MORI DREAM SPACES

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ABSTRACT. We explore the circle of ideas connecting finite generation of the Cox ring, Mori dream spaces and invariant theory.

CONTENTS

1. HILBERT'S 14TH PROBLEM

1.1. Introduction.

Problem 1.1 (Hilbert's 14th problem)**.** *Let k be a field and let $k \subset K \subset k(x_1, x_2, \dots, x_n)$.*

Is the ring $R = K \cap k[x_1, x_2, \dots, x_n]$ finitely generated?

(Here, as elsewhere, finitely generated means finitely generated as a k-algebra.) For an entertaining and more comprehensive treatment of Hilbert's 14th problem, see [23]. Even though most of what is covered in these notes will apply to any algebraically closed field of characteristic zero (and some of what is covered will apply to any field of characteristic zero or even arbitrary characteristic), we will work over $k = \mathbb{C}$ for simplicity. Hilbert's original motivation for this problem comes from

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invariant theory. Suppose that we have a linear algebraic group G, that is an algebraic subgroup $G \subset \mathrm{GL}(n, \mathbb{C})$. Then G acts on the polynomial ring $\mathbb{C}[x_1, x_2, \dots, x_n]$ and we can form the ring of invariant polynomials

$$\mathbb{C}[x_1, x_2, \dots, x_n]^G = \{\, f \in \mathbb{C}[x_1, x_2, \dots, x_n] \,|\, f^g = f \,\}.$$

It is then natural to ask if the ring of invariants is finitely generated. If we let $K = \mathbb{C}(x_1, x_2, \dots, x_n)^G$, the field of invariant rational functions, then we see that this is a special case of Hilbert's 14th problem.

In general we can start with an arbitrary finitely generated ring R (that is, a quotient of the polynomial ring $\mathbb{C}[x_1, x_2, \dots, x_n]$) and a linear algebraic group G acting on R and consider the ring of invariants R^G. It is then natural to ask if R^G is finitely generated.

Perhaps the most general result along these lines is due to Hilbert and Mumford. We recall the definition of a reductive group.

Definition 1.2. *Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a linear algebraic group. The **radical** of G is the identity component of the maximal normal solvable subgroup of G. An element $g \in G$ is **unipotent** if $g - I_n$ is **nilpotent**, that is some power is zero.*

*The **unipotent radical** of G is the set of unipotent elements in the radical of G. We say that G is **reductive** if the unipotent radical of G is the trivial subgroup.*

The main point is that if the characteristic is zero and G is a linear algebraic group acting on a finite dimensional vector space V and W is a G-invariant subspace then there is a G-invariant complement W'.

If G and H are reductive groups then so is their product. On the other hand, $\mathbb{G}_m$, the group of non-zero elements of the underlying field $\mathbb{C}$ under multiplication, is a reductive group, so that any torus $\mathbb{G}_m^k$ is a reductive group. $\mathrm{SL}(n, \mathbb{C})$ and $\mathrm{GL}(n, \mathbb{C})$ are also reductive groups, as are many of the classical groups.

By contrast, $\mathbb{G}_a$, the group of elements of $\mathbb{C}$ under addition, is not reductive. There are two ways to see this. The first is directly.

$$\mathbb{G}_a = \{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \,|\, a \in k \,\} \subset \mathrm{GL}(2, k).$$

The radical of $\mathbb{G}_a$ is $\mathbb{G}_a$ itself and given $g \in \mathbb{G}_a$,

$$g - I_2 = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix},$$

squares to zero, so that $\mathbb{G}_a$ is equal to its own unipotent radical.

On the other hand, consider the natural action of $\mathbb{G}_a$ on $\mathbb{C}^2$. Then

$$W = \{\,(x,0)\,|\,x\in\mathbb{C}\,\},$$

is an invariant subspace but there is no $\mathbb{G}_a$-invariant complement.

Theorem 1.3 (Hilbert, Mumford)**.** *Let G be a reductive group and let R be a finitely generated ring.*

Then R^G is finitely generated.

Proof. We sketch Hilbert's amazing argument. In particular, we assume for simplicity that $R=\mathbb{C}[x_1,x_2,\dots,x_n]$ is the polynomial ring. See [24] for a proof of the complete result. The key thing is that R comes with a natural grading by the degree:

$$R=\bigoplus_{n\in\mathbb{N}} R_n \qquad \text{and so} \qquad R^G=\bigoplus_{n\in\mathbb{N}} R^G_n.$$

As G is reductive we may find R'_n such that $R_n=R^G_n\oplus R'_n$. Let

$$\rho_n\colon R_n \longrightarrow R^G_n,$$

be the projection map with kernel R'_n. The important thing is that these maps glue together to give

$$\rho\colon R \longrightarrow R^G,$$

and ρ has a nice linearity property:

$$\rho(fg)=f\rho(g),$$

where $f\in R^G$ and $g\in R$. Let R^G_+ be the elements of R^G of positive degree and let I be the ideal they generate. By Hilbert's basis Theorem $I=\langle f_1,f_2,\dots,f_k\rangle$, for some polynomials $f_1,f_2,\dots,f_k$. We may assume that $f_1,f_2,\dots,f_k$ belong to R^G and that they are homogeneous. Let S be the subring generated by $f_1,f_2,\dots,f_k$. We claim that $R^G=S$.

Clearly $S\subset R^G$. Let $g\in R^G$ be homogeneous of degree d. We have to show that $g\in S$. We proceed by induction on the degree d. We suppose that any homogeneous element $h\in R^G$ of smaller degree belongs to S. As $g\in I$ we may write $g=\sum f_ig_i$, where g_i is homogeneous of degree smaller than d. Now

$$\begin{aligned} g&=\rho(g)\\ &=\rho(\sum f_ig_i)\\ &=\sum_i f_i\rho(g_i). \end{aligned}$$

As $\rho(g_i) \in R^G$ is homogeneous of degree smaller than d, by our inductive hypothesis $\rho(g_i) \in S$, and so $g \in S$. $\square$

1.2. **Nagata's Example.** Nagata was one of the first people to give a counterexample to Hilbert's 14th problem. Consider the following general situation. Let $\mathbb{G}_a^n$ act on the polynomial ring

$$R = \mathbb{C}[x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n],$$

as follows: an element $(t_1, t_2, \dots, t_n) \in \mathbb{G}_a^n$ acts by the rule

$$x_i \longrightarrow x_i \qquad \text{and} \qquad y_i \longrightarrow y_i + t_i x_i \qquad 1 \leq i \leq n.$$

Note that $\mathbb{G}_a^n$ is really nothing more than a vector space of dimension n. Let $G = \mathbb{G}_a^{n-r} \subset \mathbb{G}_a^n$ be a general linear subspace of codimension r, where r is at least three. Nagata proved that if $n = 16$ and $r = 3$ then the ring of invariants $S = R^G$ is not finitely generated. Much later, Mukai proved that the ring of invariants is not finitely generated if

$$\frac{1}{r} + \frac{1}{n-r} \leq \frac{1}{2}.$$

Mukai's proof of this result is very interesting and I follow his exposition quite closely, see for example [22]. Suppose that G is defined by linear equations of the form,

$$\sum a_{ij} t_j = 0 \qquad \text{where} \qquad 1 \leq i \leq r.$$

This gives us a matrix

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rn} \end{pmatrix}.$$

Possibly changing coordinates, we may assume that the rows and columns of this matrix are pairwise independent and that no entry is zero.

The key observation is that it is easier to consider what happens if we invert $x_1, x_2, \dots, x_n$. As G fixes $x_1, x_2, \dots, x_n$ there is an induced action on the ring

$$\mathbb{C}[x_1^{\pm}, x_2^{\pm}, \dots, x_n^{\pm}][y_1, y_2, \dots, y_n] = \mathbb{C}[x_1^{\pm}, x_2^{\pm}, \dots, x_n^{\pm}][y_1/x_1, y_2/x_2, \dots, y_n/x_n].$$

and the action fixes the Laurent polynomial ring

$$\mathbb{C}[x_1^{\pm}, x_2^{\pm}, \dots, x_n^{\pm}],$$

and otherwise acts by translation

$$y_i/x_i \longrightarrow y_i/x_i + t_i.$$

It is not hard to see that the invariant ring $R[x_1^{-1}, x_2^{-1}, \dots x_n^{-1}]^G$ is then generated by

$$\sum_{j=1}^{n} a_{ij} y_j / x_j \qquad \text{where} \qquad 1 \leq i \leq r,$$

and that

$$R^G = R[x_1^{-1}, x_2^{-1}, \dots x_n^{-1}]^G \cap R.$$

Multiplying through by $x_1 x_2 \cdots x_n$, we get $w_1, w_2, \dots, w_r$, which are invariant under the action of G and which are polynomials in $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$. The invariant ring R^G contains this polynomial ring, $\mathbb{C}[w_1, w_2, \dots, w_r]$. However the invariant ring is much bigger than this. In fact the invariant ring is generated by the polynomials g in $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$ of the form f/m where f is a polynomial in $w_1, w_2, \dots, w_r$ and m is a monomial in $x_1, x_2, \dots, x_n$. Note that the condition that g is a polynomial in $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$ is equivalent to requiring that f is divisible by m, when considered as a polynomial in $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$.

Let us consider the geometric meaning of all of this. The polynomial ring $\mathbb{C}[w_1, w_2, \dots, w_r]$ corresponds to a copy of $\mathbb{P}^{r-1}$. Let

$$p_j = [a_{1j} : a_{2j} : \cdots : a_{rj}] \in \mathbb{P}^{r-1} \qquad \text{for} \qquad 1 \leq j \leq n,$$

be the n points of $\mathbb{P}^{r-1}$ corresponding to the n columns of the matrix A. With a little bit of work, one can show that a polynomial f in $w_1, w_2, \dots, w_r$ is divisible by x_i^b if and only if f considered as a polynomial in $w_1, w_2, \dots, w_r$ vanishes to order b at p_i.

At this point, it might help to run through some standard notation and results from classical algebraic geometry. There is a basic correspondence between line bundles and divisors. Given a line bundle L one can take its first Chern class to get a divisor, a formal linear combination of codimension one subvarieties. Topologically the first Chern class takes values in $H^2(X, \mathbb{Z})$ but in algebraic geometry we prefer to work in a much larger group, that is to work with a much finer equivalence relation, linear equivalence. Normally it is important to distinguish between topological (or numerical) equivalence and linear equivalence but in these notes we will only work on varieties with the property that if two divisors are topologically equivalent then some multiples are linearly equivalent. We also don't care which multiple we need and so we will just pretend that topological, numerical and linear equivalence are all the same. In fact in the cases covered in these notes, the Picard group $\mathrm{Pic}(X)$ of all line bundles may be identified with a subgroup of $H^2(X, \mathbb{Z})$. The line bundle associated to a divisor D is denoted $\mathcal{O}_X(D)$. The space of all sections is denoted $H^0(X, \mathcal{O}_X(D))$. The zero locus of any non-zero section is a divisor $D' \geq 0$ linearly (or numerically) equivalent to D.

We now return to Nagata's example. There is a standard trick in algebraic geometry which one applies in this situation. Instead of considering functions which vanish at a collection of points $p_1, p_2, \ldots, p_n \in \mathbb{P}^{r-1}$, instead blow up these points,

$$\pi \colon X \longrightarrow \mathbb{P}^{r-1},$$

and consider functions which vanish along the exceptional divisors $E_1, E_2, \ldots, E_n$ of the blow up. There is also some standard notation which goes with this. On $\mathbb{P}^{r-1}$ the set of polynomials of degree d is denoted

$$H^0(\mathbb{P}^{r-1}, \mathcal{O}_{\mathbb{P}^{r-1}}(dH)).$$

Here H is the class of a hyperplane and $\mathcal{O}_{\mathbb{P}^{r-1}}(dH)$ is the associated line bundle. On the blow up X we have introduced new divisors $E_1, E_2, \ldots, E_n$ and new line bundles. The invariant ring is then

$$\bigoplus_{(d,a_1,a_2,\ldots,a_n)\in\mathbb{N}^{n+1}} H^0(X, \mathcal{O}_X(dH - \sum a_i E_i)).$$

Here

$$H^0(X, \mathcal{O}_X(dH - \sum a_i E_i)),$$

is simply the space of degree d polynomials which vanish as p_i to order a_i. The advantage of this approach is that if we present the invariant ring this way there is a very useful grading. We recall some basic facts about graded rings.

Definition 1.4. *Let W be an abelian monoid (so that there is an associative and commutative law of addition, together with a zero). We say that a ring R is a* ***graded ring****, graded by W, if*

$$R = \bigoplus_{w\in W} R_w,$$

where $R_w \subset R$ are additive subgroups and

$$R_w R_{w'} \subset R_{w+w'}.$$

Note that R is naturally an R_0-algebra and we say that R is finitely generated if it is a finitely generated R_0-algebra. In these talks, we will be most interested in the case when R_0 is a field and more often than not $R_0 = \mathbb{C}$. Perhaps the most natural choice for $W = \mathbb{N}$, the natural numbers under addition, but there are other very interesting choices.

Example 1.5. *Let $R = \mathbb{C}[x_1, x_2, \dots, x_n]$. Then R is naturally graded by $\mathbb{N}^n$. It is also naturally graded by $\mathbb{N}$, the degree. In fact there is a morphism of monoids,*

$$\mathbb{N}^n \longrightarrow \mathbb{N},$$

which sends $(a_1, a_2, \dots, a_n)$ to the sum $\sum a_i$ and the second grading is induced from the first by this map.

The analysis above shows that Nagata's invariant ring has a natural grading by $W = \mathbb{Z}^{n+1} = H^2(X, \mathbb{Z})$.

The key point is that the grading on the ring is very useful either to prove that the ring is finitely generated by induction on the grading or to show that the ring is not finitely generated.

Definition 1.6. *Let R be a graded ring, graded by W.*

*The **support** of R is the set*

$$M = \{\, w \in W \mid R_w \neq 0 \,\}.$$

Example 1.7. *The support has a very interesting interpretation in terms of Nagata's example. It is the set of divisors of the form $dH - \sum a_i E_i$ such that the associated line bundle has a non-zero section. The zero locus of this section is an effective divisor $D = \sum d_i D_i \geq 0$ (that is the coefficients are greater than zero) numerically equivalent to $dH - \sum a_i E_i$. Effective divisors play a very special role in algebraic geometry. Now given any monoid $M \subset \mathbb{Z}^k$ it is natural to extend scalars to the real numbers $\mathbb{R}$:*

$$\mathcal{E} \subset \mathbb{R}^k = \mathbb{Z}^k \bigotimes_{\mathbb{Z}} \mathbb{R}.$$

*Here $\mathcal{E}$ is the convex cone generated by M. Note that M is a finitely generated monoid if and only if $\mathcal{E}$ is a rational polyhedron. In the case of the monoid of effective divisors, $\mathcal{E}$ is called the **effective cone** of divisors. It has the property that it is strongly convex, that is, it does not contain any non-trivial linear subspaces.*

Lemma 1.8. *Let R be a graded ring, graded by W.*

If R is a finitely generated ring then the support M of R is a finitely generated monoid.

Proof. Suppose that $r_1, r_2, \dots, r_k$ generate R as an R_0-algebra. We may assume that $r_i \neq 0$ and that each r_i is homogeneous. In this case, if $r_i \in R_{w_i}$ then $w_1, w_2, \dots, w_k$ generate M as a monoid. $\square$

It is now easy to give examples of invariant rings which are not finitely generated. If $r = 3$ then we are looking at $\mathbb{P}^2$ blown up at n general points. Suppose that $n = 9$. Then we get 9 exceptional divisors

$E_1, E_2, \dots, E_9$. There is a classical way to characterise exceptional divisors on a smooth projective surface.

1.3. Some classical geometry.

Definition 1.9. *Let X be a smooth projective variety. The **canonical divisor** K_X is the first Chern class of the cotangent bundle, T^*_X.*

If $H^0(X, \wedge^n T^*_X) \neq 0$ then the canonical divisor K_X is the zero locus of a non-zero section.

Theorem 1.10 (Castelnuovo)**.** *Let S be a smooth projective surface.*

Then a curve E is the exceptional divisor of the blow up $\pi\colon S \longrightarrow T$ of a point if and only if any two of the following three conditions hold:

- $E \simeq \mathbb{P}^1$,
- $E^2 = -1$,
- $K_S \cdot E = -1$.

A curve E which satisfies these conditions is called a -1-curve.

Now suppose that S is $\mathbb{P}^2$ blown up at nine sufficiently general points. In this case, if α is the numerical equivalence class of a divisor such that $\alpha^2 = K_S \cdot \alpha = -1$ then there is a -1-curve E whose numerical class is α. Further, if we have a continuous family of smooth projective surfaces then the number of -1-curves is lower semi-continuous.

Proof. We skip the proof of the result that a -1-curve is contractible.

We first show that any two of the three conditions imply the third. The key point is the adjunction formula

$$(K_S + E)|_E = K_E.$$

Here K_E is the canonical divisor of the curve, which is a formal linear combination of points of the curve. If we take the degree then we get the formula

$$K_S \cdot E + E^2 = 2g - 2,$$

where g is the genus of the curve. Now note that E is isomorphic to $\mathbb{P}^1$ if and only if $g = 0$ that is if and only if $2g - 2 = -2$.

Now suppose that S is $\mathbb{P}^2$ blown up at nine points. Assume that α is a curve class such that $K_S \cdot \alpha = \alpha^2 = -1$. If we apply Riemann-Roch to α we get:

$$\chi(S, \mathcal{O}_S(\alpha)) = \alpha \cdot (\alpha - K_S)/2 + \chi(S, \mathcal{O}_S).$$

By hypothesis, $h^1(S, \mathcal{O}_S) = h^2(S, \mathcal{O}_S) = 0$ so that $\chi(S, \mathcal{O}_S) \geq 1$. As the first term of the RHS is zero, the RHS is positive. Serre duality implies that

$$h^2(S, \mathcal{O}_S(\alpha)) = h^0(S, \mathcal{O}_S(K_S - \alpha)).$$

$-K_{\mathbb{P}^2}$ is represented by a cubic. $-K_S$ is then represented by a cubic passing through the nine points we blow up; since there is a nine dimensional family of cubics, it follows that $-K_S$ is represented by an effective divisor. Further, if the points are sufficiently general then we may find an irreducible cubic through the nine points and this implies that $-K_S$ is represented by a prime divisor (both irreducible and reduced). Since $-K_S^2 = 0$ this implies that $-K_S$ is nef (recall that a divisor D is nef if the intersection number $D \cdot C \geq 0$ for every curve C). Since

$$-K_S \cdot (K_S - \alpha) = -1,$$

it follows that $h^0(S, \mathcal{O}_S(K_S - \alpha)) = 0$. So α is represented by an effective curve. Again, if the points we blow up are sufficiently general then this implies that S contains no -2-curves (copies of $\mathbb{P}^1$ of self-intersection -2) and this easily implies that the effective curve is a -1-curve.

Suppose that S_t is a smooth family of surfaces over the unit disc (parametrised by t) and that the central fibre S_0 contains a -1-curve. There are two ways to see the last part. The first is to use the fact that every smooth fibration is topologically locally trivial. The topological class α_0 extends to a class α_t on every fibre. Clearly $\alpha_t^2 = -1$ and $K_{S_t} \cdot \alpha_t < 0$ for small t. But then α_t represents a -1-curve E_t.

Or one can show that there are no obstructions to deforming the morphism $f_0 \colon \mathbb{P}^1 \longrightarrow S_0$, which is an isomorphism onto E_0, to a family of morphisms $f_t \colon \mathbb{P}^1 \longrightarrow S_t$.

The only catch here is that in both cases one might need to make a base change to kill the monodromy action. □

Lemma 1.11. *Let X be a smooth projective variety.*

If $E \geq 0$ is a divisor which is covered by curves C such that $E \cdot C < 0$ then

$$H^0(X, \mathcal{O}_X(E)) = \mathbb{C}.$$

and E appears in any list of generators of the monoid of effective divisors.

Proof. Suppose not, suppose that E is numerically equivalent to $D_1 + D_2$ where $D_1 \geq 0$ and $D_2 \geq 0$ are effective divisors and the support of neither D_1 nor D_2 contains E. Intersecting with C we get

$$0 > E \cdot C = D_1 \cdot C + D_2 \cdot C.$$

But then $D_i \cdot C < 0$ for some i. This can only happen if C belongs to the support of D_i. Since C covers E this forces E to belong to the support of D_i, a contradiction. □

To finish off we need to see that there are infinitely many -1-curves. There are two ways to do this. One can proceed directly. Using (1.10), if E is a -1-curve and $E = dH - \sum a_i E_i$ then we have

$$E^2 = -1 \qquad \text{and} \qquad K_X \cdot E = -1.$$

On the other hand, we have already seen that Riemann-Roch on a smooth surface implies that any divisor class which looks like a -1-curve in fact represents the class of a -1-curve. Now $K_{\mathbb{P}^2} = -3L$, where L is the class of a line and so $K_X = -3H + \sum E_i$. We are reduced to solving the following two Diophantine equations in 10 variables:

$$d^2 - \sum a_i^2 = -1 \qquad \text{and} \qquad -3d + \sum a_i = -1.$$

It is not hard to show that there are infinitely many solutions directly. For example:

- any line spanned by two of the nine points is a -1-curve. The self-intersection of the line is 1 and blowing up drops the self-intersection by 2. In fact the class of the strict transform of the line is $H - E_i - E_j$ and we have
$$1^2 - 1 - 1 = -1 \qquad \text{and} \qquad -3 + 1 + 1 = -1.$$
- any conic through five points is a -1-curve. The self-intersection starts off as $4 = 2^2$ and drops by five. If the conic passes through p_1, p_2, p_3, p_4 and p_5 then the class of the strict transform is $2H - E_1 - E_2 - E_3 - E_4 - E_5$ and we have

$$4 - 1 - 1 - 1 - 1 - 1 = -1 \qquad \text{and} \qquad -6 + 1 + 1 + 1 + 1 + 1 = -1,$$

and so on.

The second approach is geometrically more appealing. Note first that there is a unique cubic through nine general points. On the other hand, if one picks two cubics then they will intersect in nine points and there is a pencil of cubics through these nine points (if the cubics are defined by F and G then the curve defined by $\lambda F + \mu G$ will also contain those nine points). We get a morphism to $\mathbb{P}^1$ (with coordinates $[\lambda : \mu]$) and the fibres are cubics (the zero locus of $\lambda F + \mu G$). The nine exceptional divisors are nine sections of this fibration. Recall that a smooth cubic curve in $\mathbb{P}^2$ is an algebraic group. A typical line in $\mathbb{P}^2$ will intersect the cubic in three points. The group law is given by declaring that three collinear points sum to zero in the group. Using the group law, the two points p_1 and p_2 generate a third point p_t on the cubic C_t (where $t = \mu/\lambda$, say). In fact the point p_t is nothing more than the intersection of the line spanned by p_1 and p_2 with C_t. Then the points p_1 and p_t generate a fourth point and so. If we choose the cubics sufficiently general and we consider the difference of two sections $E_2 - E_1$ then this defines an automorphism of the fibration, which has infinite order. The translates of a section under the powers of this automorphism will then give infinitely many -1-curves. This shows that if we choose the nine points very carefully then we get infinitely many -1-curves. It remains to observe that the last part of (1.10) implies that the -1-curves can never disappear as we move the points around.

Putting all of this together we see that the ring of invariants is not finitely generated in Nagata's examples. The ring of invariants is the Cox ring of $\mathbb{P}^{r-1}$ blown up at n points. If $r = 3$ and $n = 9$ we get a surface with infinitely many -1-curves. (1.11) implies that each of these -1-curves generates an extremal ray of the cone of effective divisors (which on a surface is the same as Mori's cone of effective curves) and so the invariant ring is not finitely generated.

2. Mori dream spaces

Definition 2.1. *Let X be a smooth projective variety. Let $D_1, D_2, \dots, D_k$ be a sequence of divisors. The **multi-graded section ring** associated to $D_1, D_2, \dots, D_k$ is the ring*

$$R(X, D_1, D_2, \dots, D_k) = \bigoplus_{m \in \mathbb{Z}^k} H^0(X, \mathcal{O}_X(D)) \quad \text{where} \quad D = \sum m_i D_i.$$

*If $k = 1$ then we call $R(X, D_1)$ a **section ring**.*

Suppose that the group of line bundles $\operatorname{Pic}(X)$ *is a finitely generated abelian group. Pick a set of divisors $D_1, D_2, \dots, D_k$ so that the line bundles $\mathcal{O}_X(D_1)$, $\mathcal{O}_X(D_2)$, $\dots$, $\mathcal{O}_X(D_k)$, generate the group* $\operatorname{Pic}(X)$. *Then the **Cox ring of** X (aka the total coordinate ring), denoted $Cox(X)$, is $R(X, D_1, D_2, \dots, D_k)$.*

Lemma 2.2. *Let X be a smooth projective variety and let $C_1, C_2, \dots, C_k$ and $D_1, D_2, \dots, D_k$ be two sequences of divisors on X.*

If there are positive integers $c_1, c_2, \dots, c_k$ and $d_1, d_2, \dots, d_k$ such that $c_i C_i = d_i D_i$, $1 \leq i \leq k$ then

$$R(X, C_1, C_2, \dots, C_k),$$

is finitely generated if and only if

$$R(X, D_1, D_2, \dots, D_k),$$

is finitely generated.

Proof. It is convenient to prove the same result for rings graded by $\mathbb{N}^k$. To prove this stronger statement, by induction on k, we immediately reduce to the case of two rings R and $R_{(d)}$, graded by $\mathbb{N}$, where

$$R_{(d)} = \bigoplus_{n \in \mathbb{N}} R_{dn}.$$

Suppose that R is finitely generated. Note that there is a natural action of the cyclic group $\mathbb{Z}_d$ on R such that $R_{(d)}$ is the invariant ring. As every finite group is reductive over $\mathbb{C}$, $R_{(d)}$ is finitely generated.

Now suppose that $R_{(d)}$ is finitely generated. Note that R is integrally closed over $R_{(d)}$, so that R is finitely generated by a result of Noether. $\square$

Note that the Cox ring is naturally graded by $W = \operatorname{Pic}(X) \subset H^2(X, \mathbb{Z})$. On the other hand, note that it is necessary to pick a generating set, since the vector space $H^0(X, L)$ is only defined up to scalars, if the line bundle is only defined up to isomorphism. However, the only important question for us is whether or not the Cox ring is finitely generated, in which case we are even free to pick divisors $D_1, D_2, \dots, D_k$ which generate the Picard group modulo torsion.

We have already seen that Cox rings turn up naturally in the context of Nagata's examples. We have also seen that if X is a smooth projective surface and the Cox ring is finitely generated then X only contains finitely many -1-curves. There is a similar story in higher dimensions but with one very interesting complication.

First note that if the Cox ring is finitely generated then so is every section ring. But if the section ring is finitely generated then it is a quotient of a polynomial ring. This realises the section ring as the homogeneous coordinate ring of a projective variety $Y \subset \mathbb{P}^n$. Moreover there is an induced rational map $\phi_D \colon X \dashrightarrow Y$, where the inverse map does not contract any divisors. In fact given any effective divisor $D \geq 0$ there is always an associated rational map $f_D \colon X \dashrightarrow \mathbb{P}^n$; finite generation implies that the image of this rational map stabilises, that is the induced map to the image $f_{mD} = \phi_D$ is independent of m, provided that m is sufficiently large and divisible. In the case of surfaces ϕ_D is always a morphism but the twist here is that in higher dimensions ϕ_D is not a morphism, due to the existence of flips and flops.

Example 2.3. *The easiest example of a flop is due originally to Atiyah. Let $Q \subset \mathbb{C}^4$ be the quadric cone given by the vanishing of $xt - yz$. Then the origin is a singular point of Q. Suppose that we blow up the origin $\pi \colon Y \longrightarrow Q$. The exceptional divisor is a copy of the quadric cone E sitting inside $\mathbb{P}^3$ (E is defined by the same equation $XT - YZ = 0$). E is a copy of $\mathbb{P}^1 \times \mathbb{P}^1$. The key point is that one can choose to contract E in three different ways. One can contract E down to a point, to recover*

Q. But we can also project E down to either factor. The interesting thing is that we can realise all of this inside Y over Q, to get two birational contractions (that is, a morphism with connected fibres which is an isomorphism on an open subset) $X_1 \longrightarrow Q$ and $X_2 \longrightarrow Q$. The induced rational map $X_1 \dashrightarrow X_2$ is not everywhere defined, but it is an isomorphism outside the two curves C_1 and C_2, both copies of $\mathbb{P}^1$, which are contracted down to Q.

Anyway the Cox ring encodes the data of all possible maps $\phi\colon X \dashrightarrow Y$ where Y is embedded in projective space. To see how this works in practice, we need a little more classical geometry. Suppose that D is a divisor whose section ring $R(X, D)$ is finitely generated. Let $\phi_D\colon X \dashrightarrow Y$ be the associated rational map. Then $D = \phi_D^* H + E$ where H is the restriction to Y of a hyperplane in $\mathbb{P}^n$ and $E \geq 0$. The divisor E may be characterised by the property that if $D' \geq 0$ is numerically equivalent to D then $D' \geq E$. E is called the fixed divisor and what is left $M = D - E$ is called the **movable part**; the movable part has the property that $\phi_D = \phi_M$ (indeed the section rings are the same). We say that D is **movable** if $E = 0$, that is the fixed divisor is empty. H is an **ample divisor**; it has the property that if $C \subset Y$ then $H \cdot C > 0$. We say that D is **semiample** if D is movable and ϕ_D is a morphism. In this case D is a nef divisor; it has the property that if C is a curve on X then $D \cdot C \geq 0$. In fact $D \cdot C = 0$ if and only if C is contracted by ϕ_D.

Definition 2.4. *Let X be a smooth projective variety. We say that X is a **Mori dream space** if*

- *the group of line bundles* $\operatorname{Pic}(X)$ *is a finitely generated abelian group,*
- *there are finitely $1 \leq i \leq k$ many birational maps $f_i\colon X \dashrightarrow X_i$ which are isomorphisms in codimension one, such that if D is a movable divisor then there is an index $1 \leq i \leq k$ and a semi-ample divisor D_i on X_i such that $D = f_i^* D_i$.*

The key thing about this definition is that it allows one to run the minimal model program (commonly abbreviated to MMP). A few more definitions. Recall that the effective cone of divisors $\mathcal{E}$ is a cone inside $H^2(X, \mathbb{R})$. Its closure $\mathcal{P}$ is called the cone of pseudo-effective divisors. Note that the movable cone is a subcone. Put an equivalence relation on this cone by declaring two divisors D_1 and D_2 equivalent if and only if $\phi_{D_1} = \phi_{D_2}$. If X is a Mori dream space then the pseudo-effective cone

$\mathcal{P}$ is divided into finitely many rational polyhedra, $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_m$,

$$\mathcal{P} = \bigcup_{i=1}^{m} \mathcal{P}_i.$$

The movable cone is a union of some subset $\mathcal{Q}_1$, $\mathcal{Q}_2, \dots, \mathcal{Q}_k$ of the rational polyhedra $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_m$ and in fact the birational maps $f_1, f_2, \dots, f_k$ defined in (2.4) are precisely the maps ϕ_{D_i} associated to a big movable divisor D_i belonging to the interior of each polytope $\mathcal{Q}_i$.

The aim of the MMP is to start with a pseudo-effective divisor D and find a sequence of flips and divisorial contractions $X \dashrightarrow Y$ such that $D = f^*D'$, where D' is semiample.

So now suppose that X is a Mori dream space and suppose we start with a pseudo-effective divisor D. Let $\mathcal{P}_1$ be a cone of maximal dimension to which D belongs. Pick D' belonging to the interior of $\mathcal{P}_1$. Then D' is big, that is $\phi_{D'}$ is birational, as $\mathcal{P}_1$ has maximal dimension. Pick an ample divisor H and suppose that H belongs to the interior of the polyhedra $\mathcal{P}_2$. We may assume that H is general in $\mathcal{P}_2$. Draw the line connecting H to D'. This line (starting at H, ending at D') will cross a finite number of walls between two polyhedra, at a general point of the intersection of the two polyhedra. The wall crossing corresponds to two different birational models of X. There are two cases. If the two different birational models are isomorphic in codimension one, then the corresponding birational map $X_i \dashrightarrow X_j$ is a D-flip. Otherwise we have a divisorial contraction. At the end we are on a model $Y = X_i$ such that D is now nef and by definition of a Mori dream space the strict transform of D is now semiample.

3. Geometric Invariant Theory

It is interesting to see this story using the language of Geometric Invariant Theory. Geometric Invariant Theory provides a natural way to take a quotient of a quasi-projective variety Y under the action of a reductive group G. The Geometric Invariant Theory quotient is denoted $Y//G$.

Now the natural grading on the Cox ring determines an action of a torus of dimension r equal to the rank of $\operatorname{Pic}(X)$. In abstract terms the torus is given as

$$G = \operatorname{Hom}(\operatorname{Pic}(X), \mathbb{G}_m) \simeq \mathbb{G}_m^r.$$

If $\operatorname{Cox}(X)$ is finitely generated then we can write $\operatorname{Cox}(X)$ as a quotient of a polynomial ring

$$\phi\colon \mathbb{C}[x_1, x_2, \dots, x_n] \longrightarrow \operatorname{Cox}(X).$$

Let I be the kernel of ϕ and let $Y \subset \mathbb{C}^n$ be the affine variety corresponding to I. It is straightforward to choose ϕ so that the action of G on $\mathrm{Cox}(X)$ lifts to an action of G on the polynomial ring $\mathbb{C}[x_1, x_2, \dots, x_n]$. In this case the action of G on Y is induced by an action of G on $\mathbb{C}^n$ and $Y \subset \mathbb{C}^n$ is a G-equivariant embedding.

Now the Geometric Invariant Theory quotient is determined not only by an action of G on Y, one also has to choose an embedding of G inside $\mathrm{GL}(n, \mathbb{C})$, called a linearisation of the action of G on Y. As we vary the linearisation, we vary the quotient. In fact this is sometimes called VGIT, Variation of Geometric Invariant Theory. Having chosen a linearisation, one also has to throw out a proper closed subset (the unstable points).

The simplest case is when G is a torus and $Y = \mathbb{C}^n$. In this case the quotient is a toric variety M and the locus one needs to throw away is a finite union of linear subspaces. For example, if we start with $G = \mathbb{G}_m$ and the standard action

$$(x_1, x_2, \dots, x_n) \longrightarrow (tx_1, tx_2, \dots, tx_n),$$

then one has to throw out the origin and the Geometric Invariant Theory quotient is nothing more than $\mathbb{P}^{n-1}$. On the other hand if we start with $\mathbb{C}^4$ and an action of $\mathbb{G}_m^2$ with coordinates (s, t), and the action is

$$(a, b, c, d) \longrightarrow (sa, sb, tc, td)$$

then we need to throw out two linear subspaces

$$\{\, (0, 0, c, d) \mid c, d \in \mathbb{C} \,\} \cup \{\, (a, b, 0, 0) \mid a, b \in \mathbb{C} \,\},$$

and in this case the Geometric Invariant Theory quotient is $\mathbb{P}^1 \times \mathbb{P}^1$. The key point is that everything to do with toric varieties is reduced to some combinatorics, which more often that not is quite manageable. One interesting class of toric varieties is obtained by taking $\mathbb{C}^{n+1}$ and taking an action of $\mathbb{G}_m$, with positive weights:

$$(x_0, x_1, \dots, x_n) \longrightarrow (t^{a_0}x_0, t^{a_1}x_1, \dots, t^{a_n}x_n),$$

The quotient is a toric variety M with Picard group isomorphic to $\mathbb{Z}$ (here we throw away the origin). The twist is that M has quotient singularities. Nevertheless the varieties one obtains this way are very similar to projective space and so they are called weighted projective space. They provide a very fertile place to look for interesting examples.

We are now able to state the main result due to Hu and Keel:

Theorem 3.1. *Let X be a smooth projective variety. Suppose that* $\mathrm{Pic}(X)$ *is a finitely generated abelian group.*

The following are equivalent:

(1) X *is a Mori dream space.*
(2) $\operatorname{Cox}(X)$ *is finitely generated.*

If either condition holds then X *is a Geometric Invariant Theory quotient of the affine variety* Y *corresponding to* $\operatorname{Cox}(X)$ *by a torus of dimension the rank of the Picard group.*

Proof. We sketch the proof. For full details see [13].

Suppose that X is a Mori dream space. Then the pseudo-effective cone $\mathcal{P}$ is equal to the effective cone $\mathcal{E}$ and both are rational polyhedra (see the discussion following (2.4)). Moreover there are rational polyhedra $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m$ such that

$$\mathcal{P} = \bigcup_{i=1}^{m} \mathcal{P}_i,$$

and finitely many birational maps $\phi_i \colon X \dashrightarrow X_i$ such that if $D \in \mathcal{P}$ then there is an index $1 \leq i \leq m$ such that $D \in \mathcal{P}_i$ and $\phi_D = \phi_i$. Suppose that $D_1, D_2, \ldots, D_k$ are divisors which generate the cone $\mathcal{P}_i$. Clearly it suffices to prove that

$$R(X, D_1, D_2, \ldots, D_k),$$

is finitely generated. This ring does not change if we replace X by X_i, and in this case each D_i is semiample, in which case finite generation is well known.

Now suppose that the Cox ring is finitely generated. One can check that X is the quotient of Y, the affine variety with coordinate ring $\operatorname{Cox}(X)$, by a torus, for some natural choice of linearisation (see the discussion at the start of §3). There are two ways to see that X is a Mori dream space. The first is to use the theory of Variation of Geometric Invariant Theory. The space of all linearisations of the action of G on Y is naturally a rational polyhedron $\mathcal{L}$, which is the union of finitely many rational polyhedra $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_k$. The Geometric Invariant Theory quotient corresponding to each of these polyhedra give the finitely many birational maps $f_i \colon X \dashrightarrow X_i$ which are isomorphisms in codimension one. The rest is then straightforward.

Here is another way to proceed. We are given a G-equivariant embedding $Y \subset \mathbb{C}^n$, (see the discussion at the start of §3). Taking the geometric invariant theory quotients, we get an embedding $X \subset M$, where $M = \mathbb{C}^n//G$. M is a toric variety, as G is a torus. There are two things to observe. The first is that M is a Mori dream space. This is well-known and reduces to some interesting combinatorics. The second is to observe that there is a partial correspondence between the Mori theory on M and X. For every rational map $\phi \colon X \dashrightarrow Y$ there is a rational map $\phi' \colon M \dashrightarrow N$ of toric varieties which induces ϕ by restriction. □

The reason why the correspondence is only partial is for trivial reasons. Suppose one took a threefold X in $\mathbb{P}^4$. Then one could take $M = \mathbb{P}^4$. But one could also take $M = \mathbb{P}^4 \times \mathbb{P}^1$. In this case the contraction $M \longrightarrow \mathbb{P}^4$ induces the trivial contraction on X.

It is interesting to see examples of Mori dream spaces. Every toric variety is a Mori dream space. If X is a hypersurface in $\mathbb{P}^n$ of dimension at least 3 then X is a Mori dream space. In this case $\operatorname{Pic}(X) = \mathbb{Z}$, there are no flips or flops and one may take Y be the affine cone over X, the inverse of X inside $\mathbb{C}^{n+1}$.

In fact curves C of genus $g \geq 1$ are never Mori dream spaces, since then $\operatorname{Pic}(C)$ is never a finitely generated abelian group. So smooth curves of degree at least three in $\mathbb{P}^2$ are never Mori dream spaces. The case of surfaces S in $\mathbb{P}^3$ seems quite mysterious. For most surfaces S of degree at least four, $\operatorname{Pic}(S) = \mathbb{Z}$, generated by a hyperplane class and S is a Mori dream space. But if S contains a line (for example) the Picard group has rank at least two. In fact from the point of view of Hodge theory, the locus of surfaces of degree d whose Picard group is not generated by the class of a hyperplane would seem to be the union of countably many components. Presumably for some surfaces in $\mathbb{P}^3$ the cone of effective divisors is not a rational polyhedron, but to determine precisely when this happens seems very subtle.

It has been proved recently that the component of the Hilbert scheme which parametrises pairs of codimension two linear subspaces of $\mathbb{P}^n$ is a Mori dream space, [6].

4. Mori theory

There are two very interesting sources for Mori dream spaces which come from Mori theory itself. We recall the definition of kawamata log terminal singularities:

Definition 4.1. *Let X be a smooth quasi-projective variety and let $\Delta = \sum a_i \Delta_i \geq 0$ be a $\mathbb{Q}$-divisor, so that $a_1, a_2, \dots, a_k$ are non-negative rational numbers. We say that the log pair (X, Δ) is **log smooth** if the support of Δ has normal crossings (that is, looks locally like a subset of the coordinate hyperplanes). If (X, Δ) is a log pair, then we say that a birational map $\pi \colon Y \longrightarrow X$ is a **log resolution** if the strict transform of Δ union the exceptional locus is log smooth. We say that (X, Δ) is kawamata log terminal if when we write*

$$K_Y + \Gamma = \pi^*(K_X + \Delta),$$

then $\llcorner \Gamma \lrcorner \leq 0$ *(that is, the coefficients of* Γ *are less than one).*

One should think of the kawamata log terminal condition as meaning that the singularities of the pair (X, Δ) are mild. Kawamata log terminal singularities have very many nice properties:

Lemma 4.2. *Let X be a smooth quasi-projective variety.*

(1) *If $D \geq 0$ is an effective divisor then there is a positive rational number $\epsilon > 0$ such that the pair $(X, \Delta = \epsilon D)$ is kawamata log terminal.*
(2) *More generally, if (X, Δ) is kawamata log terminal and $\Delta + D \geq 0$ then $(X, \Delta + \epsilon D)$ is kawamata log terminal for any small $\epsilon > 0$.*
(3) *If (X, Δ) is kawamata log terminal and D is semiample then we may find D' numerically equivalent to D such that $K_X + \Delta + D'$ is kawamata log terminal.*

Proof. In all cases, we start by picking a log resolution $\pi\colon Y \longrightarrow X$ of $(X, \Delta + D)$.

We first prove (1). If we write

$$K_Y = \pi^* K_X + E,$$

then $E \geq 0$ so that X is certainly kawamata log terminal. On the other hand, if we write

$$K_Y + \Gamma_t = \pi^*(K_X + tD),$$

then the coefficients of Γ_t are linear functions of t. In particular they are continuous functions of t and since Γ_t has only finitely many components, (1) is clear.

The proof of (2) is very similar to (1).

To see (3), note that if we choose m sufficiently large then we may find B numerically equivalent to mD such that the inverse image of $B + \Delta$ union the exceptional locus is log smooth. It is then clear that $K_X + \Delta + D'$ is kawamata log terminal, where $D' = B/m$. $\square$

The first general source of examples is due to Shokurov:

Theorem 4.3. *Let X be a smooth projective variety of dimension at most three. Let $\Delta_1, \Delta_2, \dots, \Delta_k$ be a sequence of $\mathbb{Q}$-divisors such that $K_X + \Delta_i$ is kawamata log terminal. Pick m_i such that $D_i = m_i(K_X + \Delta_i)$ is an integral divisor. Then the multigraded section ring*

$$R(X, D_1, D_2, \dots, D_k),$$

is finitely generated.

Proof. See [28]. $\square$

In fact one may view (4.3) as the result of the work of very many mathematicians, Kawamata, Kollár, Miyaoka, Mori, Reid and Shokurov, to name a few. The main steps are to prove the cone theorem, to establish the existence of flips, [21] and [27], termination of flips, to prove the abundance theorem, [14] and [20] and to prove the log abundance theorem [15], all of this for threefolds. Finally we need (3.1). The main obstruction to extending all of this to higher dimensions is to prove:

Conjecture 4.4 (Abundance)**.** *If (X, Δ) is kawamata log terminal and $K_X + \Delta$ is nef then it is semiample.*

Using (4.3) one can prove:

Corollary 4.5. *Let X be a Calabi-Yau variety of dimension at most three.*

Then X is a Mori dream space if and only if the cone $\mathcal{E}$ of effective divisors is a rational polyhedron.

Proof. As X is Calabi-Yau, $\mathrm{Pic}(X) = H^2(X, \mathbb{Z})$ is a finitely generated abelian group.

One direction is clear; if X is a Mori dream space then $\mathcal{E}$ is always a rational polyhedron. Now suppose that $\mathcal{E}$ is a rational polyhedron. Pick generators $B_1, B_2, \dots, B_k$, where $B_i \geq 0$. By (4.2) we may pick $\epsilon > 0$ such that if we set $\Delta_i = \epsilon B_i$ then $K_X + \Delta_i$ is kawamata log terminal. As X is Calabi-Yau, $K_X = 0$ and $K_X + \Delta_i = \epsilon B_i$. Now apply (4.3). □

Given a Calabi-Yau variety X it is quite delicate to determine if the effective cone of divisors is a rational polyhedron. This problem is even non-trivial in the simplest possible case, a smooth K3 surface S of Picard number two, see [25]. In this case the cone of pseudo-effective divisors $\mathcal{P}$ is a rational polyhedron. On the other hand it is a quite subtle question to determine when $\mathcal{E} = \mathcal{P}$, that is when $\mathcal{E}$ is closed.

Recall that D is big if ϕ_D is birational.

Theorem 4.6 (Birkar, Cascini, Hacon, -)**.** *Let X be a smooth projective variety.*

If $\Delta_1, \Delta_2, \dots, \Delta_k$ is a sequence of big $\mathbb{Q}$-divisors such that $K_X + \Delta_i$ is kawamata log terminal and we pick $m_1, m_2, \dots, m_k$ such that $D_i = m_i(K_X + \Delta_i)$ is an integral divisor then

$$R(X, D_1, D_2, \dots, D_k),$$

is finitely generated.

To prove (4.6) one needs two results. The first is the existence of flips in all dimensions, due to Hacon and me, and uses Siu's theory of multiplier ideal sheaves and some ideas of Shokurov, [9] and [10]. In fact Siu has proved a result very similar in spirit to (4.6), see [29]. The second is due to Birkar, Cascini, Hacon and me and uses the ideas of Shokurov contained in [28] and establishes termination of a very particular sequence flips in very much the same spirit as the argument at the end of §2.

It is also interesting to note that since then Lazić has given a new proof of (4.6) which is more direct and does not use existence and termination of flips, see [19]. There is also some work due to Păun, [26], who gives another proof of one of the key steps in the proof of (4.6).

Definition 4.7. *Let X be a smooth projective variety.*

We say that X is a ***(log) Fano variety*** *if there is a divisor Δ such that $K_X + \Delta$ is kawamata log terminal and $-(K_X + \Delta)$ is ample.*

Fano varieties are a very special class of varieties. In particular $\mathrm{Pic}(X) = H^2(X, \mathbb{Z})$ is always a finitely generated abelian group. Typical examples of log Fano varieties are toric varieties and Grassmannians. Note that if $X \subset \mathbb{P}^{n+1}$ is a smooth hypersurface of degree d then X is log Fano if and only if $d \leq n + 1$.

Lemma 4.8. *Let X be a log Fano.*

If D is any divisor then there is a positive rational number $\epsilon > 0$ and a big divisor Δ such that $K_X + \Delta$ is kawamata log terminal and ϵD is numerically equivalent to $K_X + \Delta$.

Proof. Let Θ be the divisor such that $-(K_X+\Theta)$ is ample and $K_X+\Theta$ is kawamata log terminal. Ampleness is an open condition. So we may pick $\epsilon > 0$ such that $\epsilon D - (K_X + \Theta)$ is ample. Now pick m sufficiently large and divisible and a general divisor B numerically equivalent to $m(\epsilon D - (K_X + \Theta))$. Let $H = B/m$. Then $H \geq 0$ and $K_X + \Delta = K_X + \Theta + H$ is kawamata log terminal and numerically equivalent to ϵD. □

Corollary 4.9. *Let X be a log Fano.*

Then X is a Mori dream space.

Proof. We have already seen that the Picard group is a finitely generated abelian group. Moreover, the cone of effective divisor is a rational polytope, using results of [2]. Pick generators $B_1, B_2, \ldots, B_k$. As in the proof of (4.5), using (4.8), we may assume that $B_i = K_X + \Delta_i$, where Δ_i is big and $K_X + \Delta_i$ is kawamata log terminal. Now apply (4.6). □

Recall that if S is a smooth surface and $-K_S$ is ample (which are known classically as del Pezzo surfaces) then S is either $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$

blown up along at most eight points in general position. The geometry of these surfaces is tied up in an intriguing way to the classification of Dynkin diagrams and root systems. Even in this special case the algebra of the Cox ring is very rich. Batyrev and Popa [1] have calculated explicit generators and relations for del Pezzo surfaces. Levitt in his Ph.D. thesis calculated the geometry of the embedding of S into the toric variety M corresponding to the Cox ring in some non-trivial special cases. Laface and Velasco have computed some other interesting examples [17] and have also written a survey article [18] which contains a more complete list of references.

It is interesting to wonder how close Mori dream spaces are to log Fano varieties. Here is one interesting characterisation, which is essentially due to Shokurov:

Lemma 4.10. *Let X be a smooth projective variety.*

If $-K_X$ is big and movable then X is a Mori dream space if and only if X is a log Fano variety.

Proof. One direction we have already seen; if X is log Fano then X is a Mori dream space.

So suppose that X is a Mori dream space. Then run $\phi\colon X \dashrightarrow Y$ the $-K_X$-MMP until $-K_Y$ is nef. As $-K_X$ is movable, ϕ is a composition of $-K_X$-flips; in particular it is an isomorphism in codimension one. As Y is a Mori dream space and $-K_Y$ is big and nef, $-K_Y$ is semiample. Pick G a general divisor numerically equivalent to $-mK_Y$, where m is sufficiently large and divisible. Put $\Gamma = G/m$. Then $K_Y + \Gamma$ is kawamata log terminal and numerically trivial. Let $\psi\colon Y \dashrightarrow X$ be the inverse birational map to ϕ. Then ψ is an isomorphism in codimension one. ψ is a sequence of $(K_Y+\Gamma)$-flops, so that if Δ is the strict transform of Γ, then $K_X + \Delta$ is kawamata log terminal and numerically trivial. As $-K_X$ is big then so is Δ. We may find an ample divisor A and an effective divisor $B \geq 0$ such that Δ is numerically equivalent to $A+B$. Then

$$K_X + \Theta = K_X + (1-\epsilon)\Delta + \epsilon B,$$

is kawamata log terminal, for any $\epsilon > 0$ sufficiently small. But

$$-(K_X + \Theta)$$

is numerically equivalent to

$$-(K_X + \Delta) + \epsilon A = \epsilon A,$$

so that X is log Fano. □

Note also that in the course of the proof of (4.8) we proved that X is log Fano if and only if there is divisor Δ such that $K_X + \Delta$ is kawamata log terminal and numerically trivial, where Δ is big. In particular, using this, it is easy to see that every projective toric variety is log Fano. If $D = \sum D_i$ is the sum of the invariant divisors then $K_M + D$ is numerically trivial and log canonical. D is big (as M is projective and the components of D generate the Picard group) and perturbing D a little we can find Δ numerically equivalent to D such that $K_M + \Delta$ is kawamata log terminal.

Conjecture 4.11 (Hering, Mustaţă, Payne, [12])**.** *Let M be a smooth projective toric variety, let E be a toric vector bundle on M and let X be the associated projective bundle.*

Then X is a Mori dream space.

Recall that a toric vector bundle is a vector bundle for which the torus actions lifts to the vector bundle. This conjecture has been proved if the vector bundle has rank two, see [11] and [8]. Note that it is not expected that X is always a log Fano.

5. Examples from Moduli spaces

Let us return to the problem of showing that the ring of invariants is finitely generated. By the results of Mukai and Nagata, we know that the ring of invariants is not necessarily finitely generated if we take $G = \mathbb{G}_a^r$, any $r \geq 3$. On the other hand, it is an old result due to Weitzenböck, [31], that the ring of invariants is finitely generated if $G = \mathbb{G}_a$. In fact the proof of this result is very interesting. Given an action of $\mathbb{G}_a$ on a ring R, one can find another ring S, and an action of $\mathrm{SL}(2, \mathbb{C})$ on S such that the ring of invariants are the same. As the last group is reductive, finite generation follows from Hilbert's original result. This suggests that one can prove that R^G is finitely generated by relating it to the Cox ring of some quotient $X// \mathrm{SL}(2, \mathbb{C})$ for an appropriate variety X.

The unresolved case is $G = \mathbb{G}_a \oplus \mathbb{G}_a$. It is then natural to ask:

Question 5.1. *Let R be a finitely generated ring and suppose that $\mathbb{G}_a \oplus \mathbb{G}_a$ acts on R.*

Is the ring of invariants R^G finitely generated?

Mukai answered this question affirmatively in the case of Nagata's action. In this case we have $\mathbb{P}^n$ blown up in $n + 3$ points and we are asking if the Cox ring is finitely generated. It is a classic result in projective geometry that there is a unique rational curve of degree n in $\mathbb{P}^n$ containing $n+3$ points in linear general position. In fact Castravet and Tevelev, [5], proved that the Cox ring of any blow up of $\mathbb{P}^n$ along any number of points contained on a rational normal curve is finitely generated. In both cases, the authors write down explicit generators.

There is one very interesting class of projective varieties which are closely related to Nagata's example. Suppose one takes $\mathbb{P}^n$ and picks $n+2$ general points (so one less than the case above) in linear general position. In fact $n+2$ points in linear general position in $\mathbb{P}^n$ have no moduli; we may always choose coordinates so that the points are $p_1 = [1:0:0:\cdots:0]$, $p_2 = [0:1:0:\cdots:0]$, ..., $p_{n+1} = [0:0:0:\cdots:1]$, and $p_{n+2} = [1:1:1:\cdots:1]$. Now blow up these points, the lines that connect them, the planes spanned by any three points, and so on. The resulting space is $\overline{M}_{0,n+3}$, the moduli space of stable curves of genus zero with $n+3$ marked points. For obvious reasons it is convenient to shift indices and consider $\overline{M}_{0,n}$. A curve C of genus zero is a tree of $\mathbb{P}^1$'s with only nodes as singularities. The n marked points are n points $q_1, q_2, \dots, q_n$ of C, not at the nodes. Stable means that each component of C has at least three special points (a node or a marked point). $\overline{M}_{0,n}$ naturally parametrises such curves up to isomorphism.

There are many ways to see that one needs to start with $\mathbb{P}^{n-3}$ and blow up. One way is as follows: note that there is a unique rational normal curve of degree $n-3$ through the $n-1$ fixed points and one further point p, in linear general position. This rational normal curve is a stable curve with n marked points. Moving p around we get an open subset of $\overline{M}_{0,n}$. But if we choose p belonging to a proper linear subspace spanned by some subset of $p_1, p_2, \dots, p_{n-1}$ then the rational normal curve breaks into pieces and we need to blow up. For example, if $n=3$ then $\overline{M}_{0,3}$ is a point. Any three points in $\mathbb{P}^1$ are equivalent to 0, 1 and ∞. $\overline{M}_{0,4}$ is a copy of $\mathbb{P}^1$. We fix the first three points to be 0, 1 and ∞ (recall that the action of $\mathrm{PGL}(2,\mathbb{C})$ on $\mathbb{P}^1$ is precisely thrice transitive) and the fourth point is a natural parameter on $\overline{M}_{0,4}$. To construct $\overline{M}_{0,5}$ take four points in $\mathbb{P}^2$ and look at the pencil of conics containing these four points. Note that there are three degenerate conics that contain these four points; just partition the four points into two sets of two and take the corresponding lines. $\overline{M}_{0,5}$ is the blow up of $\mathbb{P}^2$ in the four points.

Question 5.2 (Hu, Keel)**.** *Is $\overline{M}_{0,n}$ a Mori dream space?*

It is shown in [16] that $\overline{M}_{0,n}$ is log Fano if and only if $n \leq 6$. On the other hand, Castravet [3] has given a direct proof of finite generation of the Cox ring, when $n=6$.

There are some very interesting related conjectures:

Conjecture 5.3 (Faber, Fulton, Mumford)**.** *Is the cone of curves the rational polyhedron spanned by the* 1*-strata of* $\overline{M}_{0,n}$*?*

The cone of curves is the closure of the cone in $H_2(X,\mathbb{R})$ spanned by the classes of the effective curves. $\overline{M}_{0,n}$ is naturally stratified by the number of components in the tree. A stable curve breaks into two pieces in codimension one. Stable curves of genus 0 with $n-3$ components therefore form a curve Σ in $\overline{M}_{0,n}$ and the conjecture asks if the components of Σ generate the cone of curves. In fact the space of curves with two components is a divisor in $\overline{M}_{0,n}$, which is called the boundary divisor. If D is the sum of the boundary divisors then the log pair $(\overline{M}_{0,n}, D)$ is log smooth. The components of the boundary divisor are naturally indexed by all ways to partition $\{1,2,\dots,n\}$ into two subsets S and T, where both S and T have at least two elements. The component $D_{S,T}$ is a product of two copies of $\overline{M}_{0,k}$ and $\overline{M}_{0,l}$ where $k=|S|+1$ and $l=|T|+1$. By way of induction, it therefore suffices to prove that any curve in $\overline{M}_{0,n}$ is numerically equivalent to a curve inside this divisor.

Keel, Gibney and Morrison [7] have shown that (5.3) implies that the analogous conjecture for $\overline{M}_{g,n}$ holds. Probably a key step to prove (5.3) would be to identify the cone of effective divisors. It is known that this is not the cone spanned by the boundary divisors. This was observed independently by Keel and Vermeire, [30]. Part of the problem is that given a rational polytope $\mathcal{P}$ it is very hard to compute the dual polytope $\mathcal{Q}$. It is known that there are very many non-trivial contractions $\phi\colon \overline{M}_{0,n} \longrightarrow Y$, but giving an exhaustive list seems quite formidable. Nevertheless Castravet and Tevelev [4] have identified a potentially exhaustive list of effective divisors on $\overline{M}_{0,n}$.

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