components of Θ_l. In particular the stable base locus of $K_S + \Theta_l$ does not contain any components of Θ_l. But we may pick $l > 0$ so that $\Theta_l \in V$ and $\|\Theta_l - \Theta\| < \delta$. It follows that no component of Θ is in the stable base locus of $K_S + \Theta$.

Let $W \subset V$ be the smallest rational affine space which contains Θ. (3) of Theorem F_{n-1} implies that there is a positive integer $r > 0$ and a positive constant $\eta > 0$ such that if $\Phi \in W$, $k\Phi/r$ is Cartier and $\|\Phi - \Theta\| < \eta$ then every component of $\mathrm{Fix}(k(K_S + \Phi))$ is in fact a component of the stable base locus of $K_S + \Theta$.

Pick a rational number $\epsilon > 0$ such that $\epsilon(K_X + \Delta) + A$ is ample. By Diophantine approximation, see (3.7.7) of [2], we may find a positive integer k, a divisor Φ on S and a prime divisor G (necessarily a component of Θ whose coefficient is irrational) such that

(1) $0 \leq \Phi \in W$,
(2) both $k\Phi/r$ and $k\Delta/r$ are Cartier,
(3) $\|\Phi - \Theta\| < \min(\delta, \eta, f\epsilon/k)$ where f is the smallest non-zero coefficient of $F \neq 0$, and
(4) $\mathrm{mult}_G \Phi > \mathrm{mult}_G \Theta$.

Claim 7.2. $\Omega \wedge \lambda F \leq \Omega - \Phi \leq \Omega$, *where* $\lambda = 1 - \epsilon/k$.

Proof of (7.2). Let P be a prime divisor on S and let ω, f, ϕ and θ be the multiplicities of Ω, F, Φ and Θ along P. We just need to check that

$$\min(\omega, \lambda f) \leq \omega - \phi. \tag{$*$}$$

There are two cases. If $\omega \leq f$, then $\theta = 0$ so that $\phi = 0$ and $(*)$ holds. If $\omega \geq f$, then $\theta = \omega - f$ and since $\|\Phi - \Theta\| < f\epsilon/k$,

$$\min(\omega, \lambda f) = \left(1 - \frac{\epsilon}{k}\right) f \leq f - (\phi - \theta) = \omega - \phi. \qquad \square$$

(7.2), (2) and (6.3) imply that

$$|k(K_S + \Phi)| + k(\Omega - \Phi) \subset |k(K_X + \Delta)|_S.$$

(4) implies that G is a component of $\mathrm{Fix}(k(K_S+\Phi))$. (2) and $\|\Phi-\Theta\| < \eta$ imply that G is a component of the stable base locus of $K_S+\Theta$, a contradiction.

Thus Θ is rational. Hence $\Omega \wedge F$ is rational, and we are done by (6.3). $\square$

8. Proof of (1.3)

Theorem 8.1. *Assume Theorem F_{n-1}.*

Let $\pi\colon X \longrightarrow Z$ be a projective morphism to a normal affine variety Z. Suppose that $(X, \Delta = S + A + B)$ is a purely log terminal pair of dimension n, $S = \llcorner \Delta \lrcorner$ is irreducible and not contained in the stable base locus of $K_X+\Delta$, $A \geq 0$ is a general ample $\mathbb{Q}$-divisor and $B \geq 0$ is a $\mathbb{Q}$-divisor.

Then there is a birational morphism $g\colon T \longrightarrow S$, a positive integer l and a kawamata log terminal pair (T, Θ) such that $K_T+\Theta$ is $\mathbb{Q}$-Cartier and

$$R_S(X, l(K_X+\Delta)) \cong R(T, l(K_T+\Theta)).$$

Proof. If $f\colon Y \longrightarrow X$ is a log resolution of (X,Δ) then we may write

$$K_Y + \Gamma' = f^*(K_X+\Delta) + E,$$

where $\Gamma' \geq 0$ and $E \geq 0$ have no common components, $f_*\Gamma' = \Delta$ and $f_*E = 0$. If T is the strict transform of S then we may choose f so that $(T, \Psi' = (\Gamma' - T)|_T)$ is terminal. Note that T is not contained in the stable base locus of $K_Y+\Gamma'$ as S is not contained in the stable base locus of $K_X+\Delta$.

Pick a $\mathbb{Q}$-divisor F such that $f^*A - F$ is ample and $(Y, \Gamma'+F)$ is purely log terminal. Pick $m > 1$ so that $m(f^*A - F)$ is very ample and pick $mC \in |m(f^*A-F)|$ very general. Then

$$(Y, \Gamma = \Gamma' - f^*A + F + C \sim_{\mathbb{Q}} \Gamma'),$$

is purely log terminal (note that since A is general, f^*A is equal to the strict transform of A) and if m is sufficiently large $(T, \Psi + C|_T)$ is terminal, where $\Psi = (\Gamma - T)|_T$.

On the other hand

$$R(X, k(K_X+\Delta)) \cong R(Y, k(K_Y+\Gamma)) \qquad \text{and}$$
$$R_S(X, k(K_X+\Delta)) \cong R_T(Y, k(K_Y+\Gamma)),$$

for any k sufficiently divisible. Now apply (7.1) to (Y, Γ). $\square$

Proof of (1.3). By (3.2) we may assume that Z is affine and by (3.5), it suffices to prove that the restricted algebra is finitely generated. As Z is affine, S is mobile and as f is birational, the divisor $\Delta - S$ is big. But then

$$\Delta - S \sim_{\mathbb{Q}} A + B,$$

where A is a general ample $\mathbb{Q}$-divisor and $B \geq 0$. As S is mobile, we may assume that the support of B does not contain S. Now

$$K_X + \Delta' = K_X + S + (1 - \epsilon)(\Delta - S) + \epsilon A + \epsilon B \sim_{\mathbb{Q}} K_X + \Delta,$$

is purely log terminal, where ϵ is any sufficiently small positive rational number. By (3.4), we may replace Δ by Δ'. We may therefore assume that $\Delta = S + A + B$, where A is a general ample $\mathbb{Q}$-divisor and $B \geq 0$. Since we are assuming Theorem F_{n-1}, (8.1) implies that the restricted algebra is finitely generated. □

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Department of Mathematics, University of Utah, 155 South 1400 East, JWB 233, Salt Lake City, UT 84112, USA
E-mail address: hacon@math.utah.edu

Department of Mathematics, University of California at Santa Barbara, Santa Barbara, CA 93106, USA
E-mail address: mckernan@math.ucsb.edu

Department of Mathematics, MIT, 77 Massachusetts Avenue, Cambridge, MA 02139, USA
E-mail address: mckernan@math.mit.edu