EXISTENCE OF MINIMAL MODELS FOR VARIETIES OF LOG GENERAL TYPE

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Abstract. We prove that the canonical ring of a smooth projective variety is finitely generated.

Contents

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1. Introduction

The purpose of this paper is to prove the following result in birational algebraic geometry:

Theorem 1.1. *Let (X, Δ) be a projective kawamata log terminal pair.*
If Δ is big and $K_X + \Delta$ is pseudo-effective then $K_X + \Delta$ has a log terminal model.

In particular, it follows that if $K_X + \Delta$ is big then it has a log canonical model and the canonical ring is finitely generated. It also follows that if X is a smooth projective variety, then the ring

$$R(X, K_X) = \bigoplus_{m \in \mathbb{N}} H^0(X, \mathcal{O}_X(mK_X)),$$

is finitely generated.

The birational classification of complex projective surfaces was understood by the Italian Algebraic Geometers in the early 20th century: If X is a smooth complex projective surface of non-negative Kodaira dimension, that is $\kappa(X, K_X) \geq 0$, then there is a unique smooth surface Y birational to X such that the canonical class K_Y is nef (that is $K_Y \cdot C \geq 0$ for any curve $C \subset Y$). Y is obtained from X simply by contracting all -1-curves, that is all smooth rational curves E with $K_X \cdot E = -1$. If, on the other hand, $\kappa(X, K_X) = -\infty$, then X is birational to either $\mathbb{P}^2$ or a ruled surface over a curve of genus $g > 0$.

The minimal model program aims to generalise the classification of complex projective surfaces to higher dimensional varieties. The main goal of this program is to show that given any n-dimensional smooth complex projective variety X, we have:

- If $\kappa(X, K_X) \geq 0$, then there exists a **minimal model**, that is a variety Y birational to X such that K_Y is nef.
- If $\kappa(X, K_X) = -\infty$, then there is a variety Y birational to X which admits a **Fano fibration**, that is a morphism $Y \longrightarrow Z$ whose general fibres F have ample anticanonical class $-K_F$.

It is possible to exhibit 3-folds which have no smooth minimal model, see for example (16.17) of [37], and so one must allow varieties X with singularities. However, these singularities cannot be arbitrary. At the very minimum, we must still be able to compute $K_X \cdot C$ for any curve $C \subset X$. So, we insist that K_X is $\mathbb{Q}$-Cartier (or sometimes we require the stronger property that X is $\mathbb{Q}$-factorial). We also require that X and the minimal model Y have the same pluricanonical forms. This condition is essentially equivalent to requiring that the induced birational map $\phi\colon X \dashrightarrow Y$ is K_X-non-positive.

There are two natural ways to construct the minimal model (it turns out that if one can construct a minimal model for pseudo-effective K_X, then we can construct Mori fibre spaces whenever K_X is not pseudo-effective). Since one of the main ideas of this paper is to blend the techniques of both methods, we describe both methods.

The first method is to use the ideas behind finite generation. If the canonical ring

$$R(X, K_X) = \bigoplus_{m \in \mathbb{N}} H^0(X, \mathcal{O}_X(mK_X)),$$

is finitely generated and K_X is big, then the canonical model Y is nothing more than the Proj of $R(X, K_X)$. It is then automatic that the induced rational map $\phi\colon X \dashrightarrow Y$ is K_X-negative.

The other natural way to ensure that ϕ is K_X-negative is to factor ϕ into a sequence of elementary steps all of which are K_X-negative. We now explain one way to achieve this factorisation.

If K_X is not nef, then, by the cone theorem, there is a rational curve $C \subset X$ such that $K_X \cdot C < 0$ and a morphism $f\colon X \longrightarrow Z$ which is surjective, with connected fibres, on to a normal projective variety and which contracts an irreducible curve D if and only if $[D] \in \mathbb{R}^+[C] \subset N_1(X)$. Note that $\rho(X/Z) = 1$ and $-K_X$ is f-ample. We have the following possibilities:

- If $\dim Z < \dim X$, this is the required Fano fibration.
- If $\dim Z = \dim X$ and f contracts a divisor, then we say that f is a divisorial contraction and we replace X by Z.

- If $\dim Z = \dim X$ and f does not contract a divisor, then we say that f is a small contraction. In this case K_Z is not $\mathbb{Q}$-Cartier, so that we can not replace X by Z. Instead, we would like to replace $f\colon X \longrightarrow Z$ by its flip $f^+\colon X^+ \longrightarrow Z$ where X^+ is isomorphic to X in codimension 1 and K_{X^+} is f^+-ample. In other words, we wish to replace some K_X-negative curves by K_{X^+}-positive curves.

The idea is to simply repeat the above procedure until we obtain either a minimal model or a Fano fibration. For this procedure to succeed, we must show that flips always exist and that they eventually terminate. Since the Picard number $\rho(X)$ drops by one after each divisorial contraction and is unchanged after each flip, there can be at most finitely many divisorial contractions. So we must show that there is no infinite sequence of flips.

This program was successfully completed for 3-folds in the 1980's by the work of Kawamata, Kollár, Mori, Reid, Shokurov and others. In particular, the existence of 3-fold flips was proved by Mori in [26].

Naturally, one would hope to extend these results to dimension 4 and higher by induction on the dimension.

Recently, Shokurov has shown the existence of flips in dimension 4 [34] and Hacon and M$^{\text{c}}$Kernan [8] have shown that assuming the minimal model program in dimension $n-1$ (or even better simply finiteness of minimal models in dimension $n-1$), then flips exist in dimension n. Thus we get an inductive approach to finite generation.

Unfortunately the problem of showing termination of an arbitrary sequence of flips seems to be a very difficult problem and in dimension ≥ 4 only some partial answers are available. Kawamata, Matsuda and Matsuki proved [18] the termination of terminal 4-fold flips, Matsuki has shown [25] the termination of terminal 4-fold flops and Fujino has shown [5] the termination of canonical 4-fold (log) flips. Alexeev, Hacon and Kawamata [1] have shown the termination of kawamata log terminal 4-fold flips when the Kodaira dimension of $-(K_X+\Delta)$ is non-negative and the existence of minimal models of kawamata log terminal 4-folds when either Δ or $K_X+\Delta$ is big by showing the termination of a certain sequence of flips (those that appear in the MMP with scaling). However, it is known that termination of flips follows from two natural conjectures on the behaviour of the log discrepancies of n-dimensional pairs (namely the ascending chain condition for minimal log discrepancies and semicontinuity of log discrepancies, cf. [35]). Moreover, if $\kappa(X, K_X+\Delta) \geq 0$, Birkar has shown [2] that it suffices to establish acc for log canonical thresholds and the MMP in dimension one less.

We now turn to the main result of the paper:

Theorem 1.2. *Let (X,Δ) be a kawamata log terminal pair, where $K_X+\Delta$ is $\mathbb{R}$-Cartier. Let $\pi\colon X\longrightarrow U$ be a projective morphism of quasi-projective varieties.*

If either Δ is π-big and $K_X+\Delta$ is π-pseudo-effective or $K_X+\Delta$ is π-big, then

1. *$K_X+\Delta$ has a log terminal model over U,*
2. *if $K_X+\Delta$ is π-big then $K_X+\Delta$ has a log canonical model over U, and*
3. *if $K_X+\Delta$ is $\mathbb{Q}$-Cartier, then the $\mathcal{O}_U$-algebra*
$$\mathfrak{R}(\pi,\Delta)=\bigoplus_{m\in\mathbb{N}}\pi_*\mathcal{O}_X(\llcorner m(K_X+\Delta)\lrcorner),$$
is finitely generated.

We now present some consequences of (1.2), most of which are known to follow from the MMP. Even though we do not prove termination of flips, we are able to derive many of the consequences of the existence of the MMP. In many cases we do not state the strongest results possible; anyone interested in further applications is directed to the references. We group these consequences under different headings.

1.1. **Minimal models.** An immediate consequence of (1.2) is:

Corollary 1.1.1. *Let X be a smooth projective variety of general type. Then*

1. *X has a minimal model,*
2. *X has a canonical model,*
3. *the ring*
$$R(X,K_X)=\bigoplus_{m\in\mathbb{N}}H^0(X,\mathcal{O}_X(mK_X)),$$
is finitely generated, and
4. *X has a model with a Kähler-Einstein metric.*

Note that (4) follows from (2) and Theorem D of [4]. Note that Siu has announced a proof of finite generation for varieties of general type, using analytic methods, see [36].

Corollary 1.1.2. *Let (X,Δ) be a projective kawamata log terminal pair, where $K_X+\Delta$ is $\mathbb{Q}$-Cartier.*

Then the ring
$$R(X,K_X+\Delta)=\bigoplus_{m\in\mathbb{N}}H^0(X,\mathcal{O}_X(\llcorner m(K_X+\Delta)\lrcorner)),$$

is finitely generated.

Let us emphasize that in (1.1.2) we make no assumption about $K_X + \Delta$ or Δ being big. Indeed Fujino and Mori, [6], proved that (1.1.2) follows from the case when $K_X + \Delta$ is big.

We will now turn our attention to the geography of minimal models. It is well known that log terminal models are not unique. The first natural question about log terminal models is to understand how any two are related. In fact there is a very simple connection:

Corollary 1.1.3. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties. Suppose that $K_X + \Delta$ is kawamata log terminal and Δ is big over U. Let $\phi_i\colon X \dashrightarrow Y_i$, $i = 1$ and 2 be two log terminal models of (X, Δ) over U. Let $\Gamma_i = \phi_{i*}\Delta$.*

Then the birational map $Y_1 \dashrightarrow Y_2$ is the composition of a sequence of $(K_{Y_1} + \Gamma_1)$-flops over U.

Note that (1.1.3) has been generalised recently to the case when Δ is not assumed big, [17]. The next natural problem is to understand how many different models there are. Even if log terminal models are not unique, in many important contexts, there are only finitely many. In fact Shokurov realised that much more is true. He realised that the dependence on Δ is well-behaved. To explain this, we need some definitions:

Definition 1.1.4. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties, and let V be a finite dimensional affine subspace of the real vector space $\mathrm{WDiv}_{\mathbb{R}}(X)$ of Weil divisors on X. Fix an $\mathbb{R}$-divisor $A \geq 0$ and define*

$$\begin{aligned}
V_A &= \{\,\Delta \,|\, \Delta = A + B, B \in V\,\},\\
\mathcal{L}_A(V) &= \{\,\Delta = A + B \in V_A \,|\, K_X + \Delta \text{ is log canonical and } B \geq 0\,\},\\
\mathcal{E}_{A,\pi}(V) &= \{\,\Delta \in \mathcal{L}_A(V) \,|\, K_X + \Delta \text{ is pseudo-effective over } U\,\},\\
\mathcal{N}_{A,\pi}(V) &= \{\,\Delta \in \mathcal{L}_A(V) \,|\, K_X + \Delta \text{ is nef over } U\,\}.
\end{aligned}$$

Given a birational contraction $\phi\colon X \dashrightarrow Y$ over U, define

$$\mathcal{W}_{\phi,A,\pi}(V) = \{\,\Delta \in \mathcal{E}_{A,\pi}(V) \,|\, \phi \text{ is a weak log canonical model for } (X,\Delta) \text{ over } U\,\},$$

and given a rational map $\psi\colon X \dashrightarrow Z$ over U, define

$$\mathcal{A}_{\psi,A,\pi}(V) = \{\,\Delta \in \mathcal{E}_{A,\pi}(V) \,|\, \psi \text{ is the ample model for } (X,\Delta) \text{ over } U\,\},$$

(cf. (3.6.7) *and* (3.6.5) *for the definitions of weak log canonical model and ample model for (X, Δ) over U).*

We will adopt the convention that $\mathcal{L}(V) = \mathcal{L}_0(V)$. If the support of A has no components in common with any element of V then the condition that $B \geq 0$ is vacuous. In many applications, A will be an ample $\mathbb{Q}$-divisor over U. In this case, we often assume that A is *general* in the sense that we fix a positive integer such that kA is very ample over U, and we assume that $A = \frac{1}{k}A'$, where $A' \sim_U kA$ is very general. With this choice of A, we have

$$\mathcal{N}_{A,\pi}(V) \subset \mathcal{E}_{A,\pi}(V) \subset \mathcal{L}_A(V) = \mathcal{L}(V) + A \subset V_A = V + A,$$

and the condition that the support of A has no common components with any element of V is then automatic. The following result was first proved by Shokurov [32] assuming the existence and termination of flips:

Corollary 1.1.5. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties. Let V be a finite dimensional affine subspace of $\operatorname{WDiv}_{\mathbb{R}}(X)$ which is defined over the rationals. Suppose there is a divisor $\Delta_0 \in V$ such that $K_X + \Delta_0$ is kawamata log terminal. Let A be a general ample $\mathbb{Q}$-divisor over U, which has no components in common with any element of V.*

(1) *There are finitely many birational contractions $\phi_i\colon X \dashrightarrow Y_i$ over U, $1 \leq i \leq p$ such that*

$$\mathcal{E}_{A,\pi}(V) = \bigcup_{i=1}^{p} \mathcal{W}_i,$$

where each $\mathcal{W}_i = \mathcal{W}_{\phi_i,A,\pi}(V)$ is a rational polytope. Moreover, if $\phi\colon X \dashrightarrow Y$ is a log terminal model of (X,Δ) over U, for some $\Delta \in \mathcal{E}_{A,\pi}(V)$, then $\phi = \phi_i$, for some $1 \leq i \leq p$.

(2) *There are finitely many rational maps $\psi_j\colon X \dashrightarrow Z_j$ over U, $1 \leq j \leq q$ which partition $\mathcal{E}_{A,\pi}(V)$ into the subsets $\mathcal{A}_j = \mathcal{A}_{\psi_j,A,\pi}(V)$.*

(3) *For every $1 \leq i \leq p$ there is a $1 \leq j \leq q$ and a morphism $f_{i,j}\colon Y_i \longrightarrow Z_j$ such that $\mathcal{W}_i \subset \bar{\mathcal{A}}_j$.*

In particular $\mathcal{E}_{A,\pi}(V)$ is a rational polytope and each $\bar{\mathcal{A}}_j$ is a finite union of rational polytopes.

Definition 1.1.6. *Let (X,Δ) be a kawamata log terminal pair and let D be a big divisor. Suppose that $K_X + \Delta$ is not pseudo-effective. The **effective log threshold** is*

$$\sigma(X,\Delta,D) = \sup\{\, t \in \mathbb{R} \mid D + t(K_X + \Delta) \text{ is pseudo-effective}\}.$$

*The **Kodaira energy** is the reciprocal of the effective log threshold.*

Following ideas of Batyrev, one can easily show that:

Corollary 1.1.7. *Let (X, Δ) be a projective kawamata log terminal pair and let D be an ample divisor. Suppose that $K_X + \Delta$ is not pseudo-effective.*

If both $K_X + \Delta$ and D are $\mathbb{Q}$-Cartier then the effective log threshold and the Kodaira energy are rational.

Definition 1.1.8. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties. Let $D^\bullet = (D_1, D_2, \dots, D_k)$ be a sequence of $\mathbb{Q}$-divisors on X. The sheaf of $\mathcal{O}_U$-modules*

$$\mathfrak{R}(\pi, D^\bullet) = \bigoplus_{m \in \mathbb{N}^k} \pi_* \mathcal{O}_X(\llcorner \sum m_i D_i \lrcorner),$$

is called the ***Cox ring*** *associated to $D^\bullet$.*

Using (1.1.5) one can show that adjoint Cox rings are finitely generated:

Corollary 1.1.9. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties. Fix $A \geq 0$ an ample $\mathbb{Q}$-divisor over U. Let $\Delta_i = A + B_i$, for some $\mathbb{Q}$-divisors $B_1, B_2, \dots, B_k \geq 0$. Assume that $D_i = K_X + \Delta_i$ is divisorially log terminal and $\mathbb{Q}$-Cartier. Then the Cox ring*

$$\mathfrak{R}(\pi, D^\bullet) = \bigoplus_{m \in \mathbb{N}^k} \pi_* \mathcal{O}_X(\llcorner \sum m_i D_i \lrcorner),$$

is a finitely generated $\mathcal{O}_U$-module.

1.2. **Moduli spaces.** At first sight (1.1.5) might seem a hard result to digest. For this reason, we would like to give a concrete, but non-trivial example. The moduli spaces $\overline{M}_{g,n}$ of n-pointed stable curves of genus g are probably the most intensively studied moduli spaces. In particular the problem of trying to understand the related log canonical models via the theory of moduli has attracted a lot of attention (e.g. see [7], [24] and [11]).

Corollary 1.2.1. *Let $X = \overline{M}_{g,n}$ the moduli space of stable curves of genus g with n marked points and let Δ_i, $1 \leq i \leq k$ denote the boundary divisors.*

Let $\Delta = \sum_i a_i \Delta_i$ be a boundary. Then $K_X + \Delta$ is log canonical and if $K_X + \Delta$ is big then there is a log canonical model $X \dashrightarrow Y$. Moreover if we fix a positive rational number δ and require that the coefficient a_i of Δ_i is at least δ for each i then the set of all log canonical models obtained this way is finite.

1.3. **Fano varieties.** The next set of applications is to Fano varieties. The key observation is that given any divisor D, a small multiple of D is linearly equivalent to a divisor of the form $K_X + \Delta$, where Δ is big and $K_X + \Delta$ is kawamata log terminal.

Definition 1.3.1. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal varieties, where U is affine.*

*We say that X is a **Mori dream space** if $h^1(X, \mathcal{O}_X) = 0$ and the Cox ring is finitely generated over the coordinate ring of U.*

Corollary 1.3.2. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal varieties, where U is affine. Suppose that X is $\mathbb{Q}$-factorial, $K_X + \Delta$ is divisorially log terminal and $-(K_X + \Delta)$ is ample over U.*

Then X is a Mori dream space.

There are many reasons why Mori dream spaces are interesting. As the name might suggest, they behave very well with respect to the minimal model program. Given any divisor D, one can run the D-MMP, and this ends with either a nef model, or a fibration, for which $-D$ is relatively ample, and in fact any sequence of D-flips terminates.

(1.3.2) was conjectured in [12] where it is also shown that Mori dream spaces are GIT quotients of affine varieties by a torus. Moreover the decomposition given in (1.1.5) is induced by all the possible ways of taking GIT quotients, as one varies the linearisation.

Finally, it was shown in [12] that if one has a Mori dream space, then the Cox Ring is finitely generated.

We next prove a result that naturally complements (1.2). We show that if $K_X + \Delta$ is not pseudo-effective, then we can run the MMP with scaling to get a Mori fibre space:

Corollary 1.3.3. *Let (X, Δ) be a $\mathbb{Q}$-factorial kawamata log terminal pair. Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties. Suppose that $K_X + \Delta$ is not π-pseudo-effective.*

Then we may run $f\colon X \dashrightarrow Y$ a $(K_X + \Delta)$-MMP over U and end with a Mori fibre space $g\colon Y \longrightarrow W$ over U.

Note that we do not claim in (1.3.3) that however we run the $(K_X + \Delta)$-MMP over U, we always end with a Mori fibre space, that is we do not claim that every sequence of flips terminates.

Finally we are able to prove a conjecture of Batyrev on the closed cone of nef curves for a Fano pair.

Definition 1.3.4. *Let X be a projective variety. A curve Σ is called **nef** if $B \cdot \Sigma \geq 0$ for all Cartier divisors $B \geq 0$. $\mathrm{NF}(X)$ denotes the cone of nef curves sitting inside $H_2(X, \mathbb{R})$ and $\overline{\mathrm{NF}}(X)$ denotes its closure.*

*Now suppose that (X, Δ) is a log pair. A $(K_X + \Delta)$-**co-extremal ray** is an extremal ray F of the closed cone of nef curves $\overline{\mathrm{NF}}(X)$ on which $K_X + \Delta$ is negative.*

Corollary 1.3.5. *Let (X, Δ) be a projective $\mathbb{Q}$-factorial kawamata log terminal pair such that $-(K_X + \Delta)$ is ample.*

Then $\overline{\mathrm{NF}}(X)$ is a rational polyhedron. If $F = F_i$ is a $(K_X + \Delta)$-co-extremal ray then there exists an $\mathbb{R}$-divisor Θ such that the pair (X, Θ) is kawamata log terminal and the $(K_X + \Theta)$-MMP $\pi \colon X \dashrightarrow Y$ ends with a Mori fibre space $f \colon Y \longrightarrow Z$ such that F is spanned by the pullback to X of the class of any curve Σ which is contracted by f.

1.4. **Birational geometry.** Another immediate consequence of (1.2) is the existence of flips:

Corollary 1.4.1. *Let (X, Δ) be a kawamata log terminal pair and let $\pi \colon X \longrightarrow Z$ be a small $(K_X + \Delta)$-extremal contraction.*

Then the flip of π exists.

As already noted, we are unable to prove the termination of flips in general. However, using (1.1.5), we can show that any sequence of flips for the MMP with scaling terminates:

Corollary 1.4.2. *Let $\pi \colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties. Let (X, Δ) be a $\mathbb{Q}$-factorial kawamata log terminal pair, where $K_X + \Delta$ is $\mathbb{R}$-Cartier and Δ is π-big. Let $C \geq 0$ be an $\mathbb{R}$-divisor.*

If $K_X + \Delta + C$ is kawamata log terminal and π-nef, then we may run the $(K_X + \Delta)$-MMP over U with scaling of C.

Another application of (1.2) is the existence of log terminal models which extract certain divisors:

Corollary 1.4.3. *Let (X, Δ) be a log canonical pair and let $f \colon W \longrightarrow X$ be a log resolution. Suppose that there is a divisor Δ_0 such that $K_X + \Delta_0$ is kawamata log terminal. Let $\mathfrak{E}$ be any set of valuations of f-exceptional divisors which satisfies the following two properties:*

(1) *$\mathfrak{E}$ contains only valuations of log discrepancy at most one, and*
(2) *the centre of every valuation of log discrepancy one in $\mathfrak{E}$ does not contain any non kawamata log terminal centres.*

Then we may find a birational morphism $\pi \colon Y \longrightarrow X$, such that Y is $\mathbb{Q}$-factorial and the exceptional divisors of π correspond to the elements of $\mathfrak{E}$.

For example, if we assume that (X, Δ) is kawamata log terminal and we let $\mathfrak{E}$ be the set of all exceptional divisors with log discrepancy at most one, then the birational morphism $\pi\colon Y \longrightarrow X$ defined in (1.4.3) above is a **terminal model** of (X,Δ). In particular there is an $\mathbb{R}$-divisor $\Gamma \geq 0$ on W such that $K_Y + \Gamma = \pi^*(K_X + \Delta)$ and the pair (Y,Γ) is terminal.

If instead we assume that (X,Δ) is kawamata log terminal but $\mathfrak{E}$ is empty, then the birational morphism $\pi\colon Y \longrightarrow X$ defined in (1.4.3) above is a log terminal model. In particular π is small, Y is $\mathbb{Q}$-factorial and there is an $\mathbb{R}$-divisor $\Gamma \geq 0$ on Y such that $K_Y + \Gamma = \pi^*(K_X + \Delta)$.

We are able to prove that every log pair admits a birational model with $\mathbb{Q}$-factorial singularities such that the non kawamata log terminal locus is a divisor:

Corollary 1.4.4. *Let (X,Δ) be a log pair.*

Then there is a birational morphism $\pi\colon Y \longrightarrow X$, where Y is $\mathbb{Q}$-factorial, such that if we write

$$K_Y + \Gamma = K_Y + \Gamma_1 + \Gamma_2 = \pi^*(K_X + \Delta),$$

where every component of Γ_1 has coefficient less than one and every component of Γ_2 has coefficient at least one, then $K_Y + \Gamma_1$ is kawamata log terminal and nef over X and no component of Γ_1 is exceptional.

Even thought the result in (1.4.3) is not optimal as it does not fully address the log canonical case, nevertheless, we are able to prove the following result (cf. [31], [21], [13]):

Corollary 1.4.5 (Inversion of adjunction)**.** *Let (X,Δ) be a log pair and let $\nu\colon S \longrightarrow S'$ be the normalisation of a component S' of Δ of coefficient one.*

If we define Θ by adjunction,

$$\nu^*(K_X + \Delta) = K_S + \Theta,$$

then the log discrepancy of $K_S + \Theta$ is equal to the minimum of the log discrepancy with respect to $K_X + \Delta$ of any valuation whose centre on X is of codimension at least two and intersects S.

One of the most compelling reasons to enlarge the category of varieties to the category of algebraic spaces (equivalently Moishezon spaces, at least in the proper case) is to allow the possibility of cut and paste operations, such as one can perform in topology. Unfortunately, it is then all too easy to construct proper smooth algebraic spaces over $\mathbb{C}$, which are not projective. In fact the appendix to [10] has two very well known examples due to Hironaka. In both examples, one exploits the fact that for two curves in a threefold which intersect in a node, the order in which one blows up the curves is important (in fact the resulting threefolds are connected by a flop).

It is then natural to wonder if this is the only way to construct such examples, in the sense that if a proper algebraic space is not projective then it must contain a rational curve. Kollár dealt with the case when X is a terminal threefold with Picard number one, see [19]. In a slightly different but related direction, it is conjectured that if a complex Kähler manifold M does not contain any rational curves then K_M is nef (see for example [30]), which would extend some of Mori's famous results from the projective case. Kollár also has some unpublished proofs of some related results.

The following result, which was proved by Shokurov assuming the existence and termination of flips, cf. [33], gives an affirmative answer to the first conjecture and at the same time connects the two conjectures:

Corollary 1.4.6. *Let $\pi\colon X \longrightarrow U$ be a proper map of normal algebraic spaces, where X is analytically $\mathbb{Q}$-factorial.*

If $K_X + \Delta$ is divisorially log terminal and π does not contract any rational curves then π is a log terminal model. In particular π is projective and $K_X + \Delta$ is π-nef.

2. Description of the proof

Theorem A (Existence of pl-flips)**.** *Let $f\colon X \longrightarrow Z$ be a pl-flipping contraction for an n-dimensional purely log terminal pair (X, Δ).*

Then the flip $f^+\colon X^+ \longrightarrow Z$ of f exists.

Theorem B (Special finiteness)**.** *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties, where X is $\mathbb{Q}$-factorial of dimension n. Let V be a finite dimensional affine subspace of $\operatorname{WDiv}_{\mathbb{R}}(X)$, which is defined over the rationals, let S be the sum of finitely many prime divisors and let A be a general ample $\mathbb{Q}$-divisor over U. Let (X, Δ_0) be a divisorially log terminal pair such that $S \leq \Delta_0$. Fix a finite set $\mathfrak{E}$ of prime divisors on X.*

Then there are finitely $1 \leq i \leq k$ many birational maps $\phi_i\colon X \dashrightarrow Y_i$ over U such that if $\phi\colon X \dashrightarrow Y$ is any $\mathbb{Q}$-factorial weak log canonical model over U of $K_X + \Delta$ where $\Delta \in \mathcal{L}_{S+A}(V)$ which only contracts elements of $\mathfrak{E}$ and which does not contract every component of S, then there is an index $1 \leq i \leq k$ such that the induced birational map $\xi\colon Y_i \dashrightarrow Y$ is an isomorphism in a neighbourhood of the strict transforms of S.

Theorem C (Existence of log terminal models)**.** *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties, where X has dimension n. Suppose that $K_X + \Delta$ is kawamata log terminal, where Δ is big over U.*

If there exists an $\mathbb{R}$-divisor D such that $K_X + \Delta \sim_{\mathbb{R},U} D \geq 0$, then $K_X + \Delta$ has a log terminal model over U.

Theorem D (Non-vanishing theorem)**.** *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties, where X has dimension n. Suppose that $K_X + \Delta$ is kawamata log terminal, where Δ is big over U.*

If $K_X + \Delta$ is π-pseudo-effective, then there exists an $\mathbb{R}$-divisor D such that $K_X + \Delta \sim_{\mathbb{R},U} D \geq 0$.

Theorem E (Finiteness of models)**.** *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties, where X has dimension n. Fix a general ample $\mathbb{Q}$-divisor $A \geq 0$ over U. Let V be a finite dimensional affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$ which is defined over the rationals. Suppose that there is a kawamata log terminal pair (X, Δ_0).*

Then there are finitely many birational maps $\psi_j\colon X \dashrightarrow Z_j$ over U, $1 \leq j \leq l$ such that if $\psi\colon X \dashrightarrow Z$ is a weak log canonical model of $K_X + \Delta$ over U, for some $\Delta \in \mathcal{L}_A(V)$ then there is an index $1 \leq j \leq l$ and an isomorphism $\xi\colon Z_j \longrightarrow Z$ such that $\psi = \xi \circ \psi_j$.

Theorem F (Finite generation)**.** *Let $\pi\colon X \longrightarrow Z$ be a projective morphism to a normal affine variety. Let $(X, \Delta = A + B)$ be a kawamata log terminal pair of dimension n, where $A \geq 0$ is an ample $\mathbb{Q}$-divisor and $B \geq 0$. If $K_X + \Delta$ is pseudo-effective, then*

(1) *The pair (X, Δ) has a log terminal model $\mu\colon X \dashrightarrow Y$. In particular if $K_X + \Delta$ is $\mathbb{Q}$-Cartier then the log canonical ring*

$$R(X, K_X + \Delta) = \bigoplus_{m \in \mathbb{N}} H^0(X, \mathcal{O}_X(\llcorner m(K_X + \Delta) \lrcorner)),$$

is finitely generated.

(2) *Let $V \subset \mathrm{WDiv}_{\mathbb{R}}(X)$ be the vector space spanned by the components of Δ. Then there is a constant $\delta > 0$ such that if G is a prime divisor contained in the stable base locus of $K_X + \Delta$ and $\Xi \in \mathcal{L}_A(V)$ such that $\|\Xi - \Delta\| < \delta$, then G is contained in the stable base locus of $K_X + \Xi$.*

(3) *Let $W \subset V$ be the smallest affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$ containing Δ, which is defined over the rationals. Then there is a constant $\eta > 0$ and a positive integer $r > 0$ such that if $\Xi \in W$ is any divisor and k is any positive integer such that $\|\Xi - \Delta\| < \eta$ and $k(K_X + \Xi)/r$ is Cartier, then every component of $\mathrm{Fix}(k(K_X + \Xi))$ is a component of the stable base locus of $K_X + \Delta$.*

The proof of Theorem A, Theorem B, Theorem C, Theorem D, Theorem E and Theorem F proceeds by induction:

- Theorem F_{n-1} implies Theorem A_n, see the main result of [9].
- Theorem E_{n-1} implies Theorem B_n, cf. (4.4).
- Theorem A_n and Theorem B_n imply Theorem C_n, cf. (5.6).
- Theorem D_{n-1}, Theorem B_n and Theorem C_n imply Theorem D_n, cf. (6.6).
- Theorem C_n and Theorem D_n imply Theorem E_n, cf. (7.3).
- Theorem C_n and Theorem D_n imply Theorem F_n, cf. (8.1).

2.1. **Sketch of the proof.** To help the reader navigate through the technical problems which naturally arise when trying to prove (1.2), we review a natural approach to proving that the canonical ring

$$R(X, K_X) = \bigoplus_{m\in\mathbb{N}} H^0(X, \mathcal{O}_X(mK_X)),$$

of a smooth projective variety X of general type is finitely generated. Even though we do not directly follow this method to prove the existence of log terminal models, instead using ideas from the MMP, many of the difficulties which arise in our approach are mirrored in trying to prove finite generation directly.

A very natural way to proceed is to pick a divisor $D \in |kK_X|$, whose existence is guaranteed as we are assuming that K_X is big, and then to restrict to D. One obtains an exact sequence

$$0 \longrightarrow H^0(X, \mathcal{O}_X((l-k)K_X)) \longrightarrow H^0(X, \mathcal{O}_X(lK_X)) \longrightarrow H^0(D, \mathcal{O}_D(mK_D)),$$

where $l = m(1+k)$ is divisible by $k+1$, and it is easy to see that it suffices to prove that the restricted algebra, given by the image of the maps

$$H^0(X, \mathcal{O}_X(m(1+k)K_X)) \longrightarrow H^0(D, \mathcal{O}_D(mK_D)),$$

is finitely generated. Various problems arise at this point. First D is neither smooth nor even reduced (which, for example, means that the symbol K_D is only formally defined; strictly speaking we ought to work with the dualising sheaf ω_D). It is natural then to pass to a log resolution, so that the support S of D has simple normal crossings, and to replace D by S. The second problem is that the kernel of the map

$$H^0(X, \mathcal{O}_X(m(1+k)K_X)) \longrightarrow H^0(S, \mathcal{O}_S(mK_S)),$$

no longer has any obvious connection with

$$H^0(X, \mathcal{O}_X((m(1+k)-k)K_X)),$$

so that even if we knew that the new restricted algebra were finitely generated, it is not immediate that this is enough. Another significant problem is to identify the restricted algebra as a subalgebra of

$$\bigoplus_{m\in\mathbb{N}} H^0(S, \mathcal{O}_S(mK_S)),$$

since it is only the latter that we can handle by induction. Yet another problem is that if C is a component of S, it is no longer the case that C is of general type, so that we need a more general induction. In this case the most significant problem to deal with is that even if K_C is pseudo-effective, it is not clear that the linear system $|kK_C|$ is non-empty for any $k > 0$. Finally, even though this aspect of the problem may not be apparent from the description above, in practise it seems as though we need to work with infinitely many different values of k and hence $D = D_k$, which entails working with infinitely many different birational models of X (since for every different value of k, one needs to resolve the singularities of D).

Let us consider one special case of the considerations above, which will hopefully throw some more light on the problem of finite generation. Suppose that to resolve the singularities of D we need to blow up a subvariety V. The corresponding divisor C will typically fibre over V and if V has codimension two then C will be close to a $\mathbb{P}^1$-bundle over V. In the best case, the projection $\pi\colon C \longrightarrow V$ will be a $\mathbb{P}^1$-bundle with two disjoint sections (this is the toroidal case) and sections of tensor powers of a line bundle on C will give sections of an algebra on V which is graded by $\mathbb{N}^2$, rather than just $\mathbb{N}$. Let us consider then the simplest possible algebras over $\mathbb{C}$ which are graded by $\mathbb{N}^2$. If we are given a submonoid $M \subset \mathbb{N}^2$ (that is a subset of $\mathbb{N}^2$ which contains the origin and is closed under addition) then we get a subalgebra $R \subset \mathbb{C}^2[x, y]$ spanned by the monomials

$$\{\, x^i y^j \mid (i, j) \in M \,\}.$$

The basic observation is that R is finitely generated iff M is a finitely generated monoid. There are two obvious cases when M is not finitely generated,

$$M = \{\, (i, j) \in \mathbb{N}^2 \mid j > 0 \,\} \cup \{(0, 0)\} \quad \text{and} \quad \{\, (i, j) \in \mathbb{N}^2 \mid i \geq \sqrt{2}j \,\}.$$

In fact, if $\mathcal{C} \subset \mathbb{R}^2$ is the convex hull of the set M, then M is finitely generated iff $\mathcal{C}$ is a rational polytope. In the general case, we will be given a convex subset $\mathcal{C}$ of a finite dimensional vector space of Weil divisors on X and a key part of the proof is to show that the set $\mathcal{C}$ is in fact a rational polytope. As naive as these examples are, hopefully they indicate why it is central to the proof of finite generation to

- consider divisors with real coefficients, and
- prove a non-vanishing result.

We now review our approach to the proof of (1.2). As is clear from the plan of the proof given in the previous subsection, the proof of (1.2) is by induction on the dimension and the proof is split into various parts. Instead of proving directly that the canonical ring is finitely generated, we try to construct a log terminal model for X. The first part is to prove the existence of pl-flips. This is proved by induction in [8], and we will not talk about the proof of this result here, since the methods used to prove this result are very different from the methods we use here. Granted the existence of pl-flips, the main issue is to prove that some MMP terminates, which means that we must show that we only need finitely many flips.

As in the scheme of the proof of finite generation sketched above, the first step is to pick $D \in |kK_X|$, and to pass to a log resolution of the support S of D. By way of induction we want to work with $K_X + S$ rather than K_X. As before this is tricky since a log terminal model for $K_X + S$ is not the same as a log terminal model for K_X. In other words having added S, we really want to subtract it as well. The trick however is to first add S, construct a log terminal model for $K_X + S$ and then subtract S (almost literally component by component). This is one of the key steps, to show that Theorem A_n and Theorem B_n imply Theorem C_n. This part of the proof splits naturally into two parts. First we have to prove that we may run the relevant minimal model programs, see §4 and the beginning of §5, then we have to prove this does indeed construct a log terminal model for K_X, see §5.

To gain intuition for how this part of the proof works, let us first consider a simplified case. Suppose that $D = S$ is irreducible. In this case it is clear that S is of general type and K_X is nef if and only if $K_X + S$ is nef and in fact a log terminal model for K_X is the same as a log terminal model for $K_X + S$. Consider running the $(K_X + S)$-MMP. Then every step of this MMP is a step of the K_X-MMP and vice-versa. Suppose that we have a $(K_X + S)$-extremal ray R. Let $\pi\colon X \longrightarrow Z$ be the corresponding contraction. Then $S \cdot R < 0$, so that every curve Σ contracted by π must be contained in S. In particular π cannot be a divisorial contraction, as S is not uniruled. Hence π is a pl-flip and by Theorem A_n, we can construct the flip of π, $\phi\colon X \dashrightarrow Y$. Consider the restriction $\psi\colon S \dashrightarrow T$ of ϕ to S, where T is the strict transform of S. Since log discrepancies increase under flips and S is irreducible, ψ

is a birational contraction. After finitely many flips, we may therefore assume that ψ does not contract any divisors, since the Picard number of S cannot keep dropping. Consider what happens if we restrict to S. By adjunction, we have

$$(K_X + S)|_S = K_S.$$

Thus $\psi\colon S \dashrightarrow T$ is K_S-negative. We have to show that this cannot happen infinitely often. If we knew that every sequence of flips on S terminates, then we would be done. In fact this is how special termination works. Unfortunately we cannot prove that every sequence of flips terminates on S, so that we have to do something slightly different. Instead we throw in an auxiliary ample divisor H on X, and consider $K_X + S + tH$, where t is a positive real number. If t is large enough then $K_X + S + tH$ is ample. Decreasing t, we may assume that there is an extremal ray R such that $(K_X + S + tH) \cdot R = 0$. If $t = 0$, then $K_X + S$ is nef and we are done. Otherwise $(K_X + S) \cdot R < 0$, so that we are still running a $(K_X + S)$-MMP, but with the additional restriction that $K_X + S + tH$ is nef and trivial on any ray we contract. This is the $(K_X + S)$-MMP with scaling of H. Let $G = H|_S$. Then $K_S + tG$ is nef and so is $K_T + tG'$, where $G' = \psi_* G$. In this case $K_T + tG'$ is a weak log canonical model for $K_S + tG$ (it is not a log terminal model, both because ϕ might contract divisors on which $K_S + tG$ is trivial and more importantly because T need not be $\mathbb{Q}$-factorial). In this case we are then done, by finiteness of weak log canonical models for (S, tG), where $t \in [0, 1]$ (cf. Theorem E_{n-1}).

We now turn to the general case. The idea is similar. First we want to use finiteness of log terminal models on S to conclude that there are only finitely many log terminal models in a neighbourhood of S. Secondly we use this to prove the existence of a very special MMP and construct log terminal models using this MMP. The intuitive idea is that if $\phi\colon X \dashrightarrow Y$ is K_X-negative then K_X is bigger than K_Y (the difference is an effective divisor on a common resolution) so that we can never return to the same neighbourhood of S. As already pointed out, in the general case we need to work with $\mathbb{R}$-divisors. This poses no significant problem at this stage of the proof, but it does make some of the proofs a little more technical. By way of induction, suppose that we have a log pair $K_X + \Delta = K_X + S + A + B$, where S is a sum of prime divisors, A is an ample divisor (with rational coefficients) and the coefficients of B are real numbers between zero and one. We are also given a divisor $D \geq 0$ such that $K_X + \Delta \sim_{\mathbb{R}} D$. The construction of log terminal models is similar to the one sketched above and breaks into two parts.

In the first part, for simplicity of exposition we assume that S is a prime divisor and $K_X + \Delta$ is purely log terminal. We fix S and A but we allow B to vary and we want to show that finiteness of log terminal models for S implies finiteness of log terminal models in a neighbourhood of S. We are free to pass to a log resolution, so we may assume that (X, Δ) is log smooth and if $B = \sum b_i B_i$ the coefficients $(b_1, b_2, \dots, b_k)$ of B lie in $[0,1]^k$. Let $\Theta = (\Delta - S)|_S$ so that $(K_X + \Delta)|_S = K_S + \Theta$.

Suppose that $f\colon X \dashrightarrow Y$ is a log terminal model of (X, Δ). There are three problems that arise, two of which are quite closely related. Suppose that $g\colon S \dashrightarrow T$ is the restriction of f to S, where T is the strict transform of S. The first problem is that g need not be a birational contraction. For example, suppose that X is a threefold and f flips a curve Σ intersecting S, which is not contained in S. Then $S \cdot \Sigma > 0$ so that $T \cdot E < 0$, where E is the flipped curve. In this case $E \subset T$ so that the induced birational map $S \dashrightarrow T$ extracts the curve E. The basic observation is that E must have log discrepancy less than one with respect to (S, Θ). Since the pair (X, Δ) is purely log terminal if we replace (X, Δ) by a fixed model which is high enough then we can ensure that the pair (S, Θ) is terminal, so that there are no such divisors E and g is then always a birational contraction. The second problem is that if E is a divisor intersecting S which is contracted to a divisor lying in T, then $E \cap S$ is not contracted by S. For this reason, g is not necessarily a weak log canonical model of (S, Θ). However we can construct a divisor $0 \leq \Xi \leq \Theta$ such that g is a weak log canonical model for (S, Ξ). Suppose that we start with a smooth threefold Y and a smooth surface $T \subset Y$ which contains a -2-curve Σ, such that $K_Y + T$ is nef. Let $f\colon X \longrightarrow Y$ be the blow up of Y along Σ with exceptional divisor E and let S be the strict transform of T. Then f is a step of the $(K_X + S + eE)$-MMP for any $e > 0$ and f is a log terminal model of $K_X + S + eE$. The restriction of f to S, $g\colon S \longrightarrow T$ is the identity, but g is not a log terminal model for $K_S + e\Sigma$, since $K_S + e\Sigma$ is negative along Σ. It is a weak log canonical model for K_T, so that in this case $\Xi = 0$. The details of the construction of Ξ are contained in (4.1).

The third problem is that the birational contraction g does not determine f. This is most transparent in the case when X is a surface and S is a curve, since in this case g is always an isomorphism. To remedy this particular part of the third problem we use the different, which is defined by adjunction,

$$(K_Y + T)|_T = K_T + \Phi.$$

The other parts of the third problem only occurs in dimension three or more. For example, suppose that Z is the cone over a smooth quadric in $\mathbb{P}^3$ and $p\colon X \longrightarrow Z$ and $q\colon Y \longrightarrow Z$ are the two small resolutions, so that the induced birational map $f\colon X \dashrightarrow Y$ is the standard flop. Let $\pi\colon W \longrightarrow Z$ blow up the maximal ideal, so that the exceptional divisor E is a copy of $\mathbb{P}^1 \times \mathbb{P}^1$. Pick a surface R which intersects E along a diagonal curve Σ. If S and T are the strict transforms of R in X and Y then the induced birational map $g\colon S \longrightarrow T$ is an isomorphism (both S and T are isomorphic to R). To get around this problem, one can perturb Δ so that f is the ample model, and one can distinguish between X and Y by using the fact that g is the ample model of (S, Ξ). Finally it is not hard to write down examples of flops which fix Ξ, but switch the individual components of Ξ. In this case one needs to keep track not only of Ξ but the individual pieces $(g_*B_i)|_T$, $1 \leq i \leq k$. We prove that an ample model f is determined in a neighbourhood of T by g, the different Φ and $(f_*B_i)|_T$, see (4.3). To finish this part, by induction we assume that there are finitely many possibilities for g and it is easy to see that there are then only finitely many possibilities for the different Φ and the divisors $(f_*B_i)|_T$ and this shows that there are only finitely many possibilities for f. This explains the implication Theorem E_{n-1} implies Theorem B_n. The details are contained in §4.

The second part consists of using finiteness of models in a neighbourhood of S to run a sequence of minimal model programs to construct a log terminal model. We may assume that X is smooth and the support of $\Delta + D$ has normal crossings.

Suppose that there is a divisor C such that

$$(*) \qquad K_X + \Delta \sim_{\mathbb{R},U} D + \alpha C,$$

where $K_X + \Delta + C$ is divisorially log terminal and nef and the support of D is contained in S. If R is an extremal ray which is $(K_X + \Delta)$-negative, then $D \cdot R < 0$, so that $S_i \cdot R < 0$ for some component S_i of S. As before this guarantees the existence of flips. It is easy to see that the corresponding step of the $(K_X + \Delta)$-MMP is not an isomorphism in a neighbourhood of S. Therefore the $(K_X + \Delta)$-MMP with scaling of C must terminate with a log terminal model for $K_X + \Delta$. To summarise, whenever the conditions above hold, we can always construct a log terminal model of $K_X + \Delta$.

We now explain how to construct log terminal models in the general case. We may write $D = D_1 + D_2$, where every component of D_1 is a component of S and no component of D_2 is a component of S. If D_2 is empty, that is every component of D is a component of S then we take C to be a sufficiently ample divisor, and the argument in the previous paragraph implies that $K_X + \Delta$ has a log terminal model. If $D_2 \neq 0$ then instead of constructing a log terminal model, we argue that we can construct a neutral model, which is exactly the same as a log terminal model, except that we drop the hypothesis on negativity. Consider $(X, \Theta = \Delta + \lambda D_2)$, where λ is the largest real number so that the coefficients of Θ are at most one. Then more components of $\llcorner \Theta \lrcorner$ are components of D. By induction (X, Θ) has a neutral model, $f\colon X \dashrightarrow Y$. It is then easy to check the conditions in the paragraph above apply, and we can construct a log terminal model $g\colon Y \dashrightarrow Z$ for $K_X + g_*\Delta$. It is then automatic that the composition $h = g \circ f\colon X \dashrightarrow Z$ is a neutral model of $K_X + \Delta$ (since f is not $(K_X + \Delta)$-negative it is not true in general that h is a log terminal model of $K_X + \Delta$). However g is automatically a log terminal model provided we only contract components of the stable base locus of $K_X + \Delta$. For this reason, we pick D so that we may write $D = M + F$ where every component of M is semiample and every component of F is a component of the stable base locus. This explains the implication Theorem A_n and Theorem B_n imply Theorem C_n. The details are contained in §5.

Now we explain how prove that if $K_X + \Delta = K_X + A + B$ is pseudo-effective, then $K_X + \Delta \sim_{\mathbb{R}} D \geq 0$. The idea is to mimic the proof of the non-vanishing theorem. As in the proof of the non-vanishing theorem and following the work of Nakayama, there are two cases. In the first case, for any ample divisor H,

$$h^0(X, \mathcal{O}_X(\llcorner m(K_X + \Delta) \lrcorner + H)),$$

is a bounded function of m. In this case it follows that $K_X + \Delta$ is numerically equivalent to the divisor $N_\sigma(K_X + \Delta) \geq 0$. It is then not hard to prove that Theorem C_n implies that $K_X + \Delta$ has a log terminal model and we are done by the base point free theorem.

In the second case we construct a non kawamata log terminal centre for

$$m(K_X + \Delta) + H,$$

when m is sufficiently large. Passing to a log resolution, and using standard arguments, we are reduced to the case when

$$K_X + \Delta = K_X + S + A + B,$$

where S is irreducible and $(K_X + \Delta)|_S$ is pseudo-effective, and the support of Δ has global normal crossings. Suppose first that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We may write

$$(K_X + S + A + B)|_S = K_S + C + D,$$

where C is ample and $D \geq 0$. By induction we know that there is a positive integer m such that $h^0(S, \mathcal{O}_S(m(K_S + C + D))) > 0$. To lift sections, we need to know that $h^1(X, \mathcal{O}_X(m(K_X+S+A+B)-S)) = 0$. Now

$$\begin{aligned} m(K_X + \Delta) - (K_X + B) - S &= (m-1)(K_X + \Delta) + A \\ &= (m-1)(K_X + \Delta + \frac{1}{m-1}A). \end{aligned}$$

As $K_X + \Delta + A/(m-1)$ is big, we can construct a log terminal model $\phi\colon X \dashrightarrow Y$ for $K_X+\Delta+A/(m-1)$, and running this argument on Y, the required vanishing holds by Kawamata-Viehweg vanishing. In the general case, $K_X + S + A + B$ is an $\mathbb{R}$-divisor. The argument is now a little more delicate as $h^0(S, \mathcal{O}_S(m(K_S+C+D)))$ does not make sense. We need to approximate $K_S + C + D$ by rational divisors, which we can do by induction. But then it is not so clear how to choose m. In practise we need to prove that the log terminal model Y constructed above does not depend on m, at least locally in a neighbourhood of T, the strict transform of S, and then the result follows by Diophantine approximation. This explains the implication Theorem D_{n-1}, Theorem B_n and Theorem C_n imply Theorem D_n. The details are in §6.

Finally, in terms of induction, we need to prove finiteness of weak log canonical models. We fix an ample divisor A and work with divisors of the form $K_X + \Delta = K_X + A + B$, where the coefficients of B are variable. For ease of exposition, we assume that the support of A and B have global normal crossings, so that $K_X + \Delta = K_X + A + \sum b_i B_i$ is log canonical if and only if $0 \leq b_i \leq 1$ for all i. The key point is that we allow the coefficients of B to be real numbers, so that the set of all possible choices of coefficients $[0,1]^k$ is a compact subset of $\mathbb{R}^k$. Thus we may check finiteness locally. In fact since A is ample, we can always perturb the coefficients of B so that none of the coefficients are equal to one or zero and so we may even assume that $K_X + \Delta$ is kawamata log terminal.

Observe that we are certainly free to add components to B (formally we add components with coefficient zero and then perturb so that their coefficients are non-zero). In particular we may assume that B is the support of an ample divisor and so working on the weak log canonical model, we may assume that we have a log canonical model for a perturbed divisor. Thus it suffices to prove that there are only finitely many log canonical models. Since the log canonical model is determined by any log terminal model, it suffices to prove that we can find a cover of $[0,1]^k$ by finitely many log terminal models. By compactness, it suffices to do this locally.

So pick $b \in [0,1]^k$. There are two cases. If $K_X + \Delta$ is not pseudo-effective, then $K_X + A + B'$ is not pseudo-effective, for B' in a neighbourhood of B, and there are no weak log canonical models at all. Otherwise we may assume that $K_X + \Delta$ is pseudo-effective. By induction we know that $K_X + \Delta \sim_{\mathbb{R}} D \geq 0$. Then we know that there is a log terminal model $\phi\colon X \dashrightarrow Y$. Replacing (X, Δ) by $(Y, \Gamma = \phi_*\Delta)$, we may assume that $K_X + \Delta$ is nef. By the base point free theorem it is semiample. Let $X \longrightarrow Z$ be the corresponding morphism. The key observation is that locally about Δ, any log terminal model over Z is an absolute log terminal model. Working over Z, we may assume that $K_X + \Delta$ is numerically trivial. In this case the problem of finding a log terminal model for $K_X + \Delta'$ only depends on the line segment spanned by Δ and Δ'. Working in a small box about Δ, we are then reduced to finding a log terminal model on the boundary of the box and we are done by induction on the dimension of the affine space containing B. Note that in practise, we need to work in slightly more generality than we have indicated; first we need to work in the relative setting and secondly we need to work with an arbitrary affine space containing B (and not just the space spanned by the components of B). This poses no significant problem. This explains the implication Theorem C_n and Theorem D_n imply Theorem E_n. The details are contained in §7.

The implication Theorem C_n, Theorem D_n and Theorem E_n imply Theorem F_n is straightforward. The details are contained in §8.

Let us end the sketch of the proof by pointing out some of the technical advantages with working with kawamata log terminal pairs (X, Δ), where Δ is big. The first observation is that since the kawamata log terminal condition is open, it is straightforward to show that Δ is $\mathbb{Q}$-linearly equivalent to $A + B$ where A is an ample $\mathbb{Q}$-divisor, $B \geq 0$ and $K_X + A + B$. The presence of the ample divisor A is very convenient for a number of reasons, two of which we have already seen in the sketch of the proof.

Firstly the restriction of an ample divisor to any divisor S is ample, so that if B does not contain S in its support then the restriction of $A + B$ to S is big. This is very useful for induction.

Secondly, as we vary the coefficients of B, the closure of the set of kawamata log terminal pairs is the set of log canonical pairs. However, we can use a small piece of A to perturb the coefficients of B so that they are bounded away from zero and $K_X + A + B$ is always kawamata log terminal.

Finally, if (X, Δ) is divisorially log terminal and $f\colon X \longrightarrow Y$ is a $(K_X + \Delta)$-trivial contraction then $K_Y + \Gamma = K_X + f_*\Delta$ is not necessarily divisorially log terminal, only log canonical. For example, suppose that

Y is a surface with a simple elliptic singularity and $f\colon X \longrightarrow Y$ is the blow up with exceptional divisor E. Then f is a weak log canonical model of K_X+E, but Y is not log terminal as it does not have rational singularities. On the other hand, if $\Delta = A+B$ where A is ample then $K_Y + \Gamma$ is always divisorially log terminal.

2.2. **Standard conjectures of the MMP.** Having sketched the proof of (1.2), we should point out the main obstruction to extending these ideas to the case when X is not of general type. The main issue seems to be the implication K_X pseudo-effective implies $\kappa(X, K_X) \geq 0$. In other words we need:

Conjecture 2.1. *Let (X,Δ) be a projective kawamata log terminal pair.*

If $K_X + \Delta$ is pseudo-effective then $\kappa(X, K_X+\Delta) \geq 0$.

We also probably need

Conjecture 2.2. *Let (X,Δ) be a projective kawamata log terminal pair.*

If $K_X + \Delta$ is pseudo-effective and

$$h^0(X, \mathcal{O}_X(\llcorner m(K_X+\Delta)\lrcorner + H)),$$

is not a bounded function of m, for some ample divisor H, then $\kappa(X, K_X+\Delta) \geq 1$.

In fact, using the methods of this paper, together with some results of Kawamata (cf. [14] and [15]), (2.1) and (2.2) would seem to imply one of the main outstanding conjectures of higher dimensional geometry:

Conjecture 2.3 (Abundance)**.** *Let (X,Δ) be a projective kawamata log terminal pair.*

If $K_X + \Delta$ is nef then it is semiample.

We remark that the following seemingly innocuous generalisation of (1.2) (in dimension $n+1$) would seem to imply (2.3) (in dimension n).

Conjecture 2.4. *Let (X,Δ) be a projective log canonical pair of dimension n.*

If $K_X + \Delta$ is big, then (X,Δ) has a log canonical model.

It also seems worth pointing out that the other remaining conjecture is:

Conjecture 2.5 (Borisov-Alexeev-Borisov)**.** *Fix a positive integer n and a positive real number $\epsilon > 0$.*

Then the set of varieties X such that $K_X + \Delta$ has log discrepancy at least ϵ and $-(K_X+\Delta)$ is ample, forms a bounded family.

3. Preliminary results

In this section we collect together some definitions and results.

3.1. **Notation and conventions.** We work over the field of complex numbers $\mathbb{C}$. We say that two $\mathbb{Q}$-divisors D_1, D_2 are $\mathbb{Q}$-*linearly equivalent* ($D_1 \sim_{\mathbb{Q}} D_2$) if there exists an integer $m > 0$ such that mD_i are linearly equivalent. We say that a $\mathbb{Q}$-divisor D is $\mathbb{Q}$-*Cartier* if some integral multiple is Cartier. We say that X is $\mathbb{Q}$-*factorial* if every Weil divisor is $\mathbb{Q}$-Cartier. We say that X is *analytically* $\mathbb{Q}$-*factorial* if every analytic Weil divisor (that is an analytic subset of codimension one) is analytically $\mathbb{Q}$-Cartier (i.e. some multiple is locally defined by a single analytic function). We recall some definitions involving divisors with real coefficients.

Definition 3.1.1. *Let $\pi\colon X \longrightarrow U$ be a proper morphism of normal algebraic spaces.*

(1) *An $\mathbb{R}$-**Weil divisor** (frequently abbreviated to $\mathbb{R}$-divisor) D on X is an $\mathbb{R}$-linear combination of prime divisors.*
(2) *An $\mathbb{R}$-**Cartier divisor** D is an $\mathbb{R}$-linear combination of Cartier divisors.*
(3) *Two $\mathbb{R}$-divisors D and D' are $\mathbb{R}$-**linearly equivalent over** U, denoted $D \sim_{\mathbb{R},U} D'$, if their difference is an $\mathbb{R}$-linear combination of principal divisors and an $\mathbb{R}$-Cartier divisor pulled back from U.*
(4) *Two $\mathbb{R}$-divisors D and D' are **numerically equivalent over** U, denoted $D \equiv_U D'$, if their difference is an $\mathbb{R}$-Cartier divisor such that $(D - D') \cdot C = 0$ for any curve C contained in a fibre of π.*
(5) *An $\mathbb{R}$-Cartier divisor D is **ample over** U (or π-ample) if it is $\mathbb{R}$-linearly equivalent to a positive linear combination of ample (in the usual sense) Cartier divisors over U.*
(6) *An $\mathbb{R}$-Cartier divisor D on X is **nef over** U (or π-nef) if $D \cdot C \geq 0$ for any curve $C \subset X$, contracted by π.*
(7) *An $\mathbb{R}$-divisor D is **big over** U (or π-big) if*

$$\limsup \frac{h^0(F, \mathcal{O}_F(\llcorner mD \lrcorner))}{m^{\dim F}} > 0,$$

for the fibre F over any generic point of U. Equivalently D is big over U if $D \sim_{\mathbb{R},U} A + B$ where A is ample over U and $B \geq 0$ (cf. [28, II 3.16]).

(8) *An $\mathbb{R}$-Cartier divisor D is* ***semiample over*** *U (or π-semiample) if there is a morphism $f\colon X \longrightarrow Y$ over U such that D is $\mathbb{R}$-linearly equivalent to the pullback of an ample $\mathbb{R}$-divisor over U.*
(9) *An $\mathbb{R}$-divisor D is π**-pseudo-effective** if the restriction of D to the generic fibre is the limit of divisors $D_i \geq 0$.*

Note that the group of Weil divisors with rational coefficients $\mathrm{WDiv}_{\mathbb{Q}}(X)$, or with real coefficients $\mathrm{WDiv}_{\mathbb{R}}(X)$, forms a vector space, with a canonical basis given by the prime divisors. Given an $\mathbb{R}$-divisor, $\|D\|$ denotes the sup norm with respect to this basis. If $A = \sum a_i C_i$ and $B = \sum b_i C_i$ are two $\mathbb{R}$-divisors,

$$A \wedge B = \sum \min(a_i, b_i) C_i.$$

Given an $\mathbb{R}$-divisor D and a subvariety Z which is not contained in the singular locus of X, $\operatorname{mult}_Z D$ denotes the multiplicity of D at the generic point of Z. If $Z = E$ is a prime divisor this is the coefficient of E in D.

A *log pair* (X, Δ) (sometimes abbreviated by $K_X + \Delta$) is a normal variety X and an $\mathbb{R}$-divisor $\Delta \geq 0$ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. We say that a log pair (X, Δ) is *log smooth*, if X is smooth and the support of Δ is a divisor with global normal crossings. A birational morphism $g\colon Y \longrightarrow X$ is a *log resolution* of the pair (X, Δ) if g is projective, Y is smooth, the exceptional locus is a divisor and $g^{-1}(\Delta)$ union the exceptional set of g is a divisor with global normal crossings support. By Hironaka's Theorem we may, and often will, assume that the exceptional locus supports an ample divisor over X. If we write

$$K_Y + \Gamma = K_Y + \sum b_i \Gamma_i = g^*(K_X + \Delta),$$

where Γ_i are distinct prime divisors then the log discrepancy $a(\Gamma_i, X, \Delta)$ of Γ_i is $1 - b_i$. The log discrepancy of (X, Δ) is then the infimum of the log discrepancy for every Γ_i and for every resolution. The image of any component of Γ of coefficient at least one (equivalently log discrepancy at most zero) is a *non kawamata log terminal centre of the pair* (X, Δ). The pair (X, Δ) is *kawamata log terminal* if for every (equivalently for one) log resolution $g\colon Y \longrightarrow X$ as above, the coefficients of Γ are strictly less than one, that is $b_i < 1$ for all i. Equivalently, the pair (X, Δ) is kawamata log terminal if there are no non kawamata log terminal centres. The *non kawamata log terminal locus* of (X, Δ) is the union of the non kawamata log terminal centres. We say that the pair (X, Δ) is *purely log terminal* if the log discrepancy of any exceptional divisor is greater than zero. We say that the pair $(X, \Delta = \sum \delta_i \Delta_i)$, where $\delta_i \in (0, 1]$, is *divisorially log terminal* if there is a log resolution such that the log discrepancy of every exceptional divisor is greater than zero. By [22, (2.40)] (X, Δ) is divisorially log terminal if and only if there is a closed subset $Z \subset X$ such that

- $(X \backslash Z, \Delta|_{X\backslash Z})$ is log smooth, and
- if $f\colon Y \longrightarrow X$ is a projective birational morphism and $E \subset Y$ is an irreducible divisor with centre contained in Z then $a(E, X, \Delta) > 0$.

We will also often write

$$K_Y + \Gamma = g^*(K_X + \Delta) + E,$$

where $\Gamma \geq 0$ and $E \geq 0$ have no common components, $g_*\Gamma = \Delta$ and E is g-exceptional. Note that this decomposition is unique.

We say that a birational map $\phi\colon X \dashrightarrow Y$ is a *birational contraction* if ϕ is proper and ϕ^{-1} does not contract any divisors. If in addition ϕ^{-1} is also a birational contraction, we say that ϕ is a *small birational map*.

3.2. **Preliminaries.**

Lemma 3.2.1. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties. Let D be an $\mathbb{R}$-Cartier divisor on X and let D' be its restriction to the generic fibre of π.*

If $D' \sim_{\mathbb{R}} B' \geq 0$ for some $\mathbb{R}$-divisor B' on the generic fibre of π, then there is a divisor B on X such that $D \sim_{\mathbb{R},U} B \geq 0$ whose restriction to the generic fibre of π is B'.

Proof. Taking the closure of the generic points of B', we may assume that there is an $\mathbb{R}$-divisor $B_1 \geq 0$ on X such that the restriction of B_1 to the generic fibre is B'. As

$$D' - B' \sim_{\mathbb{R}} 0,$$

it follows that there is an open subset U_1 of U, such that

$$(D - B_1)|_{V_1} \sim_{\mathbb{R}} 0,$$

where V_1 is the inverse image of U_1. But then there is a divisor G on X such that

$$D - B_1 \sim_{\mathbb{R}} G,$$

where $Z = \pi(\operatorname{Supp} G)$ is a proper closed subset. As U is quasi-projective, there is an ample divisor $H \geq 0$ on U which contains Z. Possibly rescaling, we may assume that $F = \pi^* H \geq -G$. But then

$$D \sim_{\mathbb{R}} (B_1 + F + G) - F,$$

so that

$$D \sim_{\mathbb{R},U} (B_1 + F + G) \geq 0. \qquad \square$$

3.3. **Nakayama-Zariski decomposition.** We will need some definitions and results from [28].

Definition-Lemma 3.3.1. *Let X be a smooth projective variety, B be a big $\mathbb{R}$-divisor and let C be a prime divisor. Let*

$$\sigma_C(B) = \inf\{\, \operatorname{mult}_C(B') \mid B' \sim_{\mathbb{Q}} B, B' \geq 0\,\}.$$

Then σ_C is a continuous function on the cone of big divisors.

Now let D be any pseudo-effective $\mathbb{R}$-divisor and let A be any ample $\mathbb{Q}$-divisor. Let

$$\sigma_C(D) = \lim_{\epsilon \to 0} \sigma_C(D + \epsilon A).$$

Then $\sigma_C(D)$ exists and is independent of the choice of A.

There are only finitely many prime divisors C such that $\sigma_C(D) > 0$ and the $\mathbb{R}$-divisor $N_\sigma(D) = \sum_C \sigma_C(D)C$ is determined by the numerical equivalence class of D. Moreover $D - N_\sigma(D)$ is pseudo-effective and $N_\sigma(D - N_\sigma(D)) = 0$.

Proof. See §III.1 of [28]. $\square$

Proposition 3.3.2. *Let X be a smooth projective variety and let D be a pseudo-effective $\mathbb{R}$-divisor. Let B be any big $\mathbb{R}$-divisor.*

If D is not numerically equivalent to $N_\sigma(D)$, then there is a positive integer k and a positive rational number β such that

$$h^0(X, \mathcal{O}_X(\llcorner mD \lrcorner + \llcorner kB \lrcorner)) > \beta m, \qquad \text{for all} \qquad m \gg 0.$$

Proof. Let A be any integral divisor. Then we may find a positive integer k such that

$$h^0(X, \mathcal{O}_X(\llcorner kB \lrcorner - A)) > 0.$$

Thus it suffices to exhibit an ample divisor A and a positive rational number β such that

$$h^0(X, \mathcal{O}_X(\llcorner mD \lrcorner + A)) > \beta m \qquad \text{for all} \qquad m \gg 0.$$

Replacing D by $D - N_\sigma(D)$, we may assume that $N_\sigma(D) = 0$. Now apply (V.1.11) of [28]. $\square$

3.4. **Adjunction.** We recall some basic facts about adjunction, see [21, §16, §17] for more details.

Definition-Lemma 3.4.1. *Let (X, Δ) be a log canonical pair, and let S be a normal component of $\llcorner \Delta \lrcorner$ of coefficient one. Then there is a divisor Θ on S such that*

$$(K_X + \Delta)|_S = K_S + \Theta.$$

1. *If (X, Δ) is divisorially log terminal then so is $K_S + \Theta$.*
2. *If (X, Δ) is purely log terminal then $K_S + \Theta$ is kawamata log terminal.*
3. *If $(X, \Delta = S)$ is purely log terminal then the coefficients of Θ have the form $(r-1)/r$, where r is the index of S at P the generic point of the corresponding divisor D on S (equivalently r is the index of $K_X + S$ at P or r is the order of the cyclic group Weil($\mathcal{O}_{X,P}$)). In particular if B is a Weil divisor on X, then the coefficient of $B|_S$ in D is an integer multiple of $1/r$.*
4. *If (X, Δ) is purely log terminal, $f \colon Y \longrightarrow X$ is a projective birational morphism and T is the strict transform of S, then $(f|_T)_*\Psi = \Theta$, where $K_Y + \Gamma = f^*(K_X + \Delta)$ and Ψ is defined by adjunction,*
$$(K_Y + \Gamma)|_T = K_T + \Psi.$$

3.5. **Stable base locus.** We need to extend the definition of the stable base locus to the case of a real divisor.

Definition 3.5.1. *Let $\pi \colon X \longrightarrow U$ be a projective morphism of normal varieties.*

*Let D be an $\mathbb{R}$-divisor on X. The **real linear system** associated to D over U is*

$$|D/U|_{\mathbb{R}} = \{\, C \geq 0 \,|\, C \sim_{\mathbb{R},U} D \,\}.$$

*The **stable base locus** of D over U is the Zariski closed set $\mathbf{B}(D/U)$ given by the intersection of the support of the elements of the real linear system $|D/U|_{\mathbb{R}}$. If $|D/U|_{\mathbb{R}} = \emptyset$, then we let $\mathbf{B}(D/U) = X$. The **stable fixed divisor** is the divisorial support of the stable base locus. The **augmented base locus** of D over U is the Zariski closed set*

$$\mathbf{B}_+(D/U) = \mathbf{B}((D - \epsilon A)/U),$$

for any ample divisor A over U and any sufficiently small rational number $\epsilon > 0$ (compare [23, Definition 10.3.2]*)*

Remark 3.5.2. *The stable base locus, the stable fixed divisor and the augmented base locus are only defined as closed subsets, they do not have any scheme structure.*

Lemma 3.5.3. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal varieties and let D be an integral Weil divisor on X.*

Then the stable base locus as defined in (3.5.1) *coincides with the usual definition of the stable base locus.*

Proof. Let
$$|D/U|_{\mathbb{Q}} = \{\, C \geq 0 \,|\, C \sim_{\mathbb{Q},U} D \,\}.$$
Let R be the intersection of the elements of $|D/U|_{\mathbb{R}}$ and let Q be the intersection of the elements of $|D/U|_{\mathbb{Q}}$. It suffices to prove that $Q = R$. As $|D/U|_{\mathbb{Q}} \subset |D/U|_{\mathbb{R}}$, it is clear that $R \subset Q$.

Suppose that $x \notin R$. We want to show that $x \notin Q$. We may find $D' \in |D/U|_{\mathbb{R}}$ such that $\operatorname{mult}_x D' = 0$. But then
$$D' = D + \sum r_i(f_i) + \pi^* E,$$
where f_i are rational functions on X, E is an $\mathbb{R}$-Cartier divisor on U and r_i are real numbers. Let V be the subspace of $\operatorname{WDiv}_{\mathbb{R}}(X)$ spanned by the components of D, D', π^*E and (f_i). We may write $E = \sum e_j E_j$, where E_i are Cartier divisors. Let W be the span of the (f_i) and the $\pi^* E_j$. Then $W \subset V$ are defined over the rationals. Set
$$\mathcal{P} = \{\, D'' \in V \,|\, D'' \geq 0,\ \operatorname{mult}_x D'' = 0,\ D'' - D \in W \,\} \subset |D/U|_{\mathbb{R}}.$$
Then $\mathcal{P}$ is a rational polyhedron. As $D' \in \mathcal{P}$, $\mathcal{P}$ is non-empty, and so it must contain a rational point D''. We may write
$$D'' = D + \sum s_i(f_i) + \sum f_j \pi^* E_j,$$
where s_i and f_j are real numbers. Since D'' and D have rational coefficients, it follows that we may find s_i and f_j which are rational. But then $D'' \in |D/U|_{\mathbb{Q}}$, and so $x \notin Q$. □

Proposition 3.5.4. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal varieties and let $D \geq 0$ be an $\mathbb{R}$-divisor. Then we may find $\mathbb{R}$-divisors M and F such that*

(1) *$M \geq 0$ and $F \geq 0$,*
(2) *$D \sim_{\mathbb{R},U} M + F$,*
(3) *every component of F is a component of $\mathbf{B}(D/U)$, and*
(4) *if B is a component of M then some multiple of B is mobile.*

We need two basic results.

Lemma 3.5.5. *Let X be a normal variety and let D and D' be two $\mathbb{R}$-divisors such that $D \sim_{\mathbb{R}} D'$.*

Then we may find rational functions $f_1, f_2, \dots, f_k$ and real numbers $r_1, r_2, \dots, r_k$ which are independent over the rationals such that

$$D = D' + \sum_i r_i(f_i).$$

In particular every component of (f_i) is either a component of D or D'.

Proof. By assumption we may find rational functions $f_1, f_2, \dots, f_k$ and real numbers $r_1, r_2, \dots, r_k$ such that

$$D = D' + \sum_{i=1}^{k} r_i(f_i).$$

Pick k minimal with this property. Suppose that the real numbers r_i are not independent over $\mathbb{Q}$. Then we can find rational numbers d_i, not all zero, such that

$$\sum_i d_i r_i = 0.$$

Possibly re-ordering we may assume that $d_k \neq 0$. Multiplying through by an integer we may assume that $d_i \in \mathbb{Z}$. Possibly replacing f_i by f_i^{-1}, we may assume that $d_i \geq 0$. Let d be the least common multiple of the non-zero d_i. If $d_i \neq 0$, we replace f_i by f_i^{d/d_i} (and hence r_i by $d_i r_i/d$) so that we may assume that either $d_i = 0$ or 1. For $1 \leq i < k$, set

$$g_i = \begin{cases} f_i/f_k & \text{if } d_i = 1 \\ f_i & \text{if } d_i = 0. \end{cases}$$

Then

$$D = D' + \sum_{i=1}^{k-1} r_i(g_i),$$

which contradicts our choice of k.

Now suppose that B is a component of (f_i). Then

$$\operatorname{mult}_B(D) = \operatorname{mult}_B(D') + \sum r_j n_j,$$

where $n_j = \operatorname{mult}_B(f_j)$ is an integer and $n_i \neq 0$. But then $\operatorname{mult}_B(D) - \operatorname{mult}_B(D') \neq 0$, so that one of $\operatorname{mult}_B(D)$ and $\operatorname{mult}_B(D')$ must be non-zero. □

Lemma 3.5.6. *Let $\pi \colon X \longrightarrow U$ be a projective morphism of normal varieties and let*

$$D' \sim_{\mathbb{R},U} D, \qquad D \geq 0, \qquad D' \geq 0,$$

be two $\mathbb{R}$-divisors on X with no common components.

Then we may find $D'' \in |D/U|_{\mathbb{R}}$ such that a multiple of every component of D'' is mobile.

Proof. Pick ample $\mathbb{R}$-divisors on U, H and H' such that $D + \pi^* H \sim_{\mathbb{R}} D' + \pi^* H'$ and $D + \pi^* H$ and $D' + \pi^* H'$ have no common components. Replacing D by $D + \pi^* H$ and D' by $D' + \pi^* H'$, we may assume that $D' \sim_{\mathbb{R}} D$.

We may write

$$D' = D + \sum r_i(f_i) = D + R,$$

where $r_i \in \mathbb{R}$ and f_i are rational functions on X. By (3.5.5) we may assume that every component of R is a component of $D + D'$.

We proceed by induction on the number of components of $D + D'$. If $q_1, q_2, \dots, q_k$ are any rational numbers then we may always write

$$C' = C + Q = C + \sum q_i(f_i),$$

where $C \geq 0$ and $C' \geq 0$ have no common components. But now if we suppose that q_i is sufficiently close to r_i then C is supported on D and C' is supported on D'. We have that $mC \sim mC'$ for some integer $m > 0$. By Bertini we may find $C'' \sim_{\mathbb{Q}} C$ such that every component of C'' has a multiple which is mobile. Pick $\lambda > 0$ maximal such that $D_1 = D - \lambda C \geq 0$ and $D_1' = D' - \lambda C' \geq 0$. Note that

$$D_1 \sim_{\mathbb{R}} D_1', \qquad D_1 \geq 0, \qquad D_1' \geq 0,$$

are two $\mathbb{R}$-divisors on X with no common components, and that $D_1 + D_1'$ has fewer components than $D + D'$. By induction we may then find

$$D_1'' \in |D_1|_{\mathbb{R}},$$

such that a multiple of every component of D_1'' is mobile. But then

$$D'' = \lambda C'' + D_1'' \in |D|_{\mathbb{R}},$$

and every component of D'' has a multiple which is mobile. $\square$

Proof of (3.5.4). We may write $D = M + F$, where every component of F is contained in $\mathbf{B}(D/U)$ and no component of M is contained in $\mathbf{B}(D/U)$. A prime divisor is **bad** if none of its multiples are mobile.

We proceed by induction on the number of bad components of M. We may assume that M has at least one bad component B. As B is a component of M, we may find $D_1 \in |D/U|_{\mathbb{R}}$ such that B is not a component of D_1. If $E = D \wedge D_1$ then $D' = D - E \geq 0$ and $D_1' = D_1 - E \geq 0$, D' and D_1' have no common components and $D' \sim_{\mathbb{R},U} D_1'$. By (3.5.6) there is a divisor $D'' \in |D'/U|_{\mathbb{R}}$ with no bad components. But then $D'' + E \in |D|_{\mathbb{R}}$, B is not a component of $D'' + E$

and the only bad components of $D'' + E$ are components of E, which are also components of D. Therefore $D''+E$ has fewer bad components than D and we are done by induction. $\square$

3.6. Types of model.

Definition 3.6.1. *Let $\phi\colon X \dashrightarrow Y$ be a proper birational contraction of normal quasi-projective varieties and let D be an $\mathbb{R}$-Cartier divisor on X such that $D' = \phi_* D$ is also $\mathbb{R}$-Cartier. We say that ϕ is D-**non-positive** (respectively D-**negative**) if for some common resolution $p\colon W \longrightarrow X$ and $q\colon W \longrightarrow Y$, we may write*

$$p^* D = q^* D' + E,$$

where $E \geq 0$ is q-exceptional (respectively $E \geq 0$ is q-exceptional and the support of E contains the strict transform of the ϕ-exceptional divisors).

We will often use the following well-known lemma.

Lemma 3.6.2 (Negativity of contraction)**.** *Let $\pi\colon Y \longrightarrow X$ be a projective birational morphism of normal quasi-projective varieties.*

1. *If $E > 0$ is an exceptional $\mathbb{R}$-Cartier divisor then there is a component F of E which is covered by curves Σ such that $E\cdot\Sigma < 0$.*
2. *If $\pi^* L \equiv M+G+E$, where L is an $\mathbb{R}$-Cartier divisor on X, M is a π-nef $\mathbb{R}$-Cartier divisor on Y, $G \geq 0$, E is π-exceptional, and G and E have no common components, then $E \geq 0$. Further if F_1 is an exceptional divisor such that there is an exceptional divisor F_2 with the same centre on X as F_1, with the restriction of M to F_2 not numerically π-trivial, then F_1 is a component of $G + E$.*
3. *If X is $\mathbb{Q}$-factorial then there is a π-exceptional divisor $E \geq 0$ such that $-E$ is ample over X. In particular the exceptional locus of π is a divisor.*

Proof. Cutting by hyperplanes in X, we reduce to the case when X is a surface, in which case (1) reduces to the Hodge Index Theorem. (2) follows easily from (1), see for example (2.19) of [21]. Let H be a general ample $\mathbb{Q}$-divisor over X. If X is $\mathbb{Q}$-factorial then

$$E = \pi^* \pi_* H - H \geq 0,$$

is π-exceptional and $-E$ is ample over X. This is (3). $\square$

Lemma 3.6.3. *Let $X \longrightarrow U$ and $Y \longrightarrow U$ be two projective morphisms of normal quasi-projective varieties. Let $\phi\colon X \dashrightarrow Y$ be a birational* *contraction over U and let D and D' be $\mathbb{R}$-Cartier divisors such that $D' = \phi_* D$ is nef over U.*

Then ϕ is D-non-positive (respectively D-negative) if given a common resolution $p\colon W \longrightarrow X$ and $q\colon W \longrightarrow Y$, we may write

$$p^*D = q^*D' + E,$$

where $p_*E \geq 0$ (respectively $p_*E \geq 0$ and the support of p_*E contains the union of all ϕ-exceptional divisors).

Further if $D = K_X + \Delta$ and $D' = K_Y + \phi_*\Delta$ then this is equivalent to requiring

$$a(F, X, \Delta) \leq a(F, Y, \phi_*\Delta) \qquad \text{(respectively } a(F, X, \Delta) < a(F, Y, \phi_*\Delta)\text{)},$$

for all ϕ-exceptional divisors $F \subset X$.

Proof. Easy consequence of (3.6.2). $\square$

Lemma 3.6.4. *Let $X \longrightarrow U$ and $Y \longrightarrow U$ be two projective morphisms of normal quasi-projective varieties. Let $\phi\colon X \dashrightarrow Y$ be a birational contraction over U and let D and D' be $\mathbb{R}$-Cartier divisors such ϕ_*D and ϕ_*D' are $\mathbb{R}$-Cartier. Let $p\colon W \longrightarrow X$ and $q\colon W \longrightarrow Y$ be common resolutions.*

If D and D' are numerically equivalent over U then

$$p^*D - q^*\phi_*D = p^*D' - q^*\phi_*D'.$$

In particular ϕ is D-non-positive (respectively D-negative) if and only if ϕ is D'-non-positive (respectively D'-negative).

Proof. Since

$$p^*(D - D') - q^*\phi_*(D - D'),$$

is q-exceptional and numerically trivial over Y this follows easily from (3.6.2). $\square$

Definition 3.6.5. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties and let D be an $\mathbb{R}$-Cartier divisor on X.*

We say that a birational contraction $f\colon X \dashrightarrow Y$ over U is a ***semiample model*** *of D over U, if f is D-non-positive, Y is normal and projective over U and $H = f_*D$ is semiample over U.*

We say that $g\colon X \dashrightarrow Z$ is the ***ample model*** *of D over U, if g is a rational map over U, Z is normal and projective over U and there is an ample divisor H over U on Z such that if $p\colon W \longrightarrow X$ and $q\colon W \longrightarrow Z$ resolve g then q is a contraction morphism and we may write $p^*D \sim_{\mathbb{R},U} q^*H + E$, where $E \geq 0$ and for every $B \in |p^*D/U|_{\mathbb{R}}$ then $B \geq E$.*

Lemma 3.6.6. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties and let D be an $\mathbb{R}$-Cartier divisor on X.*

(1) *If $g_i\colon X \dashrightarrow X_i$, $i = 1, 2$ are two ample models of D over U then there is an isomorphism $\chi\colon X_1 \longrightarrow X_2$ such that $g_2 = \chi \circ g_1$.*
(2) *Suppose that $g\colon X \dashrightarrow Z$ is the ample model of D over U and let H be the corresponding ample divisor on Z. If $p\colon W \longrightarrow X$ and $q\colon W \longrightarrow Z$ resolve g then we may write*
$$p^*D \sim_{\mathbb{R},U} q^*H + E,$$
where $E \geq 0$ and if F is any p-exceptional divisor whose centre lies in the indeterminancy locus of g then F is contained in the support of E.
(3) *If $f\colon X \dashrightarrow Y$ is a semiample model of D over U then the ample model $g\colon X \dashrightarrow Z$ of D over U exists and $g = h \circ f$, where $h\colon Y \longrightarrow Z$ is a contraction morphism and $f_*D \sim_{\mathbb{R},U} h^*H$. If B is a prime divisor contained in the stable fixed divisor of D over U then B is contracted by f.*
(4) *If $f\colon X \dashrightarrow Y$ is a birational map over U then f is the ample model of D over U if and only if f is a semiample model of D over U and f_*D is ample over U.*

Proof. Let $g\colon Y \longrightarrow X$ resolve the indeterminacy of g_i and let $f_i = g_i \circ g\colon Y \longrightarrow X_i$ be the induced contraction morphisms. By assumption $g^*D \sim_{\mathbb{R},U} f_i^*H_i + E_i$, for some divisor H_i on X_i ample over U. Since the stable fixed divisor of $f_1^*H_1$ over U is empty, $E_1 \geq E_2$. By symmetry $E_1 = E_2$ and so $f_1^*H_1 \sim_{\mathbb{R},U} f_2^*H_2$. But then f_1 and f_2 contract the same curves. This is (1).

Suppose that $g\colon X \dashrightarrow Z$ is the ample model of D over U. By assumption this means that we may write
$$p^*D \sim_{\mathbb{R},U} q^*H + E,$$
where $E \geq 0$. We may write $E = E_1 + E_2$, where every component of E_2 is exceptional for p but no component of E_1 is p-exceptional. Let $V = p(F)$. Possibly blowing up more we may assume that $p^{-1}(V)$ is a divisor. Since V is contained in the indeterminancy locus of g, there is an exceptional divisor F' with centre V such that $\dim q(F') > 0$. But then q^*H is not numerically trivial on F' and we may apply (3.6.2). This is (2).

Now suppose that $f\colon X \dashrightarrow Y$ is a semiample model of D over U. As f_*D is semiample over U, there is a contraction morphism $h\colon Y \longrightarrow Z$ over U and an ample divisor H over U on Z such that $f_*D \sim_{\mathbb{R},U} h^*H$. If $p\colon W \longrightarrow X$ and $q\colon W \longrightarrow Y$ resolve the indeterminacy of f then $p^*D \sim_{\mathbb{R},U} r^*H + E$, where $E \geq 0$ is q-exceptional and $r = h \circ q\colon W \longrightarrow$

Z. If $B \in |p^*D/U|_{\mathbb{R}}$ then $B \geq E$. But then $g = h \circ f \colon X \dashrightarrow Z$ is the ample model of D over U. This is (3).

Now suppose that $f \colon X \dashrightarrow Y$ is birational over U. If f is a semiample model of D over U then (3) implies that the ample model $g \colon X \dashrightarrow Z$ of D over U exists and there is a contraction morphism $h \colon Y \longrightarrow Z$, such that $f_*D \sim_{\mathbb{R},U} h^*H$ where H on Z is ample over U. If f_*D is ample over U then h must be the identity.

Conversely suppose that f is the ample model. Suppose that $p \colon W \longrightarrow X$ and $q \colon W \longrightarrow Y$ are projective birational morphisms which resolve f. By assumption we may write $p^*D \sim_{\mathbb{R},U} q^*H + E$, where H is ample over U. We may assume that there is a q-exceptional $\mathbb{Q}$-divisor $F \geq 0$ such that $q^*H - F$ is ample over U. Then there is a constant $\delta > 0$ such that $q^*H - F + \delta E$ is ample over U. Suppose B is a component of E. As B does not belong to the stable base locus of $q^*H - F + \delta E$ over U, B must be a component of F. It follows that E is q-exceptional. If C is a curve contracted by p then

$$0 = C \cdot p^*D = C \cdot q^*H + C \cdot E,$$

and so C is contained in the support of E. Thus if G is a divisor contracted by p it is a component of E and G is contracted by q. Therefore f is a birational contraction and f is a semiample model. Further $f_*D = H$ is ample over U. This is (4). □

Definition 3.6.7. *Let $\pi \colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties. Suppose that $K_X + \Delta$ is log canonical and let $\phi \colon X \dashrightarrow Y$ be a birational contraction of normal quasi-projective varieties over U, where Y is projective over U. Set $\Gamma = \phi_*\Delta$.*

- *Y is a **weak log canonical model** for $K_X + \Delta$ over U if ϕ is $(K_X + \Delta)$-non-positive and $K_Y + \Gamma$ is nef over U.*
- *Y is the **log canonical model** for $K_X + \Delta$ over U if ϕ is the ample model of $K_X + \Delta$ over U.*
- *Y is a **log terminal model** for $K_X+\Delta$ over U if ϕ is $(K_X+\Delta)$-negative, $K_Y+\Gamma$ is divisorially log terminal and nef over U, and Y is $\mathbb{Q}$-factorial.*

Remark 3.6.8. *Note that there is no consensus on the definitions given in* (3.6.7).

Lemma 3.6.9. *Let $\pi \colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties. Let $\phi \colon X \dashrightarrow Y$ be a birational contraction over U. Let (X, Δ) and (X, Δ') be two log pairs and set $\Gamma = \phi_*\Delta$, $\Gamma' = \phi_*\Delta'$. Let $\mu > 0$ be a positive real number.*

- *If both $K_X+\Delta$ and $K_X+\Delta'$ are log canonical and $K_X+\Delta' \sim_{\mathbb{R},U} \mu(K_X + \Delta)$ then ϕ is a weak log canonical model for $K_X + \Delta$*

over U if and only if ϕ is a weak log canonical model for $K_X+\Delta'$ over U.

- *If both $K_X+\Delta$ and $K_X+\Delta'$ are kawamata log terminal and $K_X+\Delta \equiv_U \mu(K_X+\Delta')$ then ϕ is a log terminal model for $K_X+\Delta$ over U if and only if ϕ is a log terminal model for $K_X+\Delta'$ over U.*

Proof. Note first that either $K_Y+\Gamma' \sim_{\mathbb{R},U} \mu(K_Y+\Gamma)$ or Y is $\mathbb{Q}$-factorial. In particular $K_Y+\Gamma$ is $\mathbb{R}$-Cartier if and only if $K_Y+\Gamma'$ is $\mathbb{R}$-Cartier. Therefore (3.6.4) implies that ϕ is $(K_X+\Delta)$-non-positive (respectively $(K_X+\Delta)$-negative) if and only if ϕ is $(K_X+\Delta')$-non-positive (respectively $(K_X+\Delta')$-negative).

Since $K_X+\Delta' \equiv_U \mu(K_X+\Delta)$ and $\mu E - E' \equiv_Y 0$, it follows that $K_Y+\Gamma' \equiv_U \mu(K_Y+\Gamma)$, so that $K_Y+\Gamma$ is nef over U if and only if $K_Y+\Gamma'$ is nef over U. $\square$

Lemma 3.6.10. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties. Let $\phi\colon X \dashrightarrow Y$ be a birational contraction over U, where Y is projective over U. Suppose that $K_X+\Delta$ and $K_Y+\phi_*\Delta$ are divisorially log terminal and $a(F,X,\Delta) < a(F,Y,\phi_*\Delta)$ for all ϕ-exceptional divisors $F \subset X$.*

If $\varphi\colon Y \dashrightarrow Z$ is a log terminal model of $(Y,\phi_*\Delta)$ over U, then $\eta = \varphi\circ\phi\colon X \dashrightarrow Z$ is a log terminal model of $K_X+\Delta$ over U.

Proof. Clearly η is a birational contraction, Z is $\mathbb{Q}$-factorial and $K_Z+\eta_*\Delta$ is divisorially log terminal and nef over U.

Let $p\colon W \longrightarrow X$, $q\colon W \longrightarrow Y$ and $r\colon W \longrightarrow Z$ be a common resolution. As φ is a log terminal model of $(Y,\phi_*\Delta)$ we have that $q^*(K_Y+\phi_*\Delta) - r^*(K_Z+\eta_*\Delta) = E \geq 0$ and the support of E contains the exceptional divisors of φ. By assumption $K_X+\Delta-p_*q^*(K_Y+\phi_*\Delta)$ is an effective divisor whose support is the set of all ϕ-exceptional divisors. But then

$$(K_X+\Delta) - p_*r^*(K_Z+\eta_*\Delta) = K_X+\Delta - p_*q^*(K_Y+\phi_*\Delta) + p_*E$$

contains all the η-exceptional divisors and (3.6.3) implies that η is a log terminal model of $K_X+\Delta$ over U. $\square$

Lemma 3.6.11. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties. Let (X,Δ) be a kawamata log terminal pair, where Δ is big over U. Let $f\colon Z \longrightarrow X$ be any log resolution of (X,Δ) and suppose that we write*

$$K_Z+\Phi_0 = f^*(K_X+\Delta)+E,$$

where $\Phi_0 \geq 0$ and $E \geq 0$ have no common components, $f_*\Phi_0 = \Delta$ and E is exceptional. Let $F \geq 0$ be any divisor whose support is equal to the exceptional locus of f.

If $\eta > 0$ is sufficiently small and $\Phi = \Phi_0 + \eta F$ then $K_Z + \Phi$ is kawamata log terminal and Φ is big over U. Moreover if $\phi\colon Z \dashrightarrow W$ is a log terminal model of $K_Z + \Phi$ over U then the induced birational map $\psi\colon X \dashrightarrow W$ is in fact a log terminal model of $K_X + \Delta$ over U.

Proof. Everything is clear but the last statement. Set $\Psi = \phi_*\Phi$. By (3.6.10), possibly blowing up more, we may assume that ϕ is a morphism. By assumption if we write

$$K_Z + \Phi = \phi^*(K_W + \Psi) + G,$$

then $G \geq 0$ and the support of G is the union of all the ϕ-exceptional divisors. Thus

$$f^*(K_X + \Delta) + E + \eta F = \phi^*(K_W + \Psi) + G.$$

By negativity of contraction, (3.6.2), applied to f, $G - E - \eta F \geq 0$. In particular ϕ must contract every f-exceptional divisor and so ψ is a birational contraction. But then ψ is a log terminal model over U by (3.6.3). $\square$

Lemma 3.6.12. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties, where X is $\mathbb{Q}$-factorial kawamata log terminal. Let $\phi\colon X \dashrightarrow Z$ be a birational contraction over U and let S be a sum of prime divisors. Suppose that there is a $\mathbb{Q}$-factorial quasi-projective variety Y together with a small birational projective morphism $f\colon Y \longrightarrow Z$.*

If V is any finite dimensional affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$ such that $\mathcal{L}_S(V)$ spans $\mathrm{WDiv}_{\mathbb{R}}(X)$ modulo numerical equivalence over U and $\mathcal{W}_{\phi,S,\pi}(V)$ intersects the interior of $\mathcal{L}_S(V)$ then

$$\mathcal{W}_{\phi,S,\pi}(V) = \bar{\mathcal{A}}_{\phi,S,\pi}(V).$$

Proof. By (4) of (3.6.6) $\mathcal{W}_{\phi,S,\pi}(V) \supset \mathcal{A}_{\phi,S,\pi}(V)$. Since $\mathcal{W}_{\phi,S,\pi}(V)$ is closed, it follows that

$$\mathcal{W}_{\phi,S,\pi}(V) \supset \bar{\mathcal{A}}_{\phi,S,\pi}(V).$$

To prove the reverse inclusion, it suffices to prove that a dense subset of $\mathcal{W}_{\phi,S,\pi}(V)$ is contained in $\mathcal{A}_{\phi,S,\pi}(V)$.

Pick Δ belonging to the interior of $\mathcal{W}_{\phi,S,\pi}(V)$. If $\psi\colon X \dashrightarrow Y$ is the induced birational contraction then ψ is a $\mathbb{Q}$-factorial weak log canonical model of $K_X + \Delta$ over U and

$$K_Y + \Gamma = f^*(K_Z + \phi_*\Delta),$$

where $\Gamma = \psi_*\Delta$. As $\mathcal{L}_S(V)$ spans $\mathrm{WDiv}_{\mathbb{R}}(X)$ modulo numerical equivalence over U, we may find $\Delta_0 \in \mathcal{L}_S(V)$ such that $\Delta_0 - \Delta$ is numerically equivalent over U to $\mu\Delta$ for some $\mu > 0$. Let

$$\Delta' = \Delta + \epsilon((\Delta_0 - \Delta) - \mu\Delta) = (1-\epsilon)\nu\Delta + \epsilon\Delta_0,$$

where

$$\nu = \frac{1-\epsilon-\epsilon\mu}{1-\epsilon} < 1.$$

Then Δ' is numerically equivalent to Δ over U and if $\epsilon > 0$ is sufficiently small then $\Delta' \geq 0$. As $(X, \nu\Delta)$ is kawamata log terminal it follows that (X, Δ') is kawamata log terminal. In particular (3.6.9) implies that ψ is a $\mathbb{Q}$-factorial weak log canonical model of $K_X + \Delta'$ over U and so $K_Y + \Gamma'$ is kawamata log terminal, where $\Gamma' = \psi_*\Delta'$. As Γ and Γ' are numerically equivalent over U, it follows that $K_Y + \Gamma'$ is numerically equivalent to zero over Z.

Let H be a general ample $\mathbb{Q}$-divisor over U on Z. Let $p\colon W \longrightarrow X$ and $q\colon W \longrightarrow Z$ resolve the indeterminacy locus of ϕ and let $H' = p_*q^*H$. It follows that ϕ is H'-non-positive. Pick $\Delta_1 \in \mathcal{L}_S(V)$ such that $B = \Delta_1 - \Delta$ is numerically equivalent over U to $\eta H'$ for some $\eta > 0$. Replacing H by ηH we may assume that $\eta = 1$. If $C = \psi_*B$ then C is numerically equivalent to f^*H over U. Then both C and $C - (K_Y + \Gamma')$ are numerically trivial over Z, so that $C - (K_Y + \Gamma')$ is nef and big over Z and (3.9.1) implies that $\phi_*B = f_*C$ is $\mathbb{R}$-Cartier. (3.6.4) implies that ϕ is $(K_X + \Delta + \lambda B)$-non-positive and $\phi_*(K_X + \Delta + \lambda B)$ is ample over U, for any $\lambda > 0$. On the other hand, note that

$$\Delta + \lambda B = \Delta + \lambda(\Delta_1 - \Delta) \in \mathcal{L}_S(V),$$

for any $\lambda \in [0, 1]$. Therefore ϕ is the ample model of $K_X + \Delta + \lambda B$ over U for any $\lambda \in (0, 1]$. $\square$

3.7. **Convex geometry and Diophantine approximation.**

Definition 3.7.1. *Let V be a finite dimensional real affine space. If $\mathcal{C}$ is a convex subset of V and F is a convex subset of $\mathcal{C}$, then we say that F is a **face** of $\mathcal{C}$ if whenever $\sum r_i v_i \in F$, where $r_1, r_2, \dots, r_k$ are real numbers such that $\sum r_i = 1$, $r_i \geq 0$ and $v_1, v_2, \dots, v_k$ belong to $\mathcal{C}$, then $v_i \in F$ for some i. We say that $v \in \mathcal{C}$ is an **extreme point** if $F = \{v\}$ is a face of $\mathcal{C}$.*

*A **polyhedron** $\mathcal{P}$ in V is the intersection of finitely many half spaces. The **interior** $\mathcal{P}^\circ$ of $\mathcal{P}$ is the complement of the proper faces. A **polytope** $\mathcal{P}$ in V is a compact polyhedron.*

*We say that a real vector space V_0 is **defined over the rationals**, if $V_0 = V' \underset{\mathbb{Q}}{\otimes} \mathbb{R}$, where V' is a rational vector space. We say that an* *affine subspace V of a real vector space V_0, which is defined over the rationals, is* ***defined over the rationals****, if V is spanned by a set of rational vectors of V_0. We say that a polyhedron $\mathcal{P}$ is* ***rational*** *if it is defined by rational half spaces.*

Note that a polytope is the convex hull of a finite set of points and the polytope is rational if those points can be chosen to be rational.

Lemma 3.7.2. *Let X be a normal quasi-projective variety, and let V be a finite dimensional affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$, which is defined over the rationals.*

Then $\mathcal{L}(V)$ (cf. (1.1.4) *for the definition) is a rational polytope.*

Proof. Note that the set of divisors Δ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier forms an affine subspace W of V, which is defined over the rationals, so that, replacing V by W, we may assume that $K_X + \Delta$ is $\mathbb{R}$-Cartier for every $\Delta \in V$.

Let $\pi\colon Y \longrightarrow X$ be a resolution of X, which is a log resolution of the support of any element of V. Given any divisor $\Delta \in V$, if we write

$$K_Y + \Gamma = \pi^*(K_X + \Delta),$$

then the coefficients of Γ are rational affine linear functions of the coefficients of Δ. On the other hand the condition that $K_X + \Delta$ is log canonical is equivalent to the condition that the coefficient of every component of Γ is at most one and the coefficient of every component of Δ is at least zero. $\square$

Lemma 3.7.3. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties. Let V be a finite dimensional affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$ and let $A \geq 0$ be a big $\mathbb{R}$-divisor over U. Let $\mathcal{C} \subset \mathcal{L}_A(V)$ be a polytope.*

If $\mathbf{B}_+(A/U)$ does not contain any non kawamata log terminal centres of (X, Δ), for every $\Delta \in \mathcal{C}$, then we may find a general ample $\mathbb{Q}$-divisor A' over U, a finite dimensional affine subspace V' of $\mathrm{WDiv}_{\mathbb{R}}(X)$ and a translation

$$L\colon\ \mathrm{WDiv}_{\mathbb{R}}(X) \longrightarrow \mathrm{WDiv}_{\mathbb{R}}(X),$$

by an $\mathbb{R}$-divisor T $\mathbb{R}$-linearly equivalent to zero over U such that $L(\mathcal{C}) \subset \mathcal{L}_{A'}(V')$ and $(X, \Delta - A)$ and $(X, L(\Delta))$ have the same non kawamata log terminal centres. Further, if A is a $\mathbb{Q}$-divisor then we may choose T $\mathbb{Q}$-linearly equivalent to zero over U.

Proof. Let $\Delta_1, \Delta_2, \dots, \Delta_k$ be the vertices of the polytope $\mathcal{C}$. Let $\mathfrak{Z}$ be the set of non kawamata log terminal centres of (X, Δ_i) for $1 \leq i \leq k$. Note that if $\Delta \in \mathcal{C}$ then any non kawamata log terminal centre of (X, Δ) is an element of $\mathfrak{Z}$.

By assumption, we may write $A \sim_{\mathbb{R},U} C + D$, where C is a general ample $\mathbb{Q}$-divisor over U and $D \geq 0$ does not contain any element of $\mathfrak{Z}$. Further (3.5.3) implies that if A is a $\mathbb{Q}$-divisor then we may assume that $A \sim_{\mathbb{Q},U} C + D$.

Given any rational number $\delta > 0$, let

$$L\colon \operatorname{WDiv}_{\mathbb{R}}(X) \longrightarrow \operatorname{WDiv}_{\mathbb{R}}(X) \quad \text{given by} \quad L(\Delta) = \Delta + \delta(C + D - A),$$

be the translation by the divisor $T = \delta(C + D - A) \sim_{\mathbb{R},U} 0$. Note that $T \sim_{\mathbb{Q},U} 0$ if A is a $\mathbb{Q}$-divisor. As $C + D$ does not contain any element of $\mathfrak{Z}$, if δ is sufficiently small then

$$K_X + L(\Delta_i) = K_X + \Delta_i + \delta(C + D - A) = K_X + \delta C + (\Delta_i - \delta A + \delta D),$$

is log canonical for every $1 \leq i \leq k$ and has the same non kawamata log terminal centres as $(X, \Delta_i - A)$. But then $L(\mathcal{C}) \subset \mathcal{L}_{A'}(V')$, where $A' = \delta C$ and $V' = V_{(1-\delta)A+\delta D}$ and $(X, \Delta - A)$ and $(X, L(\Delta))$ have the same non kawamata log terminal centres. $\square$

Lemma 3.7.4. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties. Let V be a finite dimensional affine subspace of $\operatorname{WDiv}_{\mathbb{R}}(X)$, which is defined over the rationals, and let A be a general ample $\mathbb{Q}$-divisor over U. Let S be a sum of prime divisors. Suppose that there is a divisorially log terminal pair (X, Δ_0), where $S = \llcorner \Delta_0 \lrcorner$, and let $G \geq 0$ be any divisor whose support does not contain any non kawamata log terminal centres of (X, Δ_0).*

Then we may find a general ample $\mathbb{Q}$-divisor A' over U, an affine subspace V' of $\operatorname{WDiv}_{\mathbb{R}}(X)$, which is defined over the rationals, and a rational affine linear isomorphism

$$L\colon V_{S+A} \longrightarrow V'_{S+A'},$$

such that

- *L preserves $\mathbb{Q}$-linear equivalence over U,*
- *$L(\mathcal{L}_{S+A}(V))$ is contained in the interior of $\mathcal{L}_{S+A'}(V')$,*
- *for any $\Delta \in L(\mathcal{L}_{S+A}(V))$, $K_X + \Delta$ is divisorially log terminal and $\llcorner \Delta \lrcorner = S$, and*
- *for any $\Delta \in L(\mathcal{L}_{S+A}(V))$, the support of Δ contains the support of G.*

Proof. Let W be the vector space spanned by the components of Δ_0. Then $\Delta_0 \in \mathcal{L}_S(W)$ and (3.7.2) implies that $\mathcal{L}_S(W)$ is a non-empty rational polytope. But then $\mathcal{L}_S(W)$ contains a rational point and so, possibly replacing Δ_0, we may assume that $K_X + \Delta_0$ is $\mathbb{Q}$-Cartier.

We first prove the result in the case that $K_X + \Delta$ is $\mathbb{R}$-Cartier for every $\Delta \in V_{S+A}$. By compactness, we may pick $\mathbb{Q}$-divisors $\Delta_1, \Delta_2, \dots, \Delta_l \in$

V_{S+A} such that $\mathcal{L}_{S+A}(V)$ is contained in the simplex spanned by $\Delta_1, \Delta_2, \dots, \Delta_l$ (we do not assume that $\Delta_i \geq 0$). Pick a rational number $\epsilon \in (0, 1/4]$ such that

$$\epsilon(\Delta_i - \Delta_0) + (1 - 2\epsilon)A,$$

is an ample $\mathbb{Q}$-divisor over U, for $1 \leq i \leq l$. Pick

$$A_i \sim_{\mathbb{Q},U} \epsilon(\Delta_i - \Delta_0) + (1 - 2\epsilon)A,$$

general ample $\mathbb{Q}$-divisors over U. Pick $A' \sim_{\mathbb{Q},U} \epsilon A$ a general ample $\mathbb{Q}$-divisor over U. If we define $L\colon V_{S+A} \longrightarrow \mathrm{WDiv}_{\mathbb{R}}(X)$ by

$$L(\Delta_i) = (1 - \epsilon)\Delta_i + A_i + \epsilon\Delta_0 + A' - (1 - \epsilon)A \sim_{\mathbb{Q},U} \Delta_i,$$

and extend to the whole of V_{S+A} by linearity, then L is an injective rational linear map which preserves $\mathbb{Q}$-linear equivalence over U. We let V' be the rational affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$ defined by $V'_{S+A'} = L(V_{S+A})$. Note also that L is the composition $L_2 \circ L_1$ of

$$L_1(\Delta_i) = \Delta_i + A_i/(1-\epsilon) + A' - A \quad \text{and} \quad L_2(\Delta) = (1-\epsilon)\Delta + \epsilon(A' + \Delta_0).$$

If $\Delta \in \mathcal{L}_{S+A}(V)$ then $K_X + \Delta + A' - A$ is log canonical, and as A_i is a general ample $\mathbb{Q}$-divisor over U it follows that $K_X + \Delta + 4/3A_i + A' - A$ is log canonical as well. As $1/(1-\epsilon) \leq 4/3$, it follows that if $\Delta \in \mathcal{L}_{S+A}(V)$ then $K_X + L_1(\Delta)$ is log canonical. Therefore, if $\Delta \in \mathcal{L}_{S+A}(V)$ then $K_X + L(\Delta)$ is divisorially log terminal and $\llcorner L(\Delta) \lrcorner = S$.

Pick a divisor G' such that $S + A' + G'$ belongs to the interior of $\mathcal{L}_{S+A'}(V')$. As $G + G'$ contains no log canonical centres of (X, Δ_0) and X is smooth at the generic point of every log canonical centre of (X, Δ_0), we may pick a $\mathbb{Q}$-Cartier divisor $H \geq G + G'$ which contains no log canonical centres of (X, Δ_0). Pick a rational number $\eta > 0$ such that $A' - \eta H$ is ample over U. Pick $A'' \sim_{\mathbb{Q},U} A' - \eta H$ a general ample $\mathbb{Q}$-divisor over U. Let $\delta > 0$ by any rational number and let

$$T\colon\ \mathrm{WDiv}_{\mathbb{R}}(X) \longrightarrow \mathrm{WDiv}_{\mathbb{R}}(X),$$

be translation by $\delta(\eta H + A'' - A') \sim_{\mathbb{Q},U} 0$. If V'' is the span of V', A' and H and $\delta > 0$ is sufficiently small then $T(L(\mathcal{L}_{S+A}(V)))$ is contained in the interior of $\mathcal{L}_{\delta A''+S}(V'')$, $K_X + T(\Delta)$ is divisorially log terminal and the support of $T(\Delta)$ contains the support of G, for all $\Delta \in L(\mathcal{L}_{S+A}(V))$. If we replace L by $T \circ L$, $V'_{S+A'}$ by $T(L(V_{S+A}))$ and A' by $\delta A''$ then this finishes the case when $K_X + \Delta$ is $\mathbb{R}$-Cartier for every $\Delta \in V_{S+A}$.

We now turn to the general case. If

$$W_0 = \{\, B \in V \mid K_X + S + A + B \text{ is } \mathbb{R}\text{-Cartier} \,\},$$

then $W_0 \subset V$ is an affine subspace of V, which is defined over the rationals. Note that $\mathcal{L}_{S+A}(V) = \mathcal{L}_{S+A}(W_0)$. By what we have already proved, there is a rational affine linear isomorphism $L_0\colon W_0 \longrightarrow W'_0$, which preserves $\mathbb{Q}$-linear equivalence over U, a general ample $\mathbb{Q}$-divisor A' over U, such that $L_0(\mathcal{L}_{S+A}(W_0))$ is contained in the interior of $\mathcal{L}_{S+A'}(W_0')$, and for every divisor $\Delta \in L_0(\mathcal{L}_{S+A}(W_0))$, $K_X + \Delta$ is divisorially log terminal and the support of Δ contains the support of G.

Let W_1 be any vector subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$, which is defined over the rationals, such that $V = W_0 + W_1$ and $W_0 \cap W_1 \subset \{0\}$. Let $V' = W_0' + W_1$. Since L_0 preserves $\mathbb{Q}$-linear equivalence over U, $W_0' \cap W_1 = W_0 \cap W_1$ and $\mathcal{L}_{S+A'}(V') = \mathcal{L}_{S+A'}(W_0')$. If we define $L\colon V_A \longrightarrow V'_{A'}$, by sending $A + B_0 + B_1$ to $L_0(A + B_0) + B_1$, where $B_i \in W_i$, then L is a rational affine linear isomorphism, which preserves $\mathbb{Q}$-linear equivalence over U and $L(\mathcal{L}_{S+A}(V))$ is contained in the interior of $\mathcal{L}_{S+A'}(V')$. $\square$

Lemma 3.7.5. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties. Let $(X, \Delta = A + B)$ be a log canonical pair, where $A \geq 0$ and $B \geq 0$.*

If A is π-big and $\mathbf{B}_+(A/U)$ does not contain any non kawamata log terminal centres of (X, Δ) and there is a kawamata log terminal pair (X, Δ_0) then we may find a kawamata log terminal pair $(X, \Delta' = A' + B')$, where $A' \geq 0$ is a general ample $\mathbb{Q}$-divisor over U, $B' \geq 0$ and $K_X + \Delta' \sim_{\mathbb{R},U} K_X + \Delta$. If in addition A is a $\mathbb{Q}$-divisor then $K_X + \Delta' \sim_{\mathbb{Q},U} K_X + \Delta$.

Proof. By (3.7.3) we may assume that A is a general ample $\mathbb{Q}$-divisor over U. If V is the vector space spanned by the components of Δ then $\Delta \in \mathcal{L}_A(V)$ and the result follows by (3.7.4). $\square$

Lemma 3.7.6. *Let V be a finite dimensional real vector space, which is defined over the rationals. Let $\Lambda \subset V$ be a lattice spanned by rational vectors. Suppose that $v \in V$ is a vector which is not contained in any proper affine subspace $W \subset V$ which is defined over the rationals.*

Then the set

$$X = \{\, mv + \lambda \,|\, m \in \mathbb{N}, \lambda \in \Lambda \,\},$$

is dense in V.

Proof. Let

$$q\colon V \longrightarrow V/\Lambda,$$

be the quotient map and let G be the closure of the image of X. As G is infinite and V/Λ is compact, G has an accumulation point. It then follows that zero is also an accumulation point and that G is a closed subgroup.

The connected component G_0 of G containing the identity is a Lie subgroup of V/Λ and so by Theorem 15.1 of [3], G_0 is a torus. Thus

$G_0 = W/\Lambda_0$, where

$$W = H_1(G_0, \mathbb{R}) = \Lambda_0 \underset{\mathbb{Z}}{\otimes} \mathbb{R} = H_1(G_0, \mathbb{Z}) \underset{\mathbb{Z}}{\otimes} \mathbb{R} \subset H_1(G, \mathbb{Z}) \underset{\mathbb{Z}}{\otimes} \mathbb{R} = H_1(G, \mathbb{R}),$$

is a subspace of V which is defined over the rationals. On the other hand, G/G_0 is finite as it is discrete and compact. Thus a translate of v by a rational vector is contained in W and so $W = V$. $\square$

Lemma 3.7.7. *Let $\mathcal{C}$ be a rational polytope contained in a real vector space V of dimension n, which is defined over the rationals. Fix a positive integer k and a positive real number α.*

If $v \in \mathcal{C}$ then we may find vectors $v_1, v_2, \dots, v_p \in \mathcal{C}$ and positive integers $m_1, m_2, \dots, m_p$, which are divisible by k, such that v is a convex linear combination of the vectors $v_1, v_2, \dots, v_p$ and

$$\|v_i - v\| < \frac{\alpha}{m_i} \qquad \text{where} \qquad \frac{m_i v_i}{k} \text{ is integral.}$$

Proof. Rescaling by k, we may assume that $k = 1$. We may assume that v is not contained in any proper affine linear subspace which is defined over the rationals. In particular v is contained in the interior of $\mathcal{C}$ since the faces of $\mathcal{C}$ are rational.

After translating by a rational vector, we may assume that $0 \in \mathcal{C}$. After fixing a suitable basis for V and possibly shrinking $\mathcal{C}$, we may assume that $\mathcal{C} = [0,1]^n \subseteq \mathbb{R}^n$ and $v = (x_1, x_2, \dots, x_n) \in (0,1)^n$. By (3.7.6), for each subset $I \subseteq \{1, 2, \dots, n\}$, we may find

$$v_I = (s_1, s_2, \dots, s_n) \in (0,1)^n \cap \mathbb{Q}^n,$$

and an integer m_I such that $m_I v_I$ is integral, such that

$$\|v - v_I\| < \frac{\alpha}{m_I} \qquad \text{and } s_j < x_j \quad \text{if and only if } j \in I.$$

In particular v is contained inside the rational polytope $\mathcal{B} \subseteq \mathcal{C}$ generated by the v_I. Thus v is a convex linear combination of a subset $v_1, v_2, \dots, v_p$ of the extreme points of $\mathcal{B}$. $\square$

3.8. **Rational curves of low degree.** We will need the following generalisation of a result of Kawamata, see Theorem 1 of [16], which is proved by Shokurov in the appendix to [29].

Theorem 3.8.1. *Let $\pi \colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties. Suppose that (X, Δ) is a log canonical pair of dimension n, where $K_X + \Delta$ is $\mathbb{R}$-Cartier. Suppose that there is a divisor Δ_0 such that $K_X + \Delta_0$ is kawamata log terminal.*

If R is an extremal ray of $\overline{\mathrm{NE}}(X/U)$ that is $(K_X + \Delta)$-negative, then there is a rational curve Σ spanning R, such that

$$0 < -(K_X + \Delta) \cdot \Sigma \leq 2n.$$

Proof. Passing to an open subset of U, we may assume that U is affine. Let V be the vector pace spanned by the components of $\Delta + \Delta_0$. By (3.7.2) the space $\mathcal{L}(V)$ of log canonical divisors is a rational polytope. Since $\Delta_0 \in \mathcal{L}(V)$, we may find $\mathbb{Q}$-divisors $\Delta_i \in V$ with limit Δ, such that $K_X + \Delta_i$ is kawamata log terminal. In particular we may assume that $(K_X + \Delta_0) \cdot R < 0$. Replacing π by the contraction defined by the extremal ray R, we may assume that $-(K_X + \Delta)$ is π-ample.

Theorem 1 of [16] implies that we can find a rational curve Σ_i contracted by π such that

$$-(K_X + \Delta_i) \cdot \Sigma_i \leq 2n.$$

Pick a π-ample $\mathbb{Q}$-divisor A such that $-(K_X + \Delta + A)$ is also π-ample. In particular $-(K_X + \Delta_i + A)$ is π-ample for $i \gg 0$. Now

$$A \cdot \Sigma_i = (K_X + \Delta_i + A) \cdot \Sigma_i - (K_X + \Delta_i) \cdot \Sigma_i < 2n.$$

It follows that the curves Σ_i belong to a bounded family. Thus, possibly passing to a subsequence, we may assume that $\Sigma = \Sigma_i$ is constant. In this case

$$-(K_X + \Delta) \cdot \Sigma = \lim_i -(K_X + \Delta_i) \cdot \Sigma \leq 2n. \qquad \square$$

Corollary 3.8.2. *Let $\pi \colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties. Suppose the pair $(X, \Delta = A + B)$ has log canonical singularities, where $A \geq 0$ is an ample $\mathbb{R}$-divisor over U and $B \geq 0$. Suppose that there is a divisor Δ_0 such that $K_X + \Delta_0$ is kawamata log terminal.*

Then there are only finitely many $(K_X + \Delta)$-negative extremal rays $R_1, R_2, \dots, R_k$ of $\overline{\mathrm{NE}}(X/U)$.

Proof. We may assume that A is a $\mathbb{Q}$-divisor. Let R be a $(K_X + \Delta)$-negative extremal ray of $\overline{\mathrm{NE}}(X/U)$. Then

$$-(K_X + B) \cdot R = -(K_X + \Delta) \cdot R + A \cdot R > 0.$$

By (3.8.1) R is spanned by a curve Σ such that

$$-(K_X + B) \cdot \Sigma \leq 2n.$$

But then

$$A \cdot \Sigma = -(K_X + B) \cdot \Sigma + (K_X + \Delta) \cdot \Sigma \leq 2n. \qquad \square$$

Therefore the curve Σ belongs to a bounded family.

3.9. **Effective base point free theorem.**

Theorem 3.9.1 (Effective Base Point Free Theorem)**.** *Fix a positive integer n. Then there is a positive integer $m > 0$ with the following property:*

Let $f\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties, and let D be a nef $\mathbb{R}$-divisor over U, such that $aD-(K_X+\Delta)$ is nef and big over U, for some positive real number a, where (X,Δ) is kawamata log terminal and X has dimension n.

Then D is semiample over U and if aD is Cartier then maD is globally generated over U.

Proof. Replacing D by aD we may assume that $a = 1$. As the property that D is either semiample or globally generated over U is local over U, we may assume that U is affine.

By assumption we may write $D - (K_X + \Delta) \sim_{\mathbb{R},U} A + B$, where A is an ample $\mathbb{Q}$-divisor and $B \geq 0$. Pick $\epsilon \in (0, 1)$ such that $(X, \Delta + \epsilon B)$ is kawamata log terminal. Then

$$D - (K_X + \Delta + \epsilon B) = (1 - \epsilon)(D - (K_X + \Delta)) + \epsilon(D - (K_X + \Delta + B)),$$

is ample. Replacing (X, Δ) by $(X, \Delta + \epsilon B)$ we may therefore assume that $D - (K_X + \Delta)$ is ample. Let V be the subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$ spanned by the components of Δ. As $\mathcal{L}(V)$ is a rational polytope, cf. (3.7.2), which contains Δ, we may find Δ' such that $K_X + \Delta'$ is $\mathbb{Q}$-Cartier and kawamata log terminal, sufficiently close to Δ so that $D-(K_X+\Delta')$ is ample. Replacing (X,Δ) by (X,Δ') we may therefore assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

The existence of the integer m is Kollár's effective version of the base point free theorem [20].

Pick a general ample $\mathbb{Q}$-divisor A such that $D - (K_X + \Delta + A)$ is ample. Replacing Δ by $\Delta + A$, we may assume that $\Delta = A + B$ where A is ample and $B \geq 0$. By (3.8.2) there are finitely many $(K_X + \Delta)$-negative extremal rays $R_1, R_2, \dots, R_k$ of $\overline{\mathrm{NE}}(X)$. Let

$$F = \{\, \alpha \in \overline{\mathrm{NE}}(X) \,|\, D \cdot \alpha = 0 \,\}.$$

Then F is a face of $\overline{\mathrm{NE}}(X)$ and if $\alpha \in F$ then $(K_X + \Delta) \cdot \alpha < 0$, and so F is spanned by a subset of the extremal rays $R_1, R_2, \dots, R_k$. Let V be the smallest affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$, which is defined over the rationals and contains D. Then

$$\mathcal{C} = \{\, C \in V \,|\, C \cdot \alpha = 0, \forall \alpha \in F \,\},$$

is a rational polyhedron. It follows that we may find positive real numbers $r_1, r_2, \dots, r_q$ and nef $\mathbb{Q}$-Cartier divisors $D_1, D_2, \dots, D_q$ such that $D = \sum r_p D_p$. Possibly re-choosing $D_1, D_2, \dots, D_q$ we may assume that $D_p - (K_X + \Delta)$ is ample. By the usual base point free theorem, D_p is semiample and so D is semiample. $\square$

Corollary 3.9.2. *Fix a positive integer n. Then there is a constant $m > 0$ with the following property:*

Let $f\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties such that $K_X + \Delta$ is nef over U and Δ is big over U, where (X, Δ) is kawamata log terminal and X has dimension n.

Then $K_X + \Delta$ is semiample over U and if r is a positive constant such that $r(K_X + \Delta)$ is Cartier, then $mr(K_X + \Delta)$ is globally generated over U.

Proof. By (3.7.5) we may find a kawamata log terminal pair

$$K_X + A + B \sim_{\mathbb{R},U} K_X + \Delta,$$

where $A \geq 0$ is a general ample $\mathbb{Q}$-divisor over U and $B \geq 0$. As

$$(K_X + \Delta) - (K_X + B) \sim_{\mathbb{R},U} A,$$

is ample over U and $K_X + B$ is kawamata log terminal, the result follows by (3.9.1). $\square$

Lemma 3.9.3. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of quasi-projective varieties Suppose that (X, Δ) is a kawamata log terminal pair, where Δ is big over U.*

If $\phi\colon X \dashrightarrow Y$ is a weak log canonical model of $K_X + \Delta$ over U then

1. *ϕ is a semiample model over U,*
2. *the ample model $\psi\colon X \dashrightarrow Z$ of $K_X + \Delta$ over U exists, and*
3. *there is a contraction morphism $h\colon Y \longrightarrow Z$ such that $K_Y + \Gamma \sim_{\mathbb{R},U} h^*H$, for some ample $\mathbb{R}$-divisor H over U, where $\Gamma = \phi_*\Delta$.*

Proof. $K_Y + \Gamma$ is semiample over U by (3.9.2). (3) of (3.6.6) implies (2) and (3). $\square$

3.10. **The MMP with scaling.** In order to run a minimal model program, two kinds of operations, known as flips and divisorial contractions are required. We begin by recalling their definitions.

Definition 3.10.1. *Let (X, Δ) be a log canonical pair and $f\colon X \longrightarrow Z$ be a projective morphism of normal varieties. Then f is a **flipping contraction** if*

1. *X is $\mathbb{Q}$-factorial and Δ is an $\mathbb{R}$-divisor,*
2. *f is a small birational morphism of relative Picard number $\rho(X/Z) = 1$, and*

(3) $-(K_X + \Delta)$ *is f-ample.*

*The **flip** $f^+ \colon X^+ \longrightarrow Z$ of a flipping contraction $f \colon X \longrightarrow Z$ is a small birational projective morphism of normal varieties $f^+ \colon X^+ \longrightarrow Z$ such that $K_{X^+} + \Delta^+$ is f^+-ample, where Δ^+ is the strict transform of Δ.*

Lemma 3.10.2. *Let (X, Δ) be a kawamata log terminal pair, let $f \colon X \longrightarrow Z$ be a flipping contraction and let $f^+ \colon X^+ \longrightarrow Z$ be the flip of f. Let $\phi \colon X \dashrightarrow X^+$ be the induced birational map.*

Then

(1) *ϕ is the log canonical model of (X, Δ) over Z,*
(2) *$(X^+, \Delta^+ = \phi_*\Delta)$ is log canonical,*
(3) *X^+ is $\mathbb{Q}$-factorial, and*
(4) *$\rho(X^+/Z) = 1$.*

Proof. As ϕ is small, (3.6.2) implies that ϕ is $(K_X + \Delta)$-negative and this implies (1) and (2).

Let B be a divisor on X^+ and let $C \subset X$ be the strict transform of B. Since $\rho(X/Z) = 1$, we may find $\lambda \in \mathbb{R}$ such $C + \lambda(K_X + \Delta)$ is numerically trivial over U. But then (3.9.1) implies that $C + \lambda(K_X + \Delta) = f^*D$, for some $\mathbb{R}$-Cartier divisor D on Z. Therefore $B + \lambda(K_{X^+} + \Delta^+) = f^{+*}D$, and this implies (3) and (4). □

Remark 3.10.3. *Results of Ambro and Fujino imply that the results of* (3.10.2) *hold in the case when (X, Δ) is log canonical.*

In terms of our induction, we will need to work with a more restrictive notion of flipping contraction.

Definition 3.10.4. *Let (X, Δ) be a purely log terminal pair and let $f \colon X \longrightarrow Z$ be a projective morphism of normal varieties. Then f is a **pl-flipping contraction** if*

(1) *X is $\mathbb{Q}$-factorial and Δ is an $\mathbb{R}$-divisor,*
(2) *f is a small birational morphism of relative Picard number $\rho(X/Z) = 1$,*
(3) *$-(K_X + \Delta)$ is f-ample, and*
(4) *$S = \llcorner \Delta \lrcorner$ is irreducible and $-S$ is f-ample.*

*A **pl-flip** is the flip of a pl-flipping contraction.*

Definition 3.10.5. *Let (X, Δ) be a log canonical pair and $f \colon X \longrightarrow Z$ be a projective morphism of normal varieties. Then f is a **divisorial contraction** if*

(1) *X is $\mathbb{Q}$-factorial and Δ is an $\mathbb{R}$-divisor,*

(2) *f is a birational morphism of relative Picard number $\rho(X/Z) = 1$ with exceptional locus a divisor, and*
(3) *$-(K_X + \Delta)$ is f-ample.*

Remark 3.10.6. *If $f\colon X \longrightarrow Z$ is a divisorial contraction, then an argument similar to* (3.10.2) *shows that $(Z, f_*\Delta)$ is log canonical and Z is $\mathbb{Q}$-factorial.*

Definition 3.10.7. *Let (X, Δ) be a log canonical pair and $f\colon X \longrightarrow Z$ be a projective morphism of normal varieties. Then f is a* ***Mori fibre space*** *if*

(1) *X is $\mathbb{Q}$-factorial and Δ is an $\mathbb{R}$-divisor,*
(2) *f is a contraction morphism, $\rho(X/Z) = 1$ and $\dim Z < \dim X$, and*
(3) *$-(K_X + \Delta)$ is f-ample.*

The objective of the MMP is to produce either a log terminal model or a Mori fibre space. Note that if $K_X + \Delta$ has a log terminal model then $K_X + \Delta$ is pseudo-effective and if $K_X + \Delta$ has a Mori fibre space then $K_X + \Delta$ is not pseudo-effective, so these two cases are mutually exclusive.

There are several versions of the MMP, depending on the singularities that are allowed (typically, one restricts to kawamata log terminal singularities or divisorially log terminal singularities or terminal singularities with $\Delta = 0$) and depending on the choices of negative extremal rays that are allowed (traditionally any choice of an extremal ray is acceptable).

In this paper, we will run the MMP with scaling for divisorially log terminal pairs satisfying certain technical assumptions. We will need the following key result.

Lemma 3.10.8. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties. Suppose that the pair $(X, \Delta = A + B)$ has kawamata log terminal singularities, where $A \geq 0$ is big over U, $B \geq 0$, D is a nef $\mathbb{R}$-Cartier divisor over U, but $K_X + \Delta$ is not nef over U. Set*

$$\lambda = \sup \{ \mu \mid D + \mu(K_X + \Delta) \text{ is nef over } U \}.$$

Then there is a $(K_X + \Delta)$-negative extremal ray R over U, such that

$$(D + \lambda(K_X + \Delta)) \cdot R = 0.$$

Proof. By (3.7.5) we may assume that A is ample over U.

By (3.8.2) there are only finitely many $(K_X + \Delta)$-negative extremal rays $R_1, R_2, \dots, R_k$ over U. For each $(K_X + \Delta)$-negative extremal ray

R_i, pick a curve Σ_i which generates R_i. Let

$$\mu = \min_i \frac{D \cdot \Sigma_i}{-(K_X + \Delta) \cdot \Sigma_i}.$$

Then $D + \mu(K_X + \Delta)$ is nef over U, since it is non-negative on each R_i, but it is zero on one of the extremal rays $R = R_i$. Thus $\lambda = \mu$. $\square$

Lemma 3.10.9. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties and let (X, Δ_0) be a kawamata log terminal pair. Suppose that $(X, \Delta = A+B)$ is a log canonical pair, where $A \geq 0$ is big over U, $B \geq 0$, $\mathbf{B}_+(A/U)$ contains no non kawamata log terminal centres of (X, Δ) and C is an $\mathbb{R}$-Cartier divisor such that $K_X + \Delta$ is not nef over U, whilst $K_X + \Delta + C$ is nef over U.*

Then there is a $(K_X+\Delta)$-negative extremal ray R and a real number $0 < \lambda \leq 1$ such that $K_X + \Delta + \lambda C$ is nef over U but trivial on R.

Proof. By (3.7.5) we may assume $K_X + \Delta$ is kawamata log terminal and that A is ample over U. Apply (3.10.8) to $D = K_X + \Delta + C$. $\square$

Remark 3.10.10. *Assuming existence and termination of the relevant flips, we may use (3.10.9) to define a special minimal model program, which we will refer to as the $(K_X + \Delta)$-**MMP with scaling of** C.*

Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties, where X is $\mathbb{Q}$-factorial, $(X, \Delta + C = S + A + B + C)$ is a divisorially log terminal pair, such that $\llcorner \Delta \lrcorner = S$, $A \geq 0$ is big over U, $\mathbf{B}_+(A/U)$ does not contain any non kawamata log terminal centres of (X, Δ), and $B \geq 0$, $C \geq 0$. We pick $\lambda \geq 0$ and a $(K_X + \Delta)$-negative extremal ray R over U as in (3.10.9) above. If $K_X + \Delta$ is nef over U we stop. Otherwise $\lambda > 0$ and we let $f\colon X \longrightarrow Z$ be the extremal contraction over U defined by R. If f is not birational, we have a Mori fibre space over U and we stop. If f is birational then either f is divisorial and we replace X by Z or f is small and assuming the existence of the flip $f^+\colon X^+ \longrightarrow Z$, we replace X by X^+. In either case $K_X+\Delta+\lambda C$ is nef over U and $K_X+\Delta$ is divisorially log terminal and so we may repeat the process.

In this way, we obtain a sequence $\phi_i\colon X_i \dashrightarrow X_{i+1}$ of $K_X + \Delta$ flips and divisorial contractions over U and real numbers $1 \geq \lambda_1 \geq \lambda_2 \geq \cdots$ such that $K_{X_i} + \Delta_i + \lambda_i C_i$ is nef over U where $\Delta_i = (\phi_{i-1})_* \Delta_{i-1}$ and $C_i = (\phi_{i-1})_* C_{i-1}$.

Note that by (3.10.11), each step of this MMP preserves the condition that (X, Δ) is divisorially log terminal and $\mathbf{B}_+(A/U)$ does not contain any non kawamata log terminal centres of (X, Δ) so that we may apply

(3.10.9). *By* (3.10.2) *and* (3.10.6), $(X, \Delta+\lambda C)$ *is log canonical. However the condition that* $\mathbf{B}_+(A/U)$ *does not contain any non kawamata log terminal centres of* $(X, \Delta+\lambda C)$ *is not necessarily preserved.*

Lemma 3.10.11. *Let* $\pi\colon X \longrightarrow U$ *be a projective morphism of normal quasi-projective varieties. Suppose that* $K_X+\Delta$ *is divisorially log terminal,* X *is* $\mathbb{Q}$*-factorial and let* $\phi\colon X \dashrightarrow Y$ *be a sequence of steps of the* $(K_X+\Delta)$*-MMP over* U.

If $\Gamma = \phi_*\Delta$ *then*

1. ϕ *is an isomorphism at the generic point of every non kawamata log terminal centre of* $K_Y+\Gamma$. *In particular* (Y,Γ) *is divisorially log terminal.*
2. *If* $\Delta = S+A+B$, *where* $S = \llcorner\Delta\lrcorner$, $A \geq 0$ *is big over* U, $\mathbf{B}_+(A/U)$ *does not contain any non kawamata log terminal centres of* $K_X+\Delta$, *and* $B \geq 0$ *then* $\phi_*S = \llcorner\Gamma\lrcorner$, ϕ_*A *is big over* U *and* $\mathbf{B}_+(\phi_*A/U)$ *does not contain any non kawamata log terminal centres of* $K_Y+\Gamma$.

   In particular $\Gamma \sim_{\mathbb{R},U} \Gamma'$ *where* (Y,Γ') *is kawamata log terminal and* Γ' *is big over* U.

Proof. We may assume that ϕ is either a flip or a divisorial contraction over U. We first prove (1). Let $p\colon W \longrightarrow X$ and $q\colon W \longrightarrow Y$ be common log resolutions, which resolve the indeterminacy of ϕ. We may write

$$p^*(K_X+\Delta) = q^*(K_Y+\Gamma)+E,$$

where E is exceptional and contains every exceptional divisor over the locus where ϕ^{-1} is not an isomorphism. In particular the log discrepancy of every valuation with centre on Y contained in the locus where ϕ^{-1} is not an isomorphism with respect to $K_Y+\Gamma$ is strictly greater than the log discrepancy with respect to $K_X+\Delta$. Hence (1).

Now suppose that $\Delta = S+A+B$, A is big over U and no non kawamata log terminal centre of (X,Δ) is contained in $\mathbf{B}_+(A/U)$. Pick a divisor C on Y which is a general ample $\mathbb{Q}$-divisor over U. Possibly replacing C by a smaller multiple, we may assume that $\mathbf{B}((A-\phi^{-1}_*C)/U)$ does not contain any log canonical centres of (X,Δ). Thus $A-\phi^{-1}_*C \sim_{\mathbb{R},U} D \geq 0$, where D does not contain any non kawamata log terminal centres of (X,Δ). If $\epsilon > 0$ is sufficiently small, then $(X, \Delta' = \Delta+\epsilon D)$ is divisorially log terminal, (X,Δ) and (X,Δ') have the same non kawamata log terminal centres and ϕ is a $K_X+\Delta'$ flip or a divisorial contraction over U. (1) implies that $(Y, \Gamma' = \phi_*\Delta')$ is divisorially log terminal and hence ϕ_*D does not contain any non kawamata log terminal centres of (Y,Γ). (2) follows as $\phi_*A - C \sim_{\mathbb{R},U} \phi_*D \geq 0$. $\square$

Lemma 3.10.12. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of quasi-projective varieties. Suppose that (X, Δ) is a divisorially log terminal pair. Let S be a prime divisor.*

Let $f_i\colon X_i \dashrightarrow X_{i+1}$ a sequence of flips and divisorial contractions over U, starting with $X_1 := X$, for the $(K_X + \Delta)$-MMP, which does not contract S. If f_i is not an isomorphism in a neighbourhood of the strict transform S_i of S, then neither is the induced birational map $X_i \dashrightarrow X_j$, $j > i$.

Proof. Since the map f_i is $(K_{X_i} + \Delta_i)$-negative and $X_{i+1} \dashrightarrow X_j$ is $(K_{X_{i+1}} + \Delta_{i+1})$-negative, there is some valuation ν whose centre intersects S_i, such that

$$a(\nu, X_i, \Delta_i) < a(\nu, X_{i+1}, \Delta_{i+1}) \quad \text{and} \quad a(\nu, X_{i+1}, \Delta_{i+1}) \leq a(\nu, X_j, \Delta_j).$$

But then $a(\nu, X_i, \Delta_i) < a(\nu, X_j, \Delta_j)$, so that $X_i \dashrightarrow X_j$ is not an isomorphism in a neighbourhood of S_i. $\square$

3.11. **Shokurov's polytopes.** We will need some results from [32]. First some notation. Given a ray $R \subset \overline{\mathrm{NE}}(X)$, let

$$R^{\perp} = \{\, \Delta \in \mathcal{L}(V) \mid (K_X + \Delta) \cdot R = 0 \,\}.$$

Theorem 3.11.1. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties. Let V be a finite dimensional affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$, which is defined over the rationals. Fix an ample $\mathbb{Q}$-divisor A over U. Suppose that there is a kawamata log terminal pair (X, Δ_0).*

Then the set of hyperplanes $R^{\perp}$ is finite in $\mathcal{L}_A(V)$, as R ranges over the set of extremal rays of $\overline{\mathrm{NE}}(X/U)$. In particular, $\mathcal{N}_{A,\pi}(V)$ is a rational polytope.

Corollary 3.11.2. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties. Let V be a finite dimensional affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$, which is defined over the rationals. Fix a general ample $\mathbb{Q}$-divisor A over U. Suppose that there is a kawamata log terminal pair (X, Δ_0). Let $\phi\colon X \dashrightarrow Y$ be any birational contraction over U.*

Then $\mathcal{W}_{\phi,A,\pi}(V)$ is a rational polytope. Moreover there are finitely many morphisms $f_i\colon Y \longrightarrow Z_i$ over U, $1 \leq i \leq k$, such that if $f\colon Y \longrightarrow Z$ is any contraction morphism over U and there is an $\mathbb{R}$-divisor D on Z, which is ample over U, such that $K_Y + \Gamma = \phi_*(K_X + \Delta) \sim_{\mathbb{R},U} f^*D$ for some $\Delta \in \mathcal{W}_{\phi,A,\pi}(V)$, then there is an index $1 \leq i \leq k$ and an isomorphism $\eta\colon Z_i \longrightarrow Z$ such that $f = \eta \circ f_i$.

Corollary 3.11.3. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties. Let V be a finite dimensional affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$, which is defined over the rationals. Fix a general ample $\mathbb{Q}$-divisor A over U. Let (X, Δ_0) be a kawamata log terminal pair, let $f\colon X \longrightarrow Z$ be a morphism over U such that $\Delta_0 \in \mathcal{L}_A(V)$ and $K_X + \Delta_0 \sim_{\mathbb{R},U} f^*H$, where H is an ample divisor over U. Let $\phi\colon X \dashrightarrow Y$ be a birational contraction over Z.*

Then there is a neighbourhood P_0 of Δ_0 in $\mathcal{L}_A(V)$ such that for all $\Delta \in P_0$, ϕ is a log terminal model for $K_X + \Delta$ over Z if and only if ϕ is a log terminal model for $K_X + \Delta$ over U.

Proof of (3.11.1). Since $\mathcal{L}_A(V)$ is compact it suffices to prove this locally about any point $\Delta \in \mathcal{L}_A(V)$. By (3.7.4) we may assume that $K_X + \Delta$ is kawamata log terminal. Fix $\epsilon > 0$ such that if $\Delta' \in \mathcal{L}_A(V)$ and $\|\Delta' - \Delta\| < \epsilon$, then $\Delta' - \Delta + A/2$ is ample over U. Let R be an extremal ray over U such that $(K_X + \Delta') \cdot R = 0$, where $\Delta' \in \mathcal{L}_A(V)$ and $\|\Delta' - \Delta\| < \epsilon$. We have

$$(K_X + \Delta - A/2) \cdot R = (K_X + \Delta') \cdot R - (\Delta' - \Delta + A/2) \cdot R < 0.$$

Finiteness then follows from (3.8.2).

$\mathcal{N}_{A,\pi}(V)$ is surely a closed subset of $\mathcal{L}_A(V)$. If $K_X + \Delta$ is not nef over U then (3.8.1) implies that $K_X + \Delta$ is negative on a rational curve Σ which generates an extremal ray R of $\overline{\mathrm{NE}}(X/U)$. Thus $\mathcal{N}_{A,\pi}(V)$ is the intersection of $\mathcal{L}_A(V)$ with the half-spaces determined by finitely many of the extremal rays of $\overline{\mathrm{NE}}(X/U)$. $\square$

Proof of (3.11.2). Since $\mathcal{L}_A(V)$ is a rational polytope its span is an affine subspace of V_A, which is defined over the rationals. Possibly replacing V, we may therefore assume that $\mathcal{L}_A(V)$ spans V_A. By compactness, to prove that $\mathcal{W}_{\phi,A,\pi}(V)$ is a rational polytope, we may work locally about a divisor $\Delta \in \mathcal{W}_{\phi,A,\pi}(V)$. By (3.7.4) we may assume that $K_X + \Delta$ is kawamata log terminal in which case $K_Y + \Gamma$ is kawamata log terminal as well. Let $W \subset \mathrm{WDiv}_{\mathbb{R}}(Y)$ be the image of V. If $C = \phi_* A$ then C is big over U and by (3.7.3) and (3.7.4), we may find a rational affine linear isomorphism $L\colon W \longrightarrow W'$ and an ample $\mathbb{Q}$-divisor C' over U such that $L(\Gamma)$ belongs to the interior of $\mathcal{L}_{C'}(W')$ and $L(\Psi) \sim_{\mathbb{Q},U} \Psi$ for any $\Psi \in W$. (3.11.1) implies that $\mathcal{N}_{C',\psi}(W')$ is a rational polytope, where $\psi\colon Y \longrightarrow U$ is the structure morphism, and so $\mathcal{N}_{C,\psi}(W)$ is a rational polytope locally about Γ.

Let $p\colon Z \longrightarrow X$ be a log resolution of (X, Δ) which resolves the indeterminacy locus of ϕ, via a birational map $q\colon Z \longrightarrow Y$. We may write

$$K_Z + \Psi = p^*(K_X + \Delta)$$
$$K_Z + \Phi = q^*(K_Y + \Gamma).$$

Note that $\Delta \in \mathcal{W}_{\phi,A,\pi}(V)$ if and only if $\Gamma = \phi_*\Delta \in \mathcal{N}_{C,\psi}(W)$ and $\Psi - \Phi \geq 0$. Since the map $L\colon V \longrightarrow W$ given by $\Delta \longrightarrow \Gamma = \phi_*\Delta$ is rational linear, the first statement is clear.

Note that if $f\colon Y \longrightarrow Z$ and $f'\colon Y \longrightarrow Z'$ are two contraction morphisms over U then there is an isomorphism $\eta\colon Z \longrightarrow Z'$ such that $f' = \eta \circ f$ if and only if the curves contracted by f and f' coincide.

Let $f\colon Y \longrightarrow Z$ be a contraction morphism over U, such that

$$K_Y + \Gamma = K_Y + \phi_*\Delta \sim_{\mathbb{R},U} f^*D,$$

where $\Delta \in \mathcal{W}_{\phi,A,\pi}(V)$ and D is an ample over U $\mathbb{R}$-divisor on Z. Γ belongs to the interior of a unique face G of $\mathcal{N}_{C,\psi}(W)$ and the curves contracted by f are determined by G. Now Δ belongs to the interior of a unique face F of $\mathcal{W}_{\phi,A,\pi}(V)$ and G is determined by F. But as $\mathcal{W}_{\phi,A,\pi}(V)$ is a rational polytope it has only finitely many faces F. $\square$

Proof of (3.11.3). By (3.11.1) we may find finitely many extremal rays $R_1, R_2, \dots, R_k$ of Y over U such that if $K_Y + \Gamma = K_Y + \phi_*\Delta$ is not nef over U, then it is negative on one of these rays. If $\Gamma_0 = \phi_*\Delta_0$ then we may write

$$K_Y + \Gamma = K_Y + \Gamma_0 + (\Gamma - \Gamma_0) \sim_{\mathbb{R},U} g^*H + \phi_*(\Delta - \Delta_0),$$

where $g\colon Y \longrightarrow Z$ is the structure morphism. Therefore there is a neighbourhood P_0 of Δ_0 in $\mathcal{L}_A(V)$ such that if $K_Y + \Gamma$ is not nef over U then it is negative on an extremal ray R_i, which is extremal over Z. In particular if ϕ is a log terminal model of $K_X + \Delta$ over Z then it is a log terminal model over U. The other direction is clear. $\square$

4. Special finiteness

Lemma 4.1. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of quasi-projective varieties. Let $(X, \Delta = S + A + B)$ be a log smooth pair, where $S = \llcorner \Delta \lrcorner$ is a prime divisor, A is a general ample $\mathbb{Q}$-divisor over U and $B \geq 0$. Let $C = A|_S$ and $\Theta = (\Delta - S)|_S$. Let $\phi\colon X \dashrightarrow Y$ be a birational map over U which does not contract S, let T be the strict transform of S and let $\tau\colon S \dashrightarrow T$ be the induced birational map. Let $\Gamma = \phi_*\Delta$ and define Ψ on T by adjunction,*

$$(K_Y + \Gamma)|_T = K_T + \Psi.$$

If (S,Θ) is terminal and ϕ is a weak log canonical model of $K_X+\Delta$ over U then there is a divisor $C \leq \Xi \leq \Theta$ such that τ is a weak log canonical model of $K_S+\Xi$ over U, where $\tau_*\Xi=\Psi$.

Proof. Let $p\colon W \longrightarrow X$ and $q\colon W \longrightarrow Y$ be log resolutions of (X,Δ) and (Y,Γ), which resolve the indeterminacy of ϕ, where $\Gamma=\phi_*\Delta$. Then we may write

$$K_W+\Delta'=p^*(K_X+\Delta)+E \quad \text{and} \quad K_W+\Gamma'=q^*(K_Y+\Gamma)+F,$$

where $\Delta'\geq 0$ and $E\geq 0$ have no common components, $\Gamma'\geq 0$ and $F\geq 0$ have no common components, $p_*\Delta'=\Delta$, $q_*\Gamma'=\Gamma$, $p_*E=0$ and $q_*F=0$. Since ϕ is $(K_X+\Delta)$-non-positive, we have $F-E+\Delta'-\Gamma'\geq 0$ and so

$$F\geq E \qquad \text{and} \qquad \Gamma'\leq\Delta'.$$

If R is the strict transform of S on W then there are two birational morphisms $f=p|_R\colon R\longrightarrow S$ and $g=q|_R\colon R\longrightarrow T$. Since (X,Δ) is purely log terminal, (Y,Γ) is purely log terminal. In particular (T,Ψ) is kawamata log terminal. It follows that

$$K_R+\Theta'=f^*(K_S+\Theta)+E' \quad \text{and} \quad K_R+\Psi'=g^*(K_T+\Psi)+F',$$

where $\Theta'=(\Delta'-R)|_R$, $E'=E|_R$, $\Psi'=(\Gamma'-R)|_R$ and $F'=F|_R$. Moreover, every component of F' is g-exceptional, by (4) of (3.4.1). As p and q are log resolutions

$$F'\geq E' \qquad \text{and} \qquad \Psi'\leq\Theta'.$$

Suppose that B is a prime divisor on R which is f-exceptional but not g-exceptional. Then B is not a component of F', and so it is not a component of E'. But then the log discrepancy of B with respect to (S,Θ) is at most one, which contradicts the fact that (S,Θ) is terminal. Thus τ is a birational contraction.

As $g_*E'=0$,

$$\tau_*\Theta=g_*\Theta'\geq g_*\Psi'=\Psi.$$

Let $\Xi\leq\Theta$ be the biggest divisor on S such that $\tau_*\Xi=\Psi$. In other words, if a component of Θ is not τ-exceptional, we replace its coefficient by the corresponding coefficient of Ψ and if a component of Θ is τ-exceptional, then we do not change its coefficient. As A is a general ample $\mathbb{Q}$-divisor over U, C is not contained in the locus where ϕ is not an isomorphism. It follows that $C\leq\Xi$. We have

$$f^*(K_S+\Xi)=g^*(K_T+\Psi)+L,$$

where $g_*L=0$. Contrast this with

$$f^*(K_S+\Theta)=g^*(K_T+\Psi)+M.$$

We have already seen that $M = F' - E' + \Theta' - \Psi' \geq 0$. By definition of Ξ, f_*L and f_*M agree on the τ-exceptional divisors. Since f_*L is τ-exceptional, it follows that $f_*L \geq 0$. But then (3.6.3) implies that $L \geq 0$ and so τ is a weak log canonical model of $K_S + \Xi$ over U. □

Lemma 4.2. *Let $X \longrightarrow U$ and $Y \longrightarrow U$ be two projective morphisms of normal quasi-projective varieties, where X and Y are $\mathbb{Q}$-factorial. Let $f\colon X \dashrightarrow Y$ be a small birational map over U. Let S be a prime divisor on X, let T be its strict transform on Y, let $g\colon S \dashrightarrow T$ be the induced birational map and let H be an ample $\mathbb{R}$-divisor on Y over U. Let D be the strict transform of H in X.*

If g is an isomorphism and $D|_S = g^*(H|_T)$ then f is an isomorphism in a neighbourhood of S and T.

Proof. If $p\colon W \longrightarrow X$ is the normalisation of the graph of f and $q\colon W \longrightarrow Y$ is the induced birational morphism then we may write

$$p^*D = q^*H + E,$$

where E is p-exceptional. Since f is small the exceptional divisors of p and q are the same. Since X and Y are $\mathbb{Q}$-factorial the exceptional locus of p and q are divisors by (3) of (3.6.2). Notice also that if F is a p-exceptional divisor then it is covered by a family of p-exceptional curves which are not contracted by q. Thus the image of F is contained in the indeterminancy locus of f and also in the support of E by (2) of (3.6.2). Putting all of this together, the support of E is equal to the inverse image of the indeterminancy locus of f and of f^{-1}.

Let R be the strict transform of S in W and let $p_1 = p|_R\colon R \longrightarrow S$ and $q_1 = q|_R\colon R \longrightarrow T$. Since

$$p_1^*(D|_S) = (p^*D)|_R = (q^*H + E)|_R = q_1^*(H|_T) + E|_R,$$

we have $E|_R = 0$. Thus S does not intersect the indeterminancy locus of f and T does not intersect the indeterminancy locus of f^{-1} and so f is an isomorphism in neighbourhood of S and T. □

Lemma 4.3. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties. Let (X, Δ_i) be two $i = 1$, 2, purely log terminal pairs, where $S = \llcorner \Delta_i \lrcorner$ is a prime divisor, which is independent of i. Let $\phi_i\colon X \dashrightarrow Y_i$ be $\mathbb{Q}$-factorial log canonical models of $K_X + \Delta_i$ over U, where the ϕ_i are birational and do not contract S for $i = 1$, 2. Let T_i be the strict transform of S and let $\tau_i\colon S \dashrightarrow T_i$ be the induced birational maps. Let Φ_i be the different*

$$(K_{Y_i} + T_i)|_{T_i} = K_{T_i} + \Phi_i.$$

If

(1) *the induced birational map* $\chi\colon Y_1 \dashrightarrow Y_2$ *is small,*
(2) *the induced birational map* $\sigma\colon T_1 \dashrightarrow T_2$ *is an isomorphism,*
(3) $\sigma^*\Phi_2 = \Phi_1$, *and*
(4) *for every component* B *of the support of* $(\Delta_2 - S)$, *we have*

$$(\phi_{1*}B)|_{T_1} = \sigma^*((\phi_{2*}B)|_{T_2}),$$

then χ *is an isomorphism in a neighbourhood of* T_1 *and* T_2.

Proof. Let $\Gamma_i = \phi_{i*}\Delta_i$ and define Ψ_i by adjunction,

$$(K_{Y_i} + \Gamma_i)|_{T_i} = K_{T_i} + \Psi_i.$$

We have

$$\begin{aligned} K_{T_i} + \Psi_i &= (K_{Y_i} + T_i + \Gamma_i - T_i)|_{T_i} \\ &= K_{T_i} + \Phi_i + (\Gamma_i - T_i)|_{T_i}, \end{aligned}$$

so that $\Psi_i = \Phi_i + (\phi_{i*}(\Delta_i - S))|_{T_i}$. Conditions (3) and (4) then imply that

$$\sigma^*\Psi_2 = \sigma^*(\Phi_2 + (\phi_{2*}(\Delta_2 - S))|_{T_2}) = \Phi_1 + (\phi_{1*}(\Delta_2 - S))|_{T_1}.$$

It follows that

$$\begin{aligned} (\chi^*(K_{Y_2} + \Gamma_2))|_{T_1} &= (\chi^*\phi_{2*}(K_X + \Delta_2))|_{T_1} \\ &= (\phi_{1*}(K_X + \Delta_2))|_{T_1} \\ &= (K_{Y_1} + T_1 + \phi_{1*}(\Delta_2 - S))|_{T_1} \\ &= K_{T_1} + \Phi_1 + (\phi_{1*}(\Delta_2 - S))|_{T_1} \\ &= \sigma^*(K_{T_2} + \Psi_2). \end{aligned}$$

Thus (4.2) implies that χ is an isomorphism in a neighbourhood of T_1 and T_2. □

Lemma 4.4. *Theorem* E_{n-1} *implies Theorem* B_n.

Proof. Suppose not. Then there would be an infinite sequence of $\mathbb{Q}$-factorial weak log canonical models over U, $\phi_i\colon X \dashrightarrow Y_i$ for $K_X + \Delta_i$ where $\Delta_i \in \mathcal{L}_{S+A}(V)$, which only contract components of $\mathfrak{E}$ and which do not contract every component of S, such that if the induced birational map $f_{ij}\colon Y_i \dashrightarrow Y_j$ is an isomorphism in a neighbourhood of the strict transforms S_i and S_j of S then $i = j$. Since $\mathfrak{E}$ is finite, possibly passing to a subsequence, we may assume that f_{ij} is small. Possibly passing to a further subsequence, we may also assume that there is a fixed component T of S such that ϕ_i does not contract T and f_{ij} is not an isomorphism in a neighbourhood of the strict transforms of T. Replacing S by T, we may therefore assume that S is irreducible.

Pick $H_1, H_2, \dots, H_k$ general ample over U $\mathbb{Q}$-divisors which span $\mathrm{WDiv}_{\mathbb{R}}(X)$ modulo numerical equivalence over U and let $H = H_1 + H_2 + \cdots + H_k$ be their sum. We may replace V by the subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$ generated by V and $H_1, H_2, \dots, H_k$. Passing to a subsequence, we may assume that $\lim \Delta_i = \Delta_\infty \in \mathcal{L}_{S+A}(V)$. By (3.7.4) we may assume that $K_X + \Delta_\infty$ is purely log terminal and Δ_∞ is in the interior of $\mathcal{L}_{S+A}(V)$. Possibly passing to a subsequence we may therefore assume that Δ_i and Δ_∞ have the same support. We may therefore assume that $K_X + \Delta_i$ is purely log terminal and Δ_i contains the support of H. By (3.6.12) we may therefore assume that ϕ_i is the ample model of $K_X + \Delta_i$ over U. In particular $\Delta_i = \Delta_j$ implies $i = j$.

Pick $\Delta \in \mathcal{L}_{S+A}(V)$ such that $K_X + \Delta$ is purely log terminal and $\Delta_i \leq \Delta$ for all $i > 0$. Let $f\colon Y \longrightarrow X$ be a log resolution of (X, Δ). Then we may write

$$K_Y + \Gamma = f^*(K_X + \Delta) + E,$$

where $\Gamma \geq 0$ and $E \geq 0$ have no common components, $f_*\Gamma = \Delta$ and $f_*E = 0$. Let T be the strict transform of S. Possibly blowing up more, we may assume that (T, Θ) is terminal, where $\Theta = (\Gamma - T)|_T$. We may find $F \geq 0$ an exceptional $\mathbb{Q}$-divisor so that $f^*A - F$ is ample over U, $(Y, \Gamma + F)$ is purely log terminal and $(T, \Theta + F|_T)$ is terminal. Let $A' \sim_{\mathbb{Q},U} f^*A - F$ be a general ample $\mathbb{Q}$-divisor over U. For every i, we may write

$$K_Y + \Gamma_i = f^*(K_X + \Delta_i) + E_i,$$

where $\Gamma_i \geq 0$ and $E_i \geq 0$ have no common components, $f_*\Gamma_i = \Delta_i$ and $f_*E_i = 0$. Let

$$\Gamma_i' = \Gamma_i - f^*A + F + A' \sim_{\mathbb{Q},U} \Gamma_i.$$

Then $K_Y + \Gamma_i'$ is purely log terminal and (3.6.9) implies that $f \circ \phi_i$ is both a weak log canonical model, and the ample model, of $K_Y + \Gamma_i$ over U.

Replacing X by Y we may therefore assume that (X, Δ) is log smooth and (S, Θ) is terminal, where $\Theta = (\Delta - S)|_S$. Let

$$C = A|_S \leq \Xi_i \leq (\Delta_i - S)|_S \leq \Theta,$$

be the divisors whose existence is guaranteed by (4.1) so that $\tau_i = \phi_i|_S\colon S \dashrightarrow S_i$ is a weak log canonical model (and the ample model) of $K_S + \Xi_i$ over U, where S_i is the strict transform of S. As we are assuming Theorem E_{n-1}, possibly passing to a subsequence we may assume that the restriction $g_{ij} = f_{ij}|_{S_i}\colon S_i \dashrightarrow S_j$ is an isomorphism.

Since Y_i is $\mathbb{Q}$-factorial, we may define two $\mathbb{R}$-divisors on S_i by adjunction,

$$(K_{Y_i}+\Gamma_i)|_{S_i} = K_{S_i}+\Psi_i \qquad \text{and} \qquad (K_{Y_i}+S_i)|_{S_i} = K_{S_i}+\Phi_i,$$

where $\Gamma_i = \phi_{i*}\Delta_i$. Then

$$0 \leq \Phi_i \leq \Psi_i = \tau_{i*}\Xi_i \leq \tau_{i*}\Theta.$$

As the coefficients of Φ_i have the form $\frac{(r-1)}{r}$ for some integer $r>0$, by (3) of (3.4.1), and the coefficients of Θ (and hence also of $\tau_{i*}\Theta$) are all less than one, it follows that there are only finitely many possibilities for Φ_i. Let B be a component of the support of $\Delta_i - S$ and let $G = (\phi_{i*}(B))|_{S_i}$. By (3) of (3.4.1), the coefficients of G are integer multiples of $1/r$. Pick a real number $\beta>0$ such that the coefficients of B in $\Delta_i - S$ are at least β. As

$$\beta G \leq \phi_{i*}(\Delta_i - S)|_{S_i} \leq \tau_{i*}\Theta,$$

there are only finitely many possibilities for G. Possibly passing to a subsequence, we may assume $\Phi_i = f^*_{ij}\Phi_j$ and that if B is a component of the support of $\Delta_j - S$ then $(\phi_{i*}(B))|_{S_i} = g^*_{ij}((\phi_{j*}B)|_{S_j})$. (4.3) implies that f_{ij} is an isomorphism in a neighbourhood of S_i and S_j, a contradiction. □

5. Log terminal models

Lemma 5.1. *Assume Theorem B_n.*

Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties, where X is $\mathbb{Q}$-factorial of dimension n. Suppose that

$$K_X+\Delta+C = K_X+S+A+B+C,$$

is divisorially log terminal and nef over U, where S is a sum of prime divisors, $\mathbf{B}_+(A/U)$ does not contain any non kawamata log terminal centres of $(X,\Delta+C)$ and $B\geq 0$, $C\geq 0$.

Then any sequence of flips and divisorial contractions for the $(K_X+\Delta)$-MMP over U with scaling of C, which does not contract S, is eventually disjoint from S.

Proof. We may assume that S is irreducible and by (3.7.5) we may assume that A is a general ample $\mathbb{Q}$-divisor over U. Let $f_i\colon X_i \dashrightarrow X_{i+1}$ be a sequence of flips and divisorial contractions over U, starting with $X_1 := X$, for the $(K_X+\Delta)$-MMP with scaling of C.

Let $\mathfrak{E}$ be the set of prime divisors in X which are contracted by any of the induced birational maps $\phi_i\colon X \dashrightarrow X_i$. Then the cardinality of $\mathfrak{E}$ is less than the relative Picard number of X over U. In particular $\mathfrak{E}$ is finite.

By assumption there is a non-increasing sequence of real numbers $\lambda_1, \lambda_2, \dots \in [0,1]$, such that $K_{X_i} + \Delta_i + \lambda_i C_i$ is nef over U, where Δ_i is the strict transform of Δ and C_i is the strict transform of C. Further the birational map f_i is $(K_{X_i} + \Delta_i)$-negative and $(K_{X_i} + \Delta_i + \lambda_i C_i)$-trivial, so that f_i is $(K_{X_i} + \Delta_i + \lambda C_i)$-non-positive if and only if $\lambda \leq \lambda_i$. By induction, the birational map ϕ_i is therefore $(K_X + \Delta + \lambda_i C)$-non-positive. In particular ϕ_i is a $\mathbb{Q}$-factorial weak log canonical model over U of $(X, \Delta + \lambda_i C)$.

Let V be the smallest affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$ containing $\Delta - S - A$ and C, which is defined over the rationals. As $\Delta + \lambda_i C \in \mathcal{L}_{S+A}(V)$, Theorem B_n implies that there is an index k and infinitely many indices l such that the induced birational map $X_k \dashrightarrow X_l$ is an isomorphism in a neighbourhood of S_k and S_l. But then (3.10.12) implies that f_i is an isomorphism in a neighbourhood of S_i for all $i \geq k$. $\square$

We use (5.1) to run a special MMP:

Lemma 5.2. *Assume Theorem A_n and Theorem B_n.*

Let $\pi \colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties, where X is $\mathbb{Q}$-factorial of dimension n. Suppose that $(X, \Delta + C = S + A + B + C)$ is a divisorially log terminal pair, such that $\llcorner \Delta \lrcorner = S$, $A \geq 0$ is big over U, $\mathbf{B}_+(A/U)$ does not contain any non kawamata log terminal centres of $(X, \Delta + C)$, and $B \geq 0$, $C \geq 0$. Suppose that there is an $\mathbb{R}$-divisor $D \geq 0$ whose support is contained in S and a real number $\alpha \geq 0$, such that

$$(*) \qquad K_X + \Delta \sim_{\mathbb{R},U} D + \alpha C.$$

If $K_X + \Delta + C$ is nef over U, then there is a log terminal model $\phi \colon X \dashrightarrow Y$ for $K_X + \Delta$ over U, where $\mathbf{B}_+(\phi_* A/U)$ does not contain any non kawamata log terminal centres of $(Y, \Gamma = \phi_* \Delta)$.

Proof. By (3.10.9) and (3.10.11) we may run the $(K_X + \Delta)$-MMP with scaling of C over U, and this will preserve the condition that $\mathbf{B}_+(A/U)$ does not contain any non kawamata log terminal centres of (X, Δ). Pick $t \in [0,1]$ minimal such that $K_X + \Delta + tC$ is nef over U. If $t = 0$ we are done. Otherwise we may find a $(K_X + \Delta)$-negative extremal ray R over U, such that $(K_X + \Delta + tC) \cdot R = 0$. Let $f \colon X \longrightarrow Z$ be the associated contraction over U. As $t > 0$, $C \cdot R > 0$ and so $D \cdot R < 0$. In particular f is always birational.

If f is divisorial, then we can replace X, S, A, B, C and D by their images in Z. Note that $(*)$ continues to hold.

Otherwise f is small. As $D \cdot R < 0$, R is spanned by a curve Σ which is contained in a component T of S, where $T \cdot \Sigma < 0$. Note that

$K_X+S+A+B-\epsilon(S-T)$ is purely log terminal for any $\epsilon \in (0,1)$, and so f is a pl-flip. As we are assuming Theorem A_n, the flip $f' \colon X' \longrightarrow Z$ of $f \colon X \longrightarrow Z$ exists. Again, if we replace X, S, A, B, C and D by their images in X', then $(*)$ continues to hold.

On the other hand this MMP is certainly not an isomorphism in a neighbourhood of S and so the MMP terminates by (5.1). $\square$

Definition 5.3. *Let $\pi \colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties. Let $(X, \Delta = A + B)$ be a $\mathbb{Q}$-factorial divisorially log terminal pair and let D be an $\mathbb{R}$-divisor, where $A \geq 0$, $B \geq 0$ and $D \geq 0$. A* ***neutral model*** *over U for (X, Δ), with respect to A and D, is any birational map $f \colon X \dashrightarrow Y$ over U, such that*

- *f is a birational contraction,*
- *the only divisors contracted by f are components of D,*
- *Y is $\mathbb{Q}$-factorial and projective over U,*
- *$\mathbf{B}_+(f_*A/U)$ does not contain any non kawamata log terminal centres of $(Y, \Gamma = f_*\Delta)$, and*
- *$K_Y + \Gamma$ is divisorially log terminal and nef over U.*

Lemma 5.4. *Assume Theorem A_n and Theorem B_n.*

Let $\pi \colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties, where X has dimension n. Let $(X, \Delta = A+B)$ be a divisorially log terminal log pair and let D be an $\mathbb{R}$-divisor, where $A \geq 0$ is big over U, $B \geq 0$ and $D \geq 0$ and D and A have no common components.

If

(i) *$K_X + \Delta \sim_{\mathbb{R},U} D$,*
(ii) *(X, G) is log smooth, where G is the support of $\Delta + D$, and*
(iii) *$\mathbf{B}_+(A/U)$ does not contain any non kawamata log terminal centres of (X, G)*

then (X, Δ) has a neutral model over U, with respect to A and D.

Proof. We may write $D = D_1 + D_2$, where every component of D_1 is a component of $\llcorner \Delta \lrcorner$ and no component of D_2 is a component of $\llcorner \Delta \lrcorner$. We proceed by induction on the number of components of D_2.

Suppose $D_2 = 0$. If H is any general ample $\mathbb{Q}$-divisor over U, which is sufficiently ample then $K_X + \Delta + H$ is divisorially log terminal and ample over U. As the support of D is contained in $\llcorner \Delta \lrcorner$, (5.2) implies that (X, Δ) has a neutral model $f \colon X \dashrightarrow Y$ over U, with respect to A and D.

Now suppose that $D_2 \neq 0$. Let

$$\lambda = \sup\{\, t \geq 0 \mid (X, \Delta + tD_2) \text{ is log canonical} \,\},$$

be the log canonical threshold of D_2. Then $\lambda > 0$ and $(X, \Theta = \Delta + \lambda D_2)$ is divisorially log terminal and log smooth, $K_X + \Theta \sim_{\mathbb{R},U} D + \lambda D_2$ and the number of components of $D + \lambda D_2$ that are not components of $\llcorner \Theta \lrcorner$ is smaller than the number of components of D_2. By induction there is a neutral model $f\colon X \dashrightarrow Y$ over U for (X, Θ), with respect to A and D.

Now

$$\begin{aligned} K_Y + f_*\Delta &\sim_{\mathbb{R},U} f_*D_1 + f_*D_2, \\ K_Y + f_*\Theta &= K_Y + f_*\Delta + \lambda f_*D_2, \end{aligned}$$

where $K_Y + f_*\Theta$ is divisorially log terminal and nef over U, and the support of f_*D_1 is contained in $\llcorner f_*\Delta \lrcorner$. Since $\mathbf{B}_+(f_*A/U)$ does not contain any non kawamata log terminal centres of $(Y, f_*\Theta)$, (5.2) implies that $(Y, f_*\Delta)$ has a neutral model $g\colon Y \dashrightarrow Z$ over U, with respect to f_*A and f_*D. The composition $g \circ f\colon X \dashrightarrow Z$ is then a neutral model over U for (X, Δ), with respect to A and D. □

Lemma 5.5. *Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties. Let $(X, \Delta = A + B)$ be a $\mathbb{Q}$-factorial divisorially log terminal log pair and let D be an $\mathbb{R}$-divisor, where $A \geq 0$ is big over U, $B \geq 0$ and $D \geq 0$.*

If every component of D is either semiample over U or a component of $\mathbf{B}((K_X + \Delta)/U)$ and $f\colon X \dashrightarrow Y$ is a neutral model over U for (X, Δ), with respect to A and D, then f is a log terminal model for (X, Δ) over U.

Proof. By hypothesis the only divisors contracted by f are components of $\mathbf{B}((K_X + \Delta)/U)$. Since the question is local over U, we may assume that U is affine. Since $\mathbf{B}_+(f_*A/U)$ does not contain any non kawamata log terminal centres of $(Y, \Gamma = f_*\Delta)$, (3.7.3) implies that we may find $K_Y + \Gamma' \sim_{\mathbb{R},U} K_Y + \Gamma$ where $K_Y + \Gamma'$ is kawamata log terminal and Γ' is big over U. (3.9.2) implies that $K_Y + \Gamma$ is semiample over U.

If $p\colon W \longrightarrow X$ and $q\colon W \longrightarrow Y$ resolve the indeterminacy of f then we may write

$$p^*(K_X + \Delta) + E = q^*(K_Y + \Gamma) + F,$$

where $E \geq 0$ and $F \geq 0$ have no common components, and both E and F are exceptional for q.

As $K_Y + \Gamma$ is semiample over U, $\mathbf{B}((q^*(K_Y + \Gamma) + F)/U)$ and F have the same support. On the other hand, every component of E is a component of $\mathbf{B}((p^*(K_X + \Delta) + E)/U)$. Thus $E = 0$ and any divisor contracted by f is contained in the support of F, and so f is a log terminal model of (X, Δ) over U. □

Lemma 5.6. *Theorem A_n and Theorem B_n imply Theorem C_n.*

Proof. By (3.7.5) we may assume that $\Delta = A+B$, where A is a general ample $\mathbb{Q}$-divisor over U and $B \geq 0$. By (3.5.4) we may assume that $D = M + F$, where every component of F is a component of $\mathbf{B}(D/U)$ and there is a positive integer m such that if L is a component of M then mL is mobile.

Pick a log resolution $f\colon Y \longrightarrow X$ of the support of D and Δ, which resolves the base locus of each linear system $|mL|$, for every component L of M. If Φ is the divisor defined in (3.6.11) then every component of the exceptional locus belongs to $\mathbf{B}((K_Y+\Phi)/U)$ and replacing Φ by an $\mathbb{R}$-linearly equivalent divisor, we may assume that Φ contains an ample divisor over U. In particular, replacing $m\pi^*L$ by a general element of the linear system $|m\pi^*L|$, we may assume that $K_Y + \Phi \sim_{\mathbb{R},U} N + G$, where every component of N is semiample, every component of G is a component of $\mathbf{B}((K_Y + \Phi)/U)$, and $(Y, \Phi + N + G)$ is log smooth. By (3.6.11), we may replace X by Y and the result follows by (5.4) and (5.5). $\square$

6. Non-vanishing

We follow the general lines of the proof of the non-vanishing theorem, see for example Chapter 3, §5 of [22]. In particular there are two cases:

Lemma 6.1. *Assume Theorem C_n. Let (X,Δ) be a projective, log smooth pair of dimension n, where $\llcorner\Delta\lrcorner = 0$, such that $K_X + \Delta$ is pseudo-effective and $\Delta - A \geq 0$ for an ample $\mathbb{Q}$-divisor A. Suppose that for every positive integer k such that kA is integral,*

$$h^0(X, \mathcal{O}_X(\llcorner mk(K_X + \Delta)\lrcorner + kA)),$$

is a bounded function of m.

Then there is an $\mathbb{R}$-divisor D such that $K_X + \Delta \sim_{\mathbb{R}} D \geq 0$.

Proof. By (3.3.2) it follows that $K_X + \Delta$ is numerically equivalent to $N_\sigma(K_X+\Delta)$. Since $N_\sigma(K_X+\Delta)-(K_X+\Delta)$ is numerically trivial and ampleness is a numerical condition, it follows that

$$A'' = A + N_\sigma(K_X + \Delta) - (K_X + \Delta),$$

is ample and numerically equivalent to A. Thus, since A'' is $\mathbb{R}$-linearly equivalent to a positive linear combination of ample $\mathbb{Q}$-divisors, there exists $0 \leq A' \sim_{\mathbb{R}} A''$ such that

$$K_X + \Delta' = K_X + A' + (\Delta - A),$$

is kawamata log terminal and numerically equivalent to $K_X + \Delta$, and

$$K_X + \Delta' \sim_{\mathbb{R}} N_\sigma(K_X + \Delta) \geq 0.$$

Thus by Theorem C_n, $K_X + \Delta'$ has a log terminal model $\phi\colon X \dashrightarrow Y$, which, by (3.6.9), is also a log terminal model for $K_X + \Delta$. Replacing (X, Δ) by (Y, Γ) we may therefore assume that $K_X + \Delta$ is nef and the result follows by the base point free theorem, cf. (3.9.2). $\square$

Lemma 6.2. *Let $(X, \Delta = A + B)$ be a projective, log smooth pair where A is a general ample $\mathbb{Q}$-divisor and $\llcorner B \lrcorner = 0$. Suppose that there is a positive integer k such that kA is integral and*

$$h^0(X, \mathcal{O}_X(\llcorner mk(K_X + \Delta) \lrcorner + kA)),$$

is an unbounded function of m.

Then we may find a projective, log smooth pair (Y, Γ) and a general ample $\mathbb{Q}$-divisor C on Y, where

- *Y is birational to X,*
- *$\Gamma - C \geq 0$,*
- *$T = \llcorner \Gamma \lrcorner$ is an irreducible divisor, and*
- *Γ and $N_\sigma(K_Y + \Gamma)$ have no common components.*

Moreover the pair (Y, Γ) has the property that $K_X + \Delta \sim_{\mathbb{R}} D \geq 0$ for some $\mathbb{R}$-divisor D if and only if $K_Y + \Gamma \sim_{\mathbb{R}} G \geq 0$ for some $\mathbb{R}$-divisor G.

Proof. Pick m large enough so that

$$h^0(X, \mathcal{O}_X(\llcorner mk(K_X + \Delta) \lrcorner + kA)) > \binom{kn + n}{n},$$

where n is the dimension of X. By standard arguments, given any point $x \in X$, we may find a divisor $H' \geq 0$ which is $\mathbb{R}$-linearly equivalent to

$$\llcorner mk(K_X + \Delta) \lrcorner + kA,$$

of multiplicity greater than kn at x. In particular, we may find an $\mathbb{R}$-Cartier divisor

$$0 \leq H \sim_{\mathbb{R}} m(K_X + \Delta) + A,$$

of multiplicity greater than n at x. Given $t \in [0, m]$, consider

$$\begin{aligned}(t + 1)(K_X + \Delta) &= K_X + \frac{m - t}{m} A + B + t(K_X + \Delta + \frac{1}{m} A) \\ &\sim_{\mathbb{R}} K_X + \frac{m - t}{m} A + B + \frac{t}{m} H \\ &= K_X + \Delta_t.\end{aligned}$$

Fix $0 < \epsilon \ll 1$, let $A' = (\epsilon/m)A$ and $u = m - \epsilon$. We have:

(1) $K_X + \Delta_0$ is kawamata log terminal,
(2) $\Delta_t \geq A'$, for any $t \in [0, u]$ and
(3) the non kawamata log terminal locus of (X, Δ_u) contains a very general point x of X.

Let $\pi\colon Y \longrightarrow X$ be a log resolution of $(X, \Delta + H)$. We may write

$$K_Y + \Psi_t = \pi^*(K_X + \Delta_t) + E_t,$$

where $E_t \geq 0$ and $\Psi_t \geq 0$ have no common components, $\pi_*\Psi_t = \Delta_t$ and E_t is exceptional. Pick an exceptional divisor $F \geq 0$ such that $\pi^*A' - F$ is ample and let $C \sim_{\mathbb{Q}} \pi^*A' - F$ be a general ample $\mathbb{Q}$-divisor. For any $t \in [0, u]$, let

$$\Phi_t = \Psi_t - \pi^*A' + C + F \sim_{\mathbb{R}} \Psi_t \qquad \text{and} \qquad \Gamma_t = \Phi_t - \Phi_t \wedge N_\sigma(K_Y + \Phi_t).$$

Then properties (1-3) above become

(1) $K_Y + \Gamma_0$ is kawamata log terminal,
(2) $\Gamma_t \geq C$, for any $t \in [0, u]$, and
(3) (Y, Γ_u) is not kawamata log terminal.

Moreover

(4) (Y, Γ_t) is log smooth, for any $t \in [0, u]$, and
(5) Γ_t and $N_\sigma(K_Y + \Gamma_t)$ do not have any common components.

Let

$$s = \sup\{\, t \in [0, u] \mid K_Y + \Gamma_t \text{ is log canonical} \,\}.$$

Note that

$$\begin{aligned} N_\sigma(K_Y + \Phi_t) &= N_\sigma(K_Y + \Psi_t) \\ &= N_\sigma(\pi^*(K_X + \Delta_t)) + E_t \\ &= N_\sigma((t+1)\pi^*(K_X + \Delta)) + E_t \\ &= (t+1)N_\sigma(\pi^*(K_X + \Delta)) + E_t. \end{aligned}$$

Thus $K_Y + \Gamma_t$ is a continuous piecewise affine linear function of t. Setting $\Gamma = \Gamma_s$, we may write

$$\Gamma = T + C + B',$$

where $T = \llcorner \Gamma \lrcorner \neq 0$, C is ample and $B' \geq 0$. Possibly perturbing Γ, we may assume that T is irreducible, so that $K_Y + \Gamma$ is purely log terminal. $\square$

We will need the following consequence of Kawamata-Viehweg vanishing:

Lemma 6.3. *Let $(X, \Delta = S + A + B)$ be a $\mathbb{Q}$-factorial projective purely log terminal pair and let $m > 1$ be an integer. Suppose that*

(1) $S = \llcorner \Delta \lrcorner$ *is irreducible,*

(2) *$m(K_X+\Delta)$ is integral,*
(3) *$m(K_X+\Delta)$ is Cartier in a neighbourhood of S,*
(4) *$h^0(S,\mathcal{O}_S(m(K_X+\Delta)))>0$,*
(5) *K_X+G+B is kawamata log terminal, where $G\geq 0$,*
(6) *$A\sim_{\mathbb{Q}}(m-1)tH+G$ for some tH, and*
(7) *$K_X+\Delta+tH$ is big and nef.*

Then $h^0(X,\mathcal{O}_X(m(K_X+\Delta)))>0$.

Proof. Considering the long exact sequence associated to the restriction exact sequence,

$$0\longrightarrow \mathcal{O}_X(m(K_X+\Delta)-S)\longrightarrow \mathcal{O}_X(m(K_X+\Delta))\longrightarrow \mathcal{O}_S(m(K_X+\Delta))\longrightarrow 0,$$

it suffices to observe that

$$H^1(X,\mathcal{O}_X(m(K_X+\Delta)-S))=0,$$

by Kawamata-Viehweg vanishing, since

$$\begin{aligned} m(K_X+\Delta)-S &= (m-1)(K_X+\Delta)+K_X+A+B \\ &\sim_{\mathbb{Q}} K_X+G+B+(m-1)(K_X+\Delta+tH), \end{aligned}$$

and $K_X+\Delta+tH$ is big and nef. □

Lemma 6.4. *Let X be a normal projective variety, let S be a prime divisor and let D_i be three, $i=1,\ 2,\ 3$, $\mathbb{R}$-Cartier divisors on X. Suppose that $f_i\colon X \dashrightarrow Z_i$ are ample models of D_i, $i=1,\ 2,\ 3$, where f_i is birational and f_i does not contract S. Suppose that Z_i is $\mathbb{Q}$-factorial, $i=1,\ 2$, and the induced birational map $Z_1\dashrightarrow Z_2$ is an isomorphism in a neighbourhood of the strict transforms of S.*

If D_3 is a positive linear combination of D_1 and D_2 then the induced birational map $Z_i\dashrightarrow Z_3$ is an isomorphism in a neighbourhood of the strict transforms of S.

Proof. By assumption $D_3=\lambda_1D_1+\lambda_2D_2$, where $\lambda_i>0$. If $g_i\colon Z\longrightarrow Z_i$ is the normalisation of the graph of $Z_1\dashrightarrow Z_2$ then g_i is by assumption an isomorphism in a neighbourhood of the strict transforms of S, for $i=1,2$. Let $f\colon X\dashrightarrow Z$ be the induced birational map. Let $g\colon W\longrightarrow X$ resolve the indeterminacy of f and f_3. Replacing X by W and D_i by g^*D_i, we may assume that f and f_3 are morphisms.

Let $H_i=f_{i*}D_i$, $i=1,\ 2,\ 3$. As the ample model is a semiample model, by (4) of (3.6.6), $E_i=D_i-f_i^*H_i\geq 0$ is f_i-exceptional. As H_i is ample, $i=1$ and 2, $H=\lambda_1g_1^*H_1+\lambda_2g_2^*H_2$ is semiample. Let $\Sigma\subset Z$ be a curve. As Σ is not contracted by both g_1 and g_2, $g_i^*H_i\cdot\Sigma\geq 0$, with equality for at most one i. Thus H is ample. Now

$$\begin{aligned} f_3^* H_3 + E_3 &= D_3 \\ &= \lambda_1 D_1 + \lambda_2 D_2 \\ &= \lambda_1 f_1^* H_1 + \lambda_2 f_2^* H_2 + \lambda_1 E_1 + \lambda_2 E_2 \\ &= f^*(\lambda_1 g_1^* H_1 + \lambda_2 g_2^* H_2) + \lambda_1 E_1 + \lambda_2 E_2 \\ &= f^* H + \lambda_1 E_1 + \lambda_2 E_2. \end{aligned}$$

Note that $E_3 \leq \lambda_1 E_1 + \lambda_2 E_2$, as $f^* H$ has no stable base locus. Let $T = f(S)$. We may write

$$f^* H = f_3^* H_3 + E_3 - (\lambda_1 E_1 + \lambda_2 E_2).$$

Since $\lambda_1 E_1 + \lambda_2 E_2$ is f-exceptional in a neighbourhood of $f^{-1}(T)$, $E_3 = \lambda_1 E_1 + \lambda_2 E_2$ in a neighbourhood of $f^{-1}(T)$, by (3.6.2). But then f and f_3 contract precisely the same curves in the same neighbourhood and so Z is isomorphic to Z_3 in a neighbourhood of the strict transforms of S. □

Lemma 6.5. *Assume Theorem B_n and Theorem C_n.*

Let $(X, \Delta_0 = S+A+B_0)$ be a log smooth projective pair of dimension n, where $A \geq 0$ is a general ample $\mathbb{Q}$-divisor, $\llcorner \Delta_0 \lrcorner = S$ is a prime divisor and $B_0 \geq 0$. Suppose that $K_X + \Delta_0$ is pseudo-effective and S is not a component of $N_\sigma(K_X + \Delta_0)$. Let V_0 be a finite dimensional affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$ *containing B_0, which is defined over the rationals.*

Then we may find a general ample $\mathbb{Q}$-divisor $H \geq 0$, a log terminal model $\phi\colon X \dashrightarrow Y$ for $K_X + \Delta_0 + H$ and a positive constant α, such that if $B \in V_0$ and

$$\|B - B_0\| < \alpha t,$$

for some $t \in (0, 1]$, then there is a log terminal model $\psi\colon X \dashrightarrow Z$ of $K_X + \Delta + tH = K_X + S + A + B + tH$, which does not contract S, such that the induced birational map $\chi\colon Y \dashrightarrow Z$ is an isomorphism in a neighbourhood of the strict transforms of S.

Proof. Pick general ample $\mathbb{Q}$-Cartier divisors $H_1, H_2, \dots, H_k$, which span $\mathrm{WDiv}_{\mathbb{R}}(X)$ modulo numerical equivalence, and let V be the affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$ spanned by V_0 and $H_1, H_2, \dots, H_k$. Let $\mathcal{C} \subset \mathcal{L}_{S+A}(V)$ be a convex subset spanning V, which does not contain Δ_0, but whose closure contains Δ_0, such that if $\Delta \in \mathcal{C}$ then $K_X + \Delta$ is purely log terminal, $\Delta - \Delta_0$ is ample and the support of $\Delta - \Delta_0$ contains the support of the sum $H_1 + H_2 + \cdots + H_k$. In particular if $\Delta \in \mathcal{C}$ then $N_\sigma(K_X + \Delta) \leq N_\sigma(K_X + \Delta_0)$. As the coefficients σ_C of N_σ are continuous on the big cone, (3.3.1), possibly replacing $\mathcal{C}$ by a subset, we may assume that if $\Delta \in \mathcal{C}$ then $N_\sigma(K_X+\Delta)$ and $N_\sigma(K_X+\Delta_0)$ share the same support. Moreover, if $\Delta \in \mathcal{C}$ then $K_X+\Delta$ is big and so, as we are assuming Theorem C_n, $K_X+\Delta$ has a log terminal model $\phi\colon X \dashrightarrow Y$, whose exceptional divisors are given by the support of $N_\sigma(K_X+\Delta_0)$. In particular ϕ does not contract S.

Given $\phi\colon X \dashrightarrow Y$, define a subset $\mathcal{S}_\phi \subset \mathcal{C}$ as follows: $\Delta \in \mathcal{S}_\phi$ if and only if there is a log terminal model $\psi\colon X \dashrightarrow Z$ of $K_X+\Delta$ such that

- the induced rational map $\chi\colon Y \dashrightarrow Z$ is an isomorphism in a neighbourhood of the strict transforms of S.

Define a subset $\mathcal{S}'_\phi \subset \mathcal{S}_\phi$ by requiring in addition that

- ϕ is isomorphic to the ample model of $K_X+\Delta$ in a neighbourhood of the strict transforms of S.

As we are assuming Theorem B_n, there are finitely many $1 \leq j \leq l$ birational maps $\phi_j\colon X \dashrightarrow Y_i$ such that

$$\mathcal{C} = \bigcup_{j=1}^{l} \mathcal{S}_{\phi_j}.$$

On the other hand, (3.6.12) implies that

$$\bigcup_{j=1}^{l} \mathcal{S}'_{\phi_j},$$

is a dense open subset of $\mathcal{C}$.

We will now show that the sets $\mathcal{S}'_\phi$ are convex. Suppose that $\Delta_i \in \mathcal{S}'_\phi$, $i = 1, 2$ and Δ is a convex linear combination of Δ_1 and Δ_2. Then $\Delta \in \mathcal{C}$ and so $K_X+\Delta$ has a log terminal model $\psi\colon X \dashrightarrow Z$. By (3) of (3.6.6) there is a birational morphism $h\colon Z \longrightarrow Z'$ to the ample model of $K_X+\Delta$. Let $\psi_i\colon X \dashrightarrow Z_i$ be the ample model of $K_X+\Delta_i$. (6.4) implies that the induced birational map $Z_i \dashrightarrow Z'$ is an isomorphism in a neighbourhood of the strict transforms of S. As Z and Z_i are isomorphic in codimension 1, $Z \longrightarrow Z'$ is small in a neighbourhood of the strict transforms of S. As Z and Z' are $\mathbb{Q}$-factorial in a neighbourhood of the strict transforms of S, the morphism $Z \longrightarrow Z'$ is also an isomorphism in a neighbourhood of the strict transforms of S. Thus $\Delta \in \mathcal{S}'_\phi$ and so $\mathcal{S}'_\phi$ is convex.

Shrinking $\mathcal{C}$, we may therefore assume that $\mathcal{C} = \mathcal{S}'_{\phi_j}$ for some $1 \leq j \leq l$.

Pick $\Delta = S + A + B \in \mathcal{C}$ and pick $H \sim_{\mathbb{Q}} \Delta - \Delta_0$ a general ample $\mathbb{Q}$-divisor. Pick a positive constant α such that if $\|B - B_0\| < \alpha t$ for some $B \in V_0$ and $t \in (0, 1]$ then $S+A+B+t(\Delta-\Delta_0) \in \mathcal{C}$. By (3.6.9), $K_X + S + A + B + t(\Delta - \Delta_0)$ and $K_X + S + A + B + tH$ have the same log terminal models. It follows that α and H have the required properties. $\square$

Lemma 6.6. *Theorem D_{n-1}, Theorem B_n and Theorem C_n imply Theorem D_n.*

Proof. By (3.2.1), it suffices to prove this result for the generic fibre of U. Thus we may assume that U is a point, so that X is a projective variety.

Let $f\colon Y \longrightarrow X$ be a log resolution of (X,Δ). We may write

$$K_Y + \Gamma = f^*(K_X + \Delta) + E,$$

where $\Gamma \geq 0$ and $E \geq 0$ have no common components, $f_*\Gamma = \Delta$ and $f_*E = 0$. If $F \geq 0$ is an $\mathbb{R}$-divisor whose support equals the union of all f-exceptional divisors then $\Gamma + F$ is big. Pick F so that $(X, \Gamma + F)$ is kawamata log terminal. Replacing (X,Δ) by $(Y, \Gamma + F)$ we may therefore assume that (X,Δ) is log smooth. By (3.7.5) we may assume that $\Delta = A + B$, where A is a general ample $\mathbb{Q}$-divisor and $B \geq 0$. By (6.1) and (6.2), we may therefore assume that $\Delta = S + A + B$, where (X,Δ) is log smooth, A is a general ample $\mathbb{Q}$-divisor, and $\llcorner \Delta \lrcorner = S$ is a prime divisor, which is not a component of $N_\sigma(K_X + \Delta)$.

Let V be the subspace of $\operatorname{WDiv}_{\mathbb{R}}(X)$ spanned by the components of B. By (6.5) we may find a constant $\alpha > 0$, a general ample $\mathbb{Q}$-divisor H on X, and a log terminal model $\phi\colon X \dashrightarrow Y$ of $K_X + \Delta + H$ such that if $B' \in V$, $t \in (0,1]$ and $\|B - B'\| < \alpha t$ then there is a log terminal model $\phi'\colon X \dashrightarrow Y'$ of $K_X + S + A + B' + tH$ such that the induced rational map $\chi\colon Y \dashrightarrow Y'$ is an isomorphism in a neighbourhood of the strict transforms of S. Pick $\epsilon > 0$ such that $A - \epsilon H$ is ample.

Let T be the strict transform of S on Y and define Φ_0 on T by adjunction

$$(K_Y + T)|_T = K_T + \Phi_0.$$

Let W be the subspace of $\operatorname{WDiv}_{\mathbb{R}}(T)$ spanned by the components of $(\phi_*B)|_T + \Phi_0$ and let $L\colon V \longrightarrow W$ be the rational affine linear map $L(B') = \Phi_0 + \phi_*B'|_T$. Let $\Gamma = \phi_*\Delta$ and let $C = \phi_*A|_T$. If we define Ψ on T by adjunction

$$(K_Y + \Gamma)|_T = K_T + \Psi,$$

then $\Psi = C + L(B)$. As $K_T + C + L(B) + t\phi_*H|_T$ is nef for any $t > 0$, it follows that $K_T + \Psi$ is nef. Since C is big and $K_T + \Psi$ is kawamata log terminal, (3.7.3) implies that there is a rational affine linear isomorphism $L'\colon \operatorname{WDiv}_{\mathbb{R}}(T) \longrightarrow \operatorname{WDiv}_{\mathbb{R}}(T)$ which preserves $\mathbb{Q}$-linear equivalence, an ample $\mathbb{Q}$-divisor G on T and a rational affine linear subspace W' of $\operatorname{WDiv}_{\mathbb{R}}(T)$ such that $L'(U) \subset \mathcal{L}_G(W')$, where

$U \subset \mathcal{L}_C(W)$ is a neighbourhood of Ψ. $\mathcal{N}_G(W')$ is a rational polytope, by (3.11.1). In particular we may find a rational polytope $\mathcal{C} \subset V$ containing B such that if $B' \in \mathcal{C}$ then $K_X + \Delta' = K_X + S + A + B'$ is purely log terminal and

$$(K_Y + \Gamma')|_T = K_T + \Psi',$$

is nef, where $\Gamma' = \phi_*\Delta'$ and $\Psi' = C + L(B')$. Pick a positive integer k such that if $r(K_Y + \Gamma')$ is integral then $rk(K_Y + \Gamma')$ is Cartier in a neighbourhood of T. By (3.9.2) there is then a constant $m > 0$ such that $mr(K_Y + \Gamma')$ is Cartier in a neighbourhood of T and $mr(K_T + \Psi')$ is base point free.

(3.7.7) implies that there are real numbers $r_i > 0$ with $\sum r_i = 1$, positive integers $p_i > 0$ and $\mathbb{Q}$-divisors $B_i \in \mathcal{C}$ such that

$$p_i(K_X + \Delta_i),$$

is integral, where $\Delta_i = S + A + B_i$,

$$K_X + \Delta = \sum r_i(K_X + \Delta_i),$$

and

$$\|B_i - B\| < \frac{\alpha\epsilon}{m_i},$$

where $m_i = mp_i$. Let $\Psi_i = C + L(B_i)$. By our choice of p_i, $m_i(K_T + \Psi_i)$ is base point free and so

$$h^0(T, \mathcal{O}_T(m_i(K_T + \Psi_i))) > 0.$$

Let $t_i = \epsilon/m_i$. By (6.5) there is a log terminal model $\phi_i \colon X \dashrightarrow Y_i$ of $K_X + \Delta_i + t_iH$, such that the induced birational map $\chi \colon Y \dashrightarrow Y_i$ is an isomorphism in a neighbourhood of T and the strict transform T_i of S. In particular if $\Gamma_i = T_i + \phi_{i*}A + \phi_{i*}B_i$ and $\tau \colon T \longrightarrow T_i$ is the induced isomorphism then

$$\tau^*(K_{Y_i} + \Gamma_i)|_{T_i} = K_T + \Psi_i,$$

and so the pair (Y_i, Γ_i) clearly satisfies conditions (1), (2), (4) and (7) of (6.3). As the induced birational map $Y_i \dashrightarrow Y$ is an isomorphism in a neighbourhood of T_i, (3) of (6.3) also holds. As

$$(m_i - 1)t_i = \left(\frac{m_i - 1}{m_i}\right)\epsilon < \epsilon,$$

$A - (m_i - 1)t_iH$ is ample and so we may pick a general ample $\mathbb{Q}$-divisor L_i such that $L_i \sim_{\mathbb{Q}} A - (m_i - 1)t_iH$. Then $K_X + S + L_i + B_i$ is purely log terminal and as ϕ_i is $(K_X + \Delta_i + t_iH)$-negative and H is ample, ϕ_i is $(K_X + S + L_i + B_i)$-negative. It follows that $K_{Y_i} + T_i + \phi_{i*}(L_i + B_i)$ is purely log terminal so that $K_{Y_i} + \phi_{i*}(L_i + B_i)$ is kawamata log terminal, and conditions (5) and (6) of (6.3) hold. Therefore (6.3) implies that

$$h^0(Y_i, \mathcal{O}_{Y_i}(m_i(K_{Y_i} + \Gamma_i))) > 0.$$

As ϕ_i is $(K_X + \Delta_i + t_i H)$-negative and H is ample, it follows that ϕ_i is $(K_X + \Delta_i)$-negative. But then

$$h^0(X, \mathcal{O}_X(m_i(K_X + \Delta_i))) = h^0(Y, \mathcal{O}_Y(m_i(K_Y + \Gamma_i))) > 0.$$

In particular there is an $\mathbb{R}$-divisor D such that

$$K_X + \Delta = \sum r_i(K_X + \Delta_i) \sim_{\mathbb{R}} D \geq 0. \qquad \square$$

7. Finiteness of models

Lemma 7.1. *Assume Theorem C_n and Theorem D_n.*

Let $\pi \colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties, where X has dimension n. Let V be a finite dimensional affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$, which is defined over the rationals. Fix a general ample $\mathbb{Q}$-divisor A over U. Let $\mathcal{C} \subset \mathcal{L}_A(V)$ be a rational polytope such that if $\Delta \in \mathcal{C}$ then $K_X + \Delta$ is kawamata log terminal.

Then there are finitely many rational maps $\phi_i \colon X \dashrightarrow Y_i$ over U, $1 \leq i \leq k$, with the property that if $\Delta \in \mathcal{C} \cap \mathcal{E}_{A,\pi}(V)$ then there is an index $1 \leq i \leq k$ such that ϕ_i is a log terminal model of $K_X + \Delta$ over U.

Proof. Possibly replacing V_A by the span of $\mathcal{C}$, we may assume that $\mathcal{C}$ spans V_A. We proceed by induction on the dimension of $\mathcal{C}$.

Suppose that $\Delta_0 \in \mathcal{C} \cap \mathcal{E}_{A,\pi}(V)$. As we are assuming Theorem D_n there is an $\mathbb{R}$-divisor $D_0 \geq 0$ such that $K_X + \Delta_0 \sim_{\mathbb{R},U} D_0$ and so, as we are assuming Theorem C_n, there is a log terminal model $\phi \colon X \dashrightarrow Y$ over U for $K_X + \Delta_0$. In particular we may assume that $\dim \mathcal{C} > 0$.

First suppose that there is a divisor $\Delta_0 \in \mathcal{C}$ such that $K_X + \Delta_0 \sim_{\mathbb{R},U} 0$. Pick $\Theta \in \mathcal{C}$, $\Theta \neq \Delta_0$. Then there is a divisor Δ on the boundary of $\mathcal{C}$ such that

$$\Theta - \Delta_0 = \lambda(\Delta - \Delta_0),$$

for some $0 < \lambda \leq 1$. Now

$$\begin{aligned} K_X + \Theta =& \lambda(K_X + \Delta) + (1 - \lambda)(K_X + \Delta_0) \\ \sim_{\mathbb{R},U}& \lambda(K_X + \Delta). \end{aligned}$$

In particular $\Delta \in \mathcal{E}_{A,\pi}(V)$ if and only if $\Theta \in \mathcal{E}_{A,\pi}(V)$ and (3.6.9) implies that $K_X + \Delta$ and $K_X + \Theta$ have the same log terminal models over U. On the other hand the boundary of $\mathcal{C}$ is contained in finitely many affine hyperplanes defined over the rationals, and we are done by induction on the dimension of $\mathcal{C}$.

We now prove the general case. By (3.7.4), we may assume that $\mathcal{C}$ is contained in the interior of $\mathcal{L}_A(V)$. Since $\mathcal{L}_A(V)$ is compact and $\mathcal{C} \cap \mathcal{E}_{A,\pi}(V)$ is closed, it suffices to prove this result locally about any divisor $\Delta_0 \in \mathcal{C} \cap \mathcal{E}_{A,\pi}(V)$. Let $\phi\colon X \dashrightarrow Y$ be a log terminal model over U for $K_X + \Delta_0$. Let $\Gamma_0 = \phi_*\Delta_0$.

Pick a neighbourhood $\mathcal{C}_0 \subset \mathcal{L}_A(V)$ of Δ_0, which is a rational polytope. As ϕ is $(K_X + \Delta_0)$-negative we may pick $\mathcal{C}_0$ such that for any $\Delta \in \mathcal{C}_0$, $a(F, K_X + \Delta) < a(F, K_Y + \Gamma)$ for all ϕ-exceptional divisors $F \subset X$, where $\Gamma = \phi_*\Delta$. Since $K_Y + \Gamma_0$ is kawamata log terminal and Y is $\mathbb{Q}$-factorial, possibly shrinking $\mathcal{C}_0$, we may assume that $K_Y + \Gamma$ is kawamata log terminal for all $\Delta \in \mathcal{C}_0$. In particular, replacing $\mathcal{C}$ by $\mathcal{C}_0$, we may assume that the rational polytope $\mathcal{C}' = \phi_*(\mathcal{C})$ is contained in $\mathcal{L}_{\phi_*A}(W)$, where $W = \phi_*(V)$. By (3.7.3), there is a rational affine linear isomorphism $L\colon W \longrightarrow V'$ and a general ample $\mathbb{Q}$-divisor A' over U such that $L(\mathcal{C}') \subset \mathcal{L}_{A'}(V')$, $L(\Gamma) \sim_{\mathbb{Q},U} \Gamma$ for all $\Gamma \in \mathcal{C}'$ and $K_Y + \Gamma$ is kawamata log terminal for any $\Gamma \in L(\mathcal{C}')$.

Note that $\dim V' \leq \dim V$. By (3.6.9) and (3.6.10), any log terminal model of $(Y, L(\Gamma))$ over U is a log terminal model of (X, Δ) over U for any $\Delta \in \mathcal{C}$. Replacing X by Y and $\mathcal{C}$ by $L(\mathcal{C}')$, we may therefore assume that $K_X + \Delta_0$ is π-nef.

By (3.9.2) $K_X + \Delta_0$ has an ample model $\psi\colon X \longrightarrow Z$ over U. In particular $K_X + \Delta_0 \sim_{\mathbb{R},Z} 0$. By what we have already proved there are finitely many birational maps $\phi_i\colon X \dashrightarrow Y_i$ over Z, $1 \leq i \leq k$, such that for any $\Delta \in \mathcal{C} \cap \mathcal{E}_{A,\psi}(V)$, there is an index i such that ϕ_i is a log terminal model of $K_X + \Delta$ over Z. Since there are only finitely many indices $1 \leq i \leq k$, possibly shrinking $\mathcal{C}$, (3.11.3) implies that if $\Delta \in \mathcal{C}$ then ϕ_i is a log terminal model for $K_X + \Delta$ over Z if and only if it is log terminal model for $K_X + \Delta$ over U.

Suppose that $\Delta \in \mathcal{C} \cap \mathcal{E}_{A,\pi}(V)$. Then $\Delta \in \mathcal{C} \cap \mathcal{E}_{A,\psi}(V)$ and so there is an index $1 \leq i \leq k$ such that ϕ_i is a log terminal model for $K_X + \Delta$ over Z. But then ϕ_i is a log terminal model for $K_X + \Delta$ over U. □

Lemma 7.2. *Assume Theorem C_n and Theorem D_n.*

Let $\pi\colon X \longrightarrow U$ be a projective morphism of normal quasi-projective varieties, where X has dimension n. Suppose that there is a kawamata log terminal pair (X, Δ_0). Fix $A \geq 0$, a general ample $\mathbb{Q}$-divisor over U. Let V be a finite dimensional affine subspace of $\operatorname{WDiv}_{\mathbb{R}}(X)$ which is defined over the rationals. Let $\mathcal{C} \subset \mathcal{L}_A(V)$ be a rational polytope.

Then there are finitely many birational maps $\psi_j\colon X \dashrightarrow Z_j$ over U, $1 \leq j \leq l$ such that if $\psi\colon X \dashrightarrow Z$ is a weak log canonical model of

$K_X + \Delta$ over U, for some $\Delta \in \mathcal{C}$ then there is an index $1 \leq j \leq l$ and an isomorphism $\xi\colon Z_j \longrightarrow Z$ such that $\psi = \xi \circ \psi_j$.

Proof. Suppose that $\Delta \in \mathcal{C}$ and $K_X + \Delta' \sim_{\mathbb{R},U} K_X + \Delta$ is kawamata log terminal. (3.6.9) implies that $\psi\colon X \dashrightarrow Z$ is a weak log canonical model of $K_X + \Delta$ over U if and only if ψ is a weak log canonical model of $K_X + \Delta'$ over U. By (3.7.4) we may therefore assume that if $\Delta \in \mathcal{C}$ then $K_X + \Delta$ is kawamata log terminal.

Let G be any divisor which contains the support of every element of V and let $f\colon Y \longrightarrow X$ be a log resolution of (X, G). Given $\Delta \in \mathcal{L}_A(V)$ we may write

$$K_Y + \Gamma = f^*(K_X + \Delta) + E,$$

where $\Gamma \geq 0$ and $E \geq 0$ have no common components, $f_*\Gamma = \Delta$ and $f_*E = 0$. If $\psi\colon X \dashrightarrow Z$ is a weak log canonical model of $K_X + \Delta$ over U then $\psi \circ f\colon Y \dashrightarrow Z$ is a weak log canonical model of $K_Y + \Gamma$ over U. If $\mathcal{C}'$ denotes the image of $\mathcal{C}$ under the map $\Delta \longrightarrow \Gamma$ then $\mathcal{C}'$ is a rational polytope, cf. (3.7.2), and if $\Gamma \in \mathcal{C}'$ then $K_Y + \Gamma$ is kawamata log terminal. In particular $\mathbf{B}_+(f^*A/U)$ does not contain any non kawamata log terminal centres of $K_Y + \Gamma$, for any $\Gamma \in \mathcal{C}'$. Let W be the subspace of $\mathrm{WDiv}_{\mathbb{R}}(Y)$ spanned by the components of the strict transform of G and the exceptional locus of f. By (3.7.3) and (3.6.9), we may assume that there is a general ample $\mathbb{Q}$-divisor A' on Y over U such that $\mathcal{C}' \subset \mathcal{L}_{A'}(W)$. Replacing X by Y and $\mathcal{C}$ by $\mathcal{C}'$, we may therefore assume that X is smooth.

Pick general ample $\mathbb{Q}$-Cartier divisors $H_1, H_2, \dots, H_p$ over U, which generate $\mathrm{WDiv}_{\mathbb{R}}(X)$ modulo relative numerical equivalence over U and let $H = H_1 + H_2 + \dots + H_p$ be their sum. By (3.7.4) we may assume that if $\Delta \in \mathcal{C}$ then Δ contains the support of H. Let W be the affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$ spanned by V and the divisors $H_1, H_2, \dots, H_p$. Pick $\mathcal{C}'$ a rational polytope in $\mathcal{L}_A(W)$ containing $\mathcal{C}$ in its interior such that if $\Delta \in \mathcal{C}'$ then $K_X + \Delta$ is kawamata log terminal.

By (7.1) there are finitely many $1 \leq i \leq k$ rational maps $\phi_i\colon X \dashrightarrow Y_i$ over U, such that given any $\Delta' \in \mathcal{C}' \cap \mathcal{E}_{A,\pi}(W)$, we may find an index $1 \leq i \leq k$ such that ϕ_i is a log terminal model of $K_X + \Delta'$ over U. By (3.11.2) for each index $1 \leq i \leq k$ there are finitely many contraction morphisms $f_{i,m}\colon Y_i \longrightarrow Z_{i,m}$ over U such that if $\Delta' \in \mathcal{W}_{\phi_i,A,\pi}(W)$ and there is a contraction morphism $f\colon Y_i \longrightarrow Z$ over U, with

$$K_{Y_i} + \Gamma_i = K_{Y_i} + \phi_{i*}\Delta' \sim_{\mathbb{R},U} f^*D,$$

for some ample over U $\mathbb{R}$-divisor D on Z, then there is an index (i, m) and an isomorphism $\xi\colon Z_{i,m} \longrightarrow Z$ such that $f = \xi \circ f_{i,m}$. Let

$\psi_j \colon X \dashrightarrow Z_j$, $1 \leq j \leq l$ be the finitely many rational maps obtained by composing every ϕ_i with every $f_{i,j}$.

Pick $\Delta \in \mathcal{C}$ and let $\psi \colon X \dashrightarrow Z$ be a weak log canonical model of $K_X + \Delta$ over U. Then $K_Z + \Theta$ is kawamata log terminal and nef over U, where $\Theta = \psi_* \Delta$. As we are assuming Theorem C$_n$, we may find a log terminal model $\eta \colon Z \dashrightarrow Y'$ of $K_Z + \Theta$ over Z. Then Y' is $\mathbb{Q}$-factorial and the structure morphism $\xi \colon Y' \longrightarrow Z$ is a small birational map, the inverse of η. By (3.6.12), we may find $\Delta' \in \mathcal{C}' \cap \mathcal{E}_{A,\pi}(W)$ such that ψ is an ample model of $K_X + \Delta'$ over U. Pick an index $1 \leq i \leq k$ such that ϕ_i is a log terminal model of $K_X + \Delta'$ over U. By (4) of (3.6.6) there is a contraction morphism $f \colon Y_i \longrightarrow Z$ such that

$$K_{Y_i} + \Gamma_i = f^*(K_Z + \Theta'),$$

where $\Gamma' = \phi_* \Delta'$ and $\Theta' = \psi_* \Delta'$. As $K_Z + \Theta'$ is ample over U, it follows that there is an index m and an isomorphism $\xi \colon Z_{i,m} \longrightarrow Z$ such that $f = \xi \circ f_{i,m}$. But then

$$\psi = f \circ \phi_i = \xi \circ f_{i,m} \circ \phi_i = \xi \circ \psi_j,$$

for some index $1 \leq j \leq l$. □

Lemma 7.3. *Theorem C_n and Theorem D_n imply Theorem E_n.*

Proof. Since $\mathcal{L}_A(V)$ is itself a rational polytope by (3.7.2), this is immediate from (7.2). □

8. Finite generation

Lemma 8.1. *Theorem C_n and Theorem D_n imply Theorem F_n.*

Proof. Theorem C$_n$ and Theorem D$_n$ imply that there is is a log terminal model $\mu \colon X \dashrightarrow Y$ of $K_X + \Delta$, and $K_Y + \Gamma = \mu_*(K_X + \Delta)$ is semiample by (1) of (3.9.3). (1) follows, as

$$R(X, K_X + \Delta) \simeq R(Y, K_Y + \Gamma).$$

As $K_Y + \Gamma$ is semiample the prime divisors contained in the stable base locus of $K_X + \Delta$ are precisely the exceptional divisors of μ. But there is a constant $\delta > 0$ such that if $\Xi \in V$ and $\|\Xi - \Delta\| < \delta$ then the exceptional divisors of μ are also $(K_X + \Xi)$-negative. Hence (2).

Note that by (3.11.2), there is a constant $\eta > 0$ such that if $\Xi \in W$ and $\|\Xi - \Delta\| < \eta$ then μ is also a log terminal model of $K_X + \Xi$. (3.9.2) implies that there is a constant $r > 0$ such that if $m(K_Y + \Gamma)$ is integral and nef, then $mr(K_Y + \Gamma)$ is base point free. It follows that if $k(K_X + \Xi)/r$ is Cartier then every component of $\operatorname{Fix}(k(K_X + \Xi))$ is contracted by μ and so every such component is in the stable base locus of $K_X + \Delta$. This is (3). □

9. Proof of Theorems

Proof of Theorems A, B, C, D, E and F. Immediate from the main result of [9], (4.4), (5.6), (6.6), (7.3) and (8.1). □

Proof of (1.2). Suppose $K_X + \Delta$ is π-big. Then we may write $K_X + \Delta \sim_{\mathbb{R},U} B \geq 0$. If $\epsilon > 0$ is sufficiently small then $K_X + \Delta + \epsilon B$ is kawamata log terminal. (3.6.9) implies that $K_X + \Delta$ and $K_X + \Delta + \epsilon B$ have the same log terminal models over U. Replacing Δ by $\Delta + \epsilon B$ we may therefore assume that Δ is big over U. (1) follows by Theorem C and Theorem D.

(2) and (3) follow from (1) and (3.9.2). □

10. Proof of Corollaries

Proof of (1.1.1). (1), (2) and (3) are immediate from (1.2). (4) is Theorem D of [4]. □

Proof of (1.1.2). Immediate by Theorem 5.2 of [6] and (3) of (1.2). □

Proof of (1.1.3). Note that Y_1 and Y_2 are isomorphic in codimension one. Replacing U by the common ample model of (X, Δ), we may assume that $K_{Y_i} + \Gamma_i$ is numerically trivial over U. Let H_2 be a divisor on Y_2, which is ample over U. Let H_1 be the strict transform on Y_1. Possibly replacing H_2 by a small multiple, we may assume that $K_{Y_i} + \Gamma_i + H_i$ is kawamata log terminal.

Suppose that $K_{Y_1} + \Gamma_1 + H_1$ is not nef over U. Then there is a $(K_{Y_1} + \Gamma_1 + H_1)$-flip over U which is automatically a $(K_{Y_1} + \Gamma_1)$-flop over U. By finiteness of log terminal models for $K_X + \Delta$ over U, this $(K_{Y_1} + \Gamma_1 + H_1)$-MMP terminates. Thus we may assume that $K_{Y_1} + \Gamma_1 + H_1$ is nef over U. But then $K_{Y_2} + \Gamma_2 + H_2$ is the corresponding ample model, and so there is a small birational morphism $f\colon Y_1 \longrightarrow Y_2$. As Y_2 is $\mathbb{Q}$-factorial, f is an isomorphism. □

Proof of (1.1.5). We first prove (1) and (2). By Theorem E and (3.11.2), and since ample models are unique by (1) of (3.6.6), it suffices to prove that if $\Delta \in \mathcal{E}_{A,\pi}(V)$ then $K_X + \Delta$ has both a log terminal model over U and an ample model over U.

By (3.7.5) and (3.6.9) we may assume that $K_X + \Delta$ is kawamata log terminal. (1.2) implies the existence of a log terminal model over U and the existence of an ample model then follows from (3.9.3).

(3) follows as in the proof of (3.11.2). □

Proof of (1.1.7). Immediate consequence of (1.1.5). □

Proof of (1.1.9). Let V_A be the affine subspace of $\operatorname{WDiv}_{\mathbb{R}}(X)$ generated by $\Delta_1, \Delta_2, \dots, \Delta_k$. (1.1.5) implies that there are finitely many $1 \leq p \leq q$ rational maps $\phi_p \colon X \dashrightarrow Y_p$ over U such that if $\Delta \in \mathcal{E}_{A,\pi}(V)$ then there is an index $1 \leq p \leq q$ such that ϕ_p is a log terminal model of $K_X + \Delta$ over U. Let $\mathcal{C} \subset \mathcal{L}_A(V)$ be the polytope spanned by $\Delta_1, \Delta_2, \dots, \Delta_k$ and let

$$\mathcal{C}_p = \mathcal{W}_{\phi_p,A,\pi}(V) \cap \mathcal{C}.$$

Then $\mathcal{C}_p$ is a rational polytope. Replacing $\Delta_1, \Delta_2, \dots, \Delta_k$ by the vertices of $\mathcal{C}_p$, we may assume that $\mathcal{C} = \mathcal{C}_p$, and we will drop the index p. Let $\pi' \colon Y \longrightarrow U$ be the induced morphism. Let $\Gamma_i = \phi_* \Delta_i$. If we pick a positive integer m so that both $G_i = m(K_Y + \Gamma_i)$ and $D_i = m(K_X + \Delta_i)$ are Cartier for every $1 \leq i \leq k$, then

$$\mathfrak{R}(\pi, D^\bullet) \simeq \mathfrak{R}(\pi', G^\bullet).$$

Replacing X by Y, we may therefore assume that $K_X + \Delta_i$ is nef over U. By (3.9.2) $K_X + \Delta_i$ is semiample over U and so the Cox ring is finitely generated.

Aliter: By (3.7.5), we may assume that each $K_X + \Delta_i$ is kawamata log terminal. Pick a log resolution $f \colon Y \longrightarrow X$ of (X, Δ), where Δ is the support of the sum $\Delta_1 + \Delta_2 + \cdots + \Delta_k$. Then we may write

$$K_Y + \Gamma_i = \pi^*(K_X + \Delta_i) + E_i,$$

where Γ_i and E_i have no common components, $f_* \Gamma_i = \Delta_i$ and $f_* E_i = 0$. We may assume that there is an exceptional $\mathbb{Q}$-divisor $F \geq 0$ such that $f^*A - F$ is ample over U and $(Y, \Gamma_i + F)$ is kawamata log terminal. Pick $A' \sim_{\mathbb{Q},U} f^*A - F$ a general ample $\mathbb{Q}$-divisor over U. Then

$$G_i = K_Y + \Gamma_i + F - f^*A + A' \sim_{\mathbb{Q},U} K_Y + \Gamma_i,$$

is kawamata log terminal and $\mathfrak{R}(\pi, D^\bullet)$ if finitely generated if and only if $\mathfrak{R}(\pi \circ f, G^\bullet)$ if finitely generated, since they have isomorphic truncations. Replacing X by Y we may therefore assume that (X, Δ) is log smooth.

Pick a positive integer m such that $D_i = m(K_X + \Delta_i)$ is integral for $1 \leq i \leq k$. Let

$$E = \bigoplus_{i=1}^{k} \mathcal{O}_X(m\Delta_i).$$

and let $Y = \mathbb{P}_X(E)$ with projection map $f \colon Y \longrightarrow X$. Pick $\sigma_i \in \mathcal{O}_X(m\Delta_i - mA)$ with zero locus $m\Delta_i$ and let $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_k) \in H^0(X, E(-mA))$. Let S be the divisor corresponding to σ in Y. Let $T_1, T_2, \dots, T_k$ be the divisors on Y, given by the summands of E, let T be their sum, and let $\Gamma = T + f^*A + S/m$. Note that $\mathcal{O}_Y(m(K_Y + \Gamma))$

is the tautological bundle associated to $E(mK_X)$; indeed if $k = 1$ this is clear and if $k > 1$ then it has degree one on the fibres and $\mathcal{O}_Y(m(K_Y + \Gamma))$ restricts to the tautological line bundle on each T_i by adjunction and induction on k. Thus

$$\mathfrak{R}(\pi, D^\bullet) \simeq \mathfrak{R}(\pi \circ f, m(K_Y + \Gamma)).$$

On the other hand (Y, Γ) is log smooth outside T and restricts to a divisorially log terminal pair on each T_i by adjunction and induction. Therefore (Y, Γ) is divisorially log terminal by inversion of adjunction. Since T is ample over X, there is a positive rational number $\epsilon > 0$ such that $f^*A + \epsilon T$ is ample over U. Let $A' \sim_{\mathbb{Q},U} f^*A + \epsilon T$ be a general ample $\mathbb{Q}$-divisor over U. Then

$$K_Y + \Gamma' = K_Y + \Gamma - \epsilon T - f^*A + A' \sim_{\mathbb{Q},U} K_Y + \Gamma,$$

is kawamata log terminal. Thus $\mathfrak{R}(\pi \circ f, m(K_Y + \Gamma))$ is finitely generated over U by (1.1.2). $\square$

We will need a well-known result on the geometry of the moduli spaces of n-pointed curves of genus g:

Lemma 10.1. *Let $X = \overline{M}_{g,n}$ and let D be the sum of the boundary divisors.*

Then

- *X is $\mathbb{Q}$-factorial,*
- *K_X is kawamata log terminal and*
- *$K_X + D$ is log canonical and ample.*

Proof. X is $\mathbb{Q}$-factorial, K_X is kawamata log terminal and $K_X + D$ is log canonical as the pair (X, D) is locally a quotient of a normal crossings pair.

It is proved in [27] that $K_X + D$ is ample when $n = 0$. To prove the general case, consider the natural map

$$\pi \colon \overline{M}_{g,n+1} \longrightarrow \overline{M}_{g,n},$$

which drops the last point. Let $Y = \overline{M}_{g,n+1}$ and G be the sum of the boundary divisors. If $\overline{\mathcal{M}}_{g,n}$ is the moduli stack of stable curves of genus g, then $\overline{M}_{g,n}$ is the coarse moduli space and so there is a representable morphism

$$f \colon \overline{\mathcal{M}}_{g,n} \longrightarrow \overline{M}_{g,n},$$

which only ramifies over the locus of stable curves with automorphisms. If $g = 1$ and $n \leq 1$ then $K_Y + G$ is obviously π-ample. Otherwise the locus of smooth curves with extra automorphisms has codimension at least two and in this case $K_{\overline{\mathcal{M}}_{g,n}} + \Delta = f^*(K_X + D)$, where Δ is the sum of the boundary divisors. On the other hand there is a fibre square

$$\begin{tikzcd}
\overline{\mathcal{M}}_{g,n+1} \arrow[r] \arrow[d, "\psi"'] & \overline{M}_{g,n+1} \arrow[d, "\pi"] \\
\overline{\mathcal{M}}_{g,n} \arrow[r, "f"] & \overline{M}_{g,n},
\end{tikzcd}$$

where ψ is the universal morphism. Since the stack is a fine moduli space, ψ is the universal curve. If Γ is the sum of the boundary divisors then $K_{\overline{\mathcal{M}}_{g,n+1}} + \Gamma$ has positive degree on the fibres of ψ, by adjunction and the definition of a stable pair. In particular $K_Y + G$ is ample on the fibres of π.

On the other hand, we may write

$$K_Y + G = \pi^*(K_X + D) + \psi,$$

for some $\mathbb{Q}$-divisor ψ. It is proved in [7] that ψ is nef. We may assume that $K_X + D$ is ample, by induction on n. If $\epsilon > 0$ is sufficiently small, it follows that $\epsilon(K_Y + G) + (1-\epsilon)\pi^*(K_X + D)$ is ample. But then

$$\begin{aligned} K_Y + G &= \epsilon(K_Y + G) + (1-\epsilon)(K_Y + G) \\ &= \epsilon(K_Y + G) + (1-\epsilon)\pi^*(K_X + D) + (1-\epsilon)\psi, \end{aligned}$$

is also ample and so the result follows by induction on n. □

Proof of (1.2.1). By (10.1) K_X+D is ample and log canonical, where D is the sum of the boundary divisors. In particular $K_X+\Delta$ is kawamata log terminal, provided none of the a_i are equal to one.

Pick a general ample $\mathbb{Q}$-divisor $A \sim_{\mathbb{Q}} \delta(K_X + D)$. Note that

$$\begin{aligned} (1+\delta)(K_X + \Delta) &= K_X + \delta(K_X + D) + (1+\delta)\Delta - \delta D \\ &\sim_{\mathbb{Q}} K_X + A + B. \end{aligned}$$

Now

$$0 \leq (\Delta - \delta D) + \delta\Delta = B = \Delta + \delta(\Delta - D) \leq D.$$

Thus the result is an immediate consequence of Theorem C, Theorem D and Theorem E. □

Proof of (1.3.2). Immediate by (1.1.9) and the main result of [12] (note that $h^1(X, \mathcal{O}_X) = 0$ by Kawamata-Viehweg vanishing). □

Proof of (1.3.3). Pick any π-ample divisor A such that $K_X + \Delta + A$ is π-ample and kawamata log terminal. We may find $\epsilon > 0$ such that $K_X+\Delta+\epsilon A$ is not π-pseudo-effective. We run the $(K_X+\Delta+\epsilon A)$-MMP over U with scaling of A. Since every step of this MMP is A-positive, it is automatically a $(K_X + \Delta)$-MMP as well. But as $K_X + \Delta + \epsilon A$

is not π-pseudo-effective, this MMP must terminate with a Mori fibre space $g\colon Y \longrightarrow W$ over U. $\square$

Proof of (1.3.5). As X is a Mori dream space, the cone of pseudo-effective divisors is a rational polyhedron. It follows that $\overline{\mathrm{NF}}(X)$ is also a rational polyhedron, as it is the dual of the cone of pseudo-effective divisors.

Now suppose that F is a co-extremal ray. As $\overline{\mathrm{NF}}(X)$ is polyhedral, we may pick a pseudo-effective divisor D which supports F. Pick $\epsilon > 0$ such that $D - \epsilon(K_X + \Delta)$ is ample. Pick a general ample $\mathbb{Q}$-divisor $A \sim_{\mathbb{Q}} \frac{1}{\epsilon}D - (K_X + \Delta)$ such that $K_X + \Delta + kA$ is ample and kawamata log terminal for some $k > 1$. Then $(X, \Theta = \Delta + A)$ is kawamata log terminal and $K_X + \Theta \sim_{\mathbb{Q}} \frac{1}{\epsilon}D$ supports F. As in the proof of (1.3.3), the $(K_X + \Delta)$-MMP with scaling of A ends with a $(K_X + \Theta)$-trivial Mori fibre space $f\colon Y \longrightarrow Z$, and it is easy to see that the pullback to X of a general curve in the fibre of f generates F. $\square$

Proof of (1.4.1). The flip of π is precisely the log canonical model, so that this result follows from Theorem C. $\square$

Proof of (1.4.2). Immediate from (3.10.12) and Theorem E. $\square$

Proof of (1.4.3). Pick an ample $\mathbb{Q}$-divisor $A \geq 0$ which contains the centre of every element of $\mathfrak{E}$ of log discrepancy one, but no non kawamata log terminal centres. If $\epsilon > 0$ is sufficiently small, then $(X, \Delta + \epsilon A)$ is log canonical and so replacing Δ by $\Delta + \epsilon A$, we may assume that $\mathfrak{E}$ contains no valuations of log discrepancy one. Replacing Δ by $(1-\eta)\Delta + \eta\Delta_0$, where $\eta > 0$ is sufficiently small, we may assume that $K_X + \Delta$ is kawamata log terminal.

We may write

$$K_W + \Psi = f^*(K_X + \Delta) + E,$$

where $\Psi \geq 0$ and $E \geq 0$ have no common components, $f_*\Psi = \Delta$ and E is exceptional. Let F be the sum of all the exceptional divisors which are neither components of E nor correspond to elements of $\mathfrak{E}$.

Pick $\epsilon > 0$ such that $K_W + \Phi = K_Y + \Psi + \epsilon F$ is kawamata log terminal. As f is birational, Φ is big over X and so by (1.2) we may find a log terminal model $g\colon W \dashrightarrow Y$ for $K_W + \Phi$ over X. Let $\pi\colon Y \longrightarrow X$ be the induced morphism. If $\Gamma = g_*\Phi$ and $E' = g_*(E + \epsilon F)$ then

$$K_Y + \Gamma = \pi^*(K_X + \Delta) + E',$$

where $K_Y + \Gamma$ is nef over X. Negativity of contraction implies that $E' = 0$ so that

$$K_Y + \Gamma = \pi^*(K_X + \Delta).$$

But then we must have contracted every exceptional divisor which does correspond to an element of $\mathfrak{E}$. $\square$

Proof of (1.4.4). Pick a log resolution $f\colon Z \longrightarrow X$ of (X, Δ). We may write

$$K_Z + \Psi = f^*(K_X + \Delta) + E_0,$$

where $\Psi \geq 0$ and $E_0 \geq 0$ have no common components, E_0 is f-exceptional and $f_*\Psi = \Delta$. We may write $\Psi = \Psi_1 + \Psi_2$ where every component of Ψ_1 has coefficient less than one and every component of Ψ_2 has coefficient at least one. Let E_1 be the sum of the components of Ψ_1 which are exceptional. Pick $\delta > 0$ such that $K_Z + \Psi_3$ is kawamata log terminal, where $\Psi_3 = \Psi_1 + \delta E_1$. Let $g\colon Z \dashrightarrow Y$ be a log terminal model of $K_Z + \Psi_3$ over X, whose existence is guaranteed by (1.2). If $\pi\colon Y \longrightarrow X$ is the induced birational morphism then we may write

$$K_Y + \Gamma = \pi^*(K_X + \Delta) + E,$$

where $\Gamma = g_*(\Psi + \delta E_1) \geq 0$, $E = g_*(E_0 + \delta E_1) \geq 0$ and $K_Y + \Gamma_3$ is nef over X, where $\Gamma_3 = g_*\Psi_3$. If $E \neq 0$ then by (3.6.2) there is a family of curves $\Sigma \subset Y$ contracted by π, which sweeps out a component of E such that $E \cdot \Sigma < 0$. Since E and $\Gamma - \Gamma_3$ have no common components, $(\Gamma - \Gamma_3) \cdot \Sigma \geq 0$, so that $(K_Y + \Gamma_3) \cdot \Sigma < 0$, a contradiction. Thus $E = 0$, in which case $\Gamma_3 = \Gamma_1$ and no component of Γ_1 is exceptional. $\square$

Lemma 10.2. *Let (X, Δ) be a quasi-projective divisorially log terminal pair and let S be a component of the support of Δ.*

Then there is a small projective birational morphism $\pi\colon Y \longrightarrow X$, where Y is $\mathbb{Q}$-factorial and $-T$ is nef over X, where T is the strict transform of S.

In particular if Σ is a curve in Y which is contracted by π and Σ intersects T then Σ is contained in T.

Proof. By (2.43) of [22] we may assume that (X, Δ) is kawamata log terminal. Let $f\colon Z \longrightarrow X$ be a log terminal model of (X, Δ) over X, let Ψ be the strict transform of Δ and let R be the strict transform of S. Let $g\colon Z \dashrightarrow Y$ be a log terminal model of $K_Z + \Psi - \epsilon R$ over X, for any $\epsilon > 0$ sufficiently small. As $\Psi - \epsilon R$ is big over X and $K_Z + \Psi$ is kawamata log terminal, (3.7.3) and (1.1.5) imply that we may assume that Y is independent of ϵ. In particular $-T$ is π-nef, where $\pi\colon Y \longrightarrow X$ is the induced morphism. $\square$

Proof of (1.4.5). We work locally about S. Let a be the log discrepancy of $K_S + \Theta$ and let b be the minimum of the log discrepancy with respect to $K_X + \Delta$ of any valuation whose centre on X is of codimension at least two. It is straightforward to prove that $b \leq a$, cf. (17.2) of [21].

If B is a prime divisor on S which is not a component of Θ then the log discrepancy of B with respect to (S, Θ) is one, so that $a \leq 1$. In particular we may assume that $b < 1$.

By the main theorem of [13], (X, Δ) is log canonical near the image of S if and only if (S, Θ) is log canonical and so we may assume that (X, Δ) is log canonical and hence $b \geq 0$.

Suppose that (X, Δ) is not purely log terminal. By (1.4.4) we may find a birational projective morphism $\pi \colon Y \longrightarrow X$ which only extracts divisors of log discrepancy zero and if we write $K_Y + \Gamma = \pi^*(K_X + \Delta)$, then Y is $\mathbb{Q}$-factorial and K_Y is kawamata log terminal. By connectedness, see (17.4) of [21], the non kawamata log terminal locus of $K_Y + \Gamma$ contains the strict transform of S and is connected (indeed the fibres are connected and we work locally about S). Thus we are free to replace X by Y and so we may assume that X is $\mathbb{Q}$-factorial and K_X is kawamata log terminal.

Suppose that (X, Δ) is purely log terminal. By (10.2) there is a birational projective morphism $\pi \colon Y \longrightarrow X$ such that Y is $\mathbb{Q}$-factorial and the exceptional locus over S is contained in the strict transform of S. Replacing X by Y, we may assume that X is $\mathbb{Q}$-factorial.

We may therefore assume that X is $\mathbb{Q}$-factorial and K_X is kawamata log terminal. Let ν be any valuation of log discrepancy b. Suppose that the centre of ν is not a divisor. By (1.4.3) there is a birational projective morphism $\pi \colon Y \longrightarrow X$ which extracts a single exceptional divisor E corresponding to ν. Since X is $\mathbb{Q}$-factorial, the exceptional locus of π is equal to the support of E and so E intersects the strict transform of S. Let $\Gamma = \Delta' + (1-b)E$, where Δ' is the strict transform of Δ. Then $K_Y + \Gamma = \pi^*(K_X + \Delta)$.

Replacing X by Y we may therefore assume there is a prime divisor $D \neq S$ on X, whose coefficient Δ is $1 - b$. By (3.4.1) some component B of Θ has coefficient at least $1 - b$, that is log discrepancy b and so $a \leq b$. $\square$

Proof of (1.4.6). Since this result is local in the étale topology, we may assume that U is affine. Let $f \colon Y \longrightarrow X$ be a log resolution of (X, Δ), so that the composition $\psi \colon Y \longrightarrow U$ of f and π is projective. We may write

$$K_Y + \Gamma' = f^*(K_X + \Delta) + E,$$

where $\Gamma' \geq 0$ and $E \geq 0$ have no common components and E is f-exceptional. If $F \geq 0$ is a $\mathbb{Q}$-divisor whose support equals the exceptional locus of f, then for any $0 < \epsilon \ll 1$, $(Y, \Gamma = \Gamma' + \epsilon F)$ is kawamata log terminal.

Pick a ψ-ample divisor H such that $K_Y + \Gamma + H$ is ψ-ample. We run the $(K_Y + \Gamma)$-MMP over U with scaling of H. Since X contains no rational curves contracted by π, this MMP is automatically a MMP over X. Since Γ is big over X and termination is local in the étale topology, this MMP terminates by (1.4.2).

Thus we may assume that $K_Y + \Gamma$ is f-nef, so that $E + \epsilon F$ is empty, and hence f is small. As X is analytically $\mathbb{Q}$-factorial it follows that f is an isomorphism. But then π is a log terminal model. □

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