# RATIONAL CURVES ON ALGEBRAIC SPACES

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*To Yujiro Kawamata on the occasion of his sixtieth birthday*

Abstract. We prove a conjecture of Starr [11] that the conclusion of bend and break holds for proper algebraic spaces over an algebraically closed field of characteristic zero. The proof uses the minimal model program, rather than reduction modulo $p$ and it even applies to spaces with kawamata log terminal singularities.

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## 1. Introduction

We work over an algebraically closed field of characteristic zero.

**Theorem 1.1.** *Let $S$ be a proper algebraic space and let $(S, \Theta)$ be a kawamata log terminal pair. Suppose that $\Theta$ is big.*

*For every point $c$ of the stable base locus of $K_S + \Theta$ there is a rational curve $c \in M \subset S$.*

**Corollary 1.2** (Bend and Break)**.** *Let $S$ be a $\mathbb{Q}$-factorial proper algebraic space and let $(S, \Theta)$ be a kawamata log terminal pair.*

*If $C \subset S$ is a curve such that $(K_S + \Theta) \cdot C < 0$ then for every point $c \in C$ there is a rational curve $c \in M \subset S$.*

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(1.2) was proved by Miyaoka and Mori, [10], when $S$ is a smooth projective variety (or more generally when $S$ is a projective variety and $C$ belongs to the smooth locus of $S$) and $\Theta$ is empty; the proof uses deformation theory and reduction modulo $p$ in an essential way, so that the argument only applies to smooth projective varieties (or at least to varieties whose singularities are sufficiently mild so that one can apply deformation theory). On the other hand Takayama, [12], proves results similar to (1.1) when $S$ is a smooth projective variety, using methods from multiplier ideal sheaves; subsequently Boucksom, Broustet, and Pacienza [2] proved (1.1) when $S$ is projective and $(S, \Theta)$ is kawamata log terminal, using the MMP (minimal model program). Although their proof does not apply directly to the case of algebraic spaces, our proof is an extension of their argument.

In fact in the papers by Takayama and by Boucksom, Broustet, and Pacienza, the authors establish the stronger result that the stable base locus is uniruled. It is natural to ask if the same is true when $X$ is an algebraic space. Unfortunately, the methods of this paper don't seem to be able to prove this stronger result; I am very grateful to the referee for pointing out a mistake in a previous version of this paper in this direction.

It is interesting to compare these results with the corresponding results for compact Kähler manifolds. Over $\mathbb{C}$, a smooth algebraic space is the same as a Moishezon manifold, that is, a complex manifold whose space of meromorphic functions has transcendence degree equal to the dimension. If a compact complex manifold is both Moishezon and Kähler then it is projective.

Suppose that $M$ is a compact Kähler manifold of dimension three. If $K_M$ is not pseudo-effective then Brunella, [4], proved that $M$ is uniruled. Quite recently, Höring and Peternell, [8], proved a cone and contraction theorem for Kähler threefolds, as well as the MMP, so that if $K_M$ is not nef then there is a rational curve. Since then, Campana, Höring and Peternell, [5], proved that abundance holds for Kähler threefolds, that is, if $K_M$ is nef it is semiample (curiously abundance for algebraic spaces does not appear to be interesting, in the sense that abundance for projective varieties will automatically imply abundance for algebraic spaces).

The idea of the proof of (1.2) is to use the MMP. Note that if $(K_S + \Theta) \cdot C < 0$ then $C$ lies in the stable base locus. The idea then is to prove (1.1). To fix ideas, assume that $S$ is smooth and $\Theta$ is a smooth prime divisor. By Chow's Lemma (see Chapter 3, Theorem 3.1 of [9] for the case of an algebraic space) and passing to a log resolution, we may assume that we have a birational morphism $\pi\colon X \longrightarrow S$, where

$X$ is smooth and projective. If $X$ is uniruled, then so is $S$ and there is nothing to prove. Otherwise, by results of Boucksom, Demailly, Păun and Peternell, [3] or by [1] (and either way, implicitly bend and break, that is, reduction modulo $p$) $K_X$ is pseudo-effective. Therefore there is a minimal model $X \dashrightarrow Y$, so that $K_Y + \Gamma$ is semiample (for this and the existence of the minimal model we need that $\Theta$ is big), where $\Gamma$ is the strict transform of $\Theta$. Consider the indeterminancy locus $Z$ of the birational map $Y \dashrightarrow S$. As $C$ belongs to the stable base locus of $K_S + \Theta$ the only possibility is that the image of the induced morphism $Y - Z \longrightarrow S$ does not contain $C$; the result then follows by [7, 1.6] (note that $Y$ is in general not smooth, even if $S$ is smooth).

## 2. Notation

Let $S$ be a normal algebraic space. An $\mathbb{R}$-*divisor* $D$ is a Weil divisor $D$ with real coefficients. We say $D$ is $\mathbb{R}$-*Cartier* if $D$ is an $\mathbb{R}$-linear combination of Cartier divisors. We say that $S$ is $\mathbb{Q}$-*factorial* if every prime divisor is $\mathbb{R}$-Cartier. We say that $(S, \Theta)$ is *kawamata log terminal*, if $\Theta \geq 0$ is an $\mathbb{R}$-divisor, $K_S + \Theta$ is $\mathbb{R}$-Cartier and there is a projective birational morphism $f\colon T \longrightarrow S$ such that $T$ is smooth, the support of the strict transform of $\Theta$ and the exceptional locus is a divisor with global normal crossings and if we write

$$K_T + \Phi = f^*(K_S + \Theta)$$

then $\lfloor \Phi \rfloor \leq 0$, where we take the round down prime component by prime component.

Let $\phi\colon S \dashrightarrow Y$ be a birational map of normal algebraic spaces and let $p\colon W \longrightarrow S$ and $q\colon W \longrightarrow Y$ be proper birational morphisms resolving $\phi$. We say $\phi\colon S \dashrightarrow Y$ is $D$-*negative*, if every exceptional divisor for $p$ is exceptional for $q$ (that is, $\phi$ does not extract any divisors), $\phi_* D$ is $\mathbb{R}$-Cartier, $p_* E \geq 0$, and the support of $p_* E$ contains every $\phi$-exceptional divisor, where $E = p^* D - q^* \phi_* D$.

We say that a birational map $\phi\colon S \dashrightarrow Y$ is a *log terminal model* of a kawamata log terminal pair $(S, \Theta)$ if $\phi$ is $(K_S + \Theta)$-negative, $Y$ is a $\mathbb{Q}$-factorial projective variety and $K_Y + \Gamma$ is nef, where $\Gamma = \phi_* \Theta$.

**Lemma 2.1.** *Let $(S, \Theta)$ be a kawamata log terminal pair, where $S$ is a proper algebraic space and $\Theta$ is big.*

*Let $\pi\colon X \longrightarrow S$ be any log resolution of $(S, \Theta)$ and suppose that we write*

$$K_X + \Delta_0 = \pi^*(K_S + \Theta) + E,$$

*where $\Delta_0 \geq 0$ and $E \geq 0$ have no common components, $f_*\Delta_0 = \Theta$ and $E$ is exceptional. Let $F \geq 0$ be any divisor whose support is equal to the exceptional locus of $\pi$.*

*If $\eta > 0$ is sufficiently small and $\Delta = \Delta_0 + \eta F$ then $K_X + \Delta$ is kawamata log terminal and $\Delta$ is big. Moreover if $f\colon X \dashrightarrow Y$ is a log terminal model of $K_X + \Delta$ then the induced birational map $\psi\colon S \dashrightarrow Y$ is in fact a log terminal model of $K_S + \Theta$.*

*Proof.* This is a simple restatement of [1, 3.6.11], where we have replaced the projective variety $X$ by the proper algebraic space $S$. The proof of [1, 3.6.11] applies mutatis mutandis. □

We need to extend the definition of the stable base locus to the case of an algebraic space.

**Definition 2.2.** *Let $D$ be an $\mathbb{R}$-divisor on a proper normal algebraic space $S$. The* ***real linear system*** *associated to $D$ is*

$$|D|_{\mathbb{R}} = \{\, C \geq 0 \mid C \sim_{\mathbb{R}} D \,\}.$$

*The* ***stable base locus*** *of $D$ is the Zariski closed set $\mathbf{B}(D)$ given by the intersection of the support of the elements of the real linear system $|D|_{\mathbb{R}}$. If $|D|_{\mathbb{R}} = \emptyset$, then we let $\mathbf{B}(D) = X$.*

**Remark 2.3.** *The stable base locus is only defined as a closed subset, it does not have any scheme structure.*

**Lemma 2.4.** *Let $D$ be an integral divisor on a proper normal algebraic space $S$.*

*Then the stable base locus as defined in* (2.2) *coincides with the usual definition of the stable base locus.*

*Proof.* The proof of [1, 3.5.3] applies mutatis mutandis. □

## 3. Proofs

**Lemma 3.1.** *Let $S$ and $Y$ be proper normal algebraic spaces, let $D$ be an $\mathbb{R}$-Cartier divisor on $S$ and let $C \subset S$ be a curve. Let $\phi\colon S \dashrightarrow Y$ be a $D$-negative birational map. Let $p\colon W \longrightarrow S$ and $q\colon W \longrightarrow Y$ be proper birational morphisms which resolve $\phi$. Let $Z$ be the indeterminancy locus of $\phi^{-1}$.*

*At least one of the following holds:*

*(1) $C \subset p(q^{-1}(Z))$, or*
*(2) every curve $\Sigma \subset Y$ such that $C \subset p(q^{-1}(\Sigma))$ belongs to the stable base locus of $\phi_* D$, or*
*(3) $C$ does not belong to the stable base locus of $D$.*

*Proof.* Suppose that (1) and (2) do not hold.

As (2) doesn't hold, we may find a curve $\Sigma$ such that $C \subset p(q^{-1}(\Sigma))$ and $\Sigma$ does not belong to the stable base locus of $\phi_* D$. Then we may find a divisor

$$0 \leq D' \sim_{\mathbb{R}} \phi_* D,$$

which does not contain $\Sigma$. As $\phi$ is $D$-negative,

$$B = \phi^* D' + p_* E \geq 0.$$

On the other hand

$$\begin{aligned} B = \phi^* D' + p_* E &= p_*(q^* D' + E) \\ &\sim_{\mathbb{R}} p_*(q^* \phi_* D + E) \\ &= p_* p^* D = D. \end{aligned}$$

Let $U = Y - Z$, $\Sigma_0 = \Sigma \cap U$, $D'_0 = D'|_U$ and $\pi_0 = \phi^{-1}|_U \colon U \longrightarrow S$. Then $D'_0$ does not contain $\Sigma_0$, $D'_0 \sim_{\mathbb{R}} \pi_0^* D$ and moreover $\Sigma_0$ dominates $C$, since (1) does not hold. It follows that $D'_0$ does not contain $\pi_0^{-1}(C)$. But then $\pi_0(D')$ does not contain $C$. On the other hand, $p_* E$ does not contain $C$ as (1) does not hold. Therefore $B = \phi^* D' + p_* E$ does not contain $C$ and so (3) holds. □

*Proof of* (1.1). Let $\pi \colon X \longrightarrow S$ be a log resolution of $(S, \Theta)$, so that $X$ is a smooth projective variety and the sum of the strict transform of $\Theta$ and every exceptional divisor has global normal crossings. As $(S, \Theta)$ is kawamata log terminal, we may write

$$K_X + \Delta = \pi^*(K_S + \Theta) + E,$$

where $E \geq 0$ is exceptional, $\pi_* \Delta = \Theta$ and $\lfloor \Delta \rfloor = 0$. In particular $(X, \Delta)$ is kawamata log terminal. Adding a small multiple of the sum of the exceptional divisors to both sides, we may assume that both the support of $\Delta$ and the support of $E$ contains every exceptional divisor. In particular $\Delta$ is big.

Suppose first that $K_X + \Delta$ is not pseudo-effective. Then $K_X$ is not pseudo-effective either, [3] implies that $X$ is covered by curves on which $K_X$ is negative and so by [10], $X$ is uniruled.

*Aliter*: [1, 1.3.3] implies that we may run the $(K_X + \Delta)$-MMP until we arrive at a Mori fibre space, $Y \longrightarrow Z$. The general fibre of $Y \longrightarrow Z$ is a Fano variety, so that $Y$ is uniruled.

Either way, $S$ is uniruled. In this case there is a rational curve through every point of $S$ and the result is clear.

Therefore we may assume that $K_X + \Delta$ is pseudo-effective. [1, 1.2.1] implies that $(X, \Delta)$ has a log terminal model, $f \colon X \dashrightarrow Y$. As $\Gamma$ is big, it follows that $K_Y + \Gamma$ is semiample, [1, 3.9.1]. (2.1) implies that

if $\phi\colon S \dashrightarrow Y$ is the induced birational map then $\phi$ is a log terminal model of $K_S + \Theta$.

Let $Z$ be the indeterminancy locus of $Y \dashrightarrow S$. By the theorem of Zariski-Fujita, [6, 1.14] we may find a curve $c \in C$ contained in the stable base locus of $K_S + \Theta$. (3.1) implies that $C \subset p(q^{-1}(Z))$. As $Y$ has kawamata log terminal singularities, [7, 1.6] implies that there is a rational curve $c \in M \subset S$. $\square$

*Proof of* (1.2). As $S$ is birational to a projective variety there is a big divisor $B \geq 0$ on $S$. As $S$ is $\mathbb{Q}$-factorial, $B$ is $\mathbb{Q}$-Cartier. Possibly replacing $B$ by a small multiple, we may assume that $(S, \Theta + B)$ is kawamata log terminal and $(K_S + \Theta + B) \cdot C < 0$. Then $\Theta + B$ is big and $C$ belongs to the stable base locus of $K_S + \Theta + B$, so that we can apply (1.1). $\square$

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