BOUNDEDNESS OF LOG TERMINAL FANO PAIRS OF BOUNDED INDEX

§1 Introduction and statement of results

A fundamental problem in classifying varieties is to determine natural subsets whose moduli is bounded. The difficulty of this problem is partially measured by the behaviour of the canonical class of the variety. Three extreme cases are especially of interest: either the canonical class is ample, that is the variety is of general type, or the canonical class is trivial, for example the variety is an abelian variety or Calabi-Yau, or minus the canonical class is ample, that is the variety is Fano.

Let us first suppose that the variety is smooth. It is well known that varieties of general type do not form a bounded family unless one fixes some invariants. For example a curve of genus g is of general type iff $g \geq 2$. Typically even varieties with trivial canonical class do not form bounded families; for example the moduli space of abelian varieties has infinitely many components corresponding to the type of polarisation. Even though the same is true of K3 surfaces, to the best of the author's knowledge it is not known whether the family of Calabi-Yau threefolds is bounded. The picture for smooth Fano varieties however is much brighter; indeed there is a complete classification up to dimension three. Such a classification in general seem unfeasible, but on the other hand it was proved by Kollár, Miyaoka and Mori [15] that smooth Fano varieties of fixed dimension form a bounded family.

One reason for focusing on the three extreme cases is that roughly speaking it is expected that up to birational equivalence, any variety is either of general type or admits a fibration to another variety, whose fibres are either Fano or have trivial canonical class. However one can only achieve this birational factorisation if one allows singularities. For example for a surface

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of general type one needs to contract all the -2-curves on the surface to make the canonical divisor ample.

Unfortunately it is no longer true that singular Fano varieties form a bounded family. For example cones over a rational normal curve of degree d form an unbounded family of Fano surfaces; to form a bounded family one needs to impose some restrictions on the singularities. One natural restriction to impose is that some fixed multiple of the canonical class is Cartier. The smallest such positive integer we will call the index. (Note that the index is also used to designate other invariants in this context; unfortunately there does not seem to be a standard notation.) However even this is not enough, since cones over elliptic curves have index one, yet form an unbounded family of Fano surfaces, simply varying the degree as before.

One natural class of singularities to work with are kawamata log terminal singularities. They form a large class of singularities which are closed under many standard geometric operations; for example the finite quotient or unramified cover of any kawamata log terminal singularity is kawamata log terminal. Moreover they seem to be the largest class where one might expect boundedness; for example cones over an elliptic curve are log canonical but not kawamata log terminal. Fortunately to achieve the factorisation alluded to above it is expected that one only needs to allow the presence of canonical singularities which are automatically kawamata log terminal.

It is also natural to enlarge the category we are working in and include a boundary divisor with fractional coefficients. These appear naturally when one takes a quotient and there is ramification in codimension one or when one passes to a resolution. For example suppose S is the cone over a rational normal curve of degree d. Then the minimal resolution T is isomorphic to $\mathbb{F}_n$, the unique $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ with a section C of self-intersection $-n$ and $-K_S - \frac{n-1}{n}C$ is ample. The morphism $T \longrightarrow S$ simply contracts C.

1.1 Definition. We say that the pair (X, Δ) is a **log pair** if X is a normal variety and Δ is a $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

If $\pi\colon Y \longrightarrow X$ is a morphism the **log pullback** Γ of Δ is defined by the formula

$$K_Y + \Gamma = \pi^*(K_X + \Delta).$$

We say that Δ is a **sub-boundary** if all of its coefficients are less than one; if in addition the coefficients of Δ are all positive then we say that Δ is a **boundary**.

We say that the pair (X, Δ) is **kawamata log terminal** if $\llcorner \Delta \lrcorner$ is empty and the log pullback is a sub-boundary, for any birational morphism.

It is impractical to give all the standard definitions and results of higher dimensional geometry; the reader is referred to the excellent introduction of [17] for those definitions we have omitted. We work over an algebraically closed field of characteristic zero. We prove the following conjecture of Batryev:

1.2 Theorem. *Let r and n be integers. Then the family of all log pairs (X, Δ) such that*

(1) *X has dimension n,*

(2) *$-(K_X + \Delta)$ is ample,*

(3) *$K_X + \Delta$ is kawamata log terminal, and*

(4) *$r(K_X + \Delta)$ is Cartier*

is bounded.

As pointed out above, (1.2) includes Fano varieties with quotient singularities. To prove (1.2), by a result of Kollár [11], it suffices to bound the top self-intersection of $-(K_X + \Delta)$, which we will refer to as the degree d. Thus (1.2) follows from

1.3 Theorem. *Let r and n be integers. Then there is a real number M such that if we have a log pair (X, Δ) where*

(1) *X has dimension n,*

(2) *$-(K_X + \Delta)$ is big and nef,*

(3) *$K_X + \Delta$ is kawamata log terminal, and*

(4) *$r(K_X + \Delta)$ is Cartier,*

then $d < M$.

One interesting feature of smooth Fano varieties is that they have trivial fundamental group. The same is true of Fano varieties with kawamata log terminal singularities but actually in this case a more natural and more general question is to consider the fundamental group of the complement of the singular points. In general this group is larger but it is believed that the fundamental group ought still to be finite. More generally still it is even more natural, at least in the context of orbifolds, to consider the orbifold fundamental group and orbifold algebraic fundamental group.

1.4 Definition. Let (X, Δ) be a log pair and let U be the complement of the support of Δ.

The **orbifold algebraic fundamental group** $\pi_1^{\text{orb-alg}}(X, \Delta)$ is by definition the quotient of the algebraic fundamental group $\pi_1^{\text{alg}}(U)$ of U consisting of those finite covers $Y \longrightarrow X$ such that the log pullback Γ of Δ is effective.

Now suppose further that (X, Δ) is log smooth, that is X is smooth and the support of Δ has normal crossings. Suppose also that the coefficients of Δ have the form $\frac{m-1}{m}$. Then the **orbifold fundamental group** $\pi_1^{\text{alg}}(X, \Delta)$ is the quotient of the fundamental group $\pi_1(U)$ of U corresponding to those topological Galois covers $Y \longrightarrow X$ whose ramification over each component D of Δ divides m, where the coefficient of m in Δ is of the form $\frac{m-1}{m}$.

It is easy to check that if the pair (X, Δ) is log smooth and G is a finite group, then G is a finite quotient of the orbifold fundamental group iff G is a quotient of the orbifold algebraic fundamental group.

1.5 Example. Suppose that $X = \mathbb{P}^1$ and $\Delta = \sum_i \frac{m_i - 1}{m_i} p_i$. Then it is well-known result that the orbifold fundamental group of the pair (X, Δ) is finite iff $\sum_i \frac{m_i-1}{m_i} < 2$. The latter condition is equivalent to the condition that $-(K_X + \Delta)$ is ample.

Using (1.3) we are able to prove that there is a bound on the maximum degree of a finite cover so that at least the orbifold algebraic fundamental group is finite.

1.6 Corollary. *Let (X, Δ) be a kawamata log terminal log pair such that $-(K_X + \Delta)$ is big and nef.*

Then the orbifold algebraic fundamental group of the pair (X, Δ) is finite.

In particular let U be any open subset of X whose complement is of codimension at least two. Then the algebraic fundamental group of U is finite.

Problems centered around boundedness of Fano varieties have received considerable attention. Nikulin [20] proved (1.2) in dimension two and Borisov [4] proved (1.2) in dimension three. Nadel [19] proved the boundedness of smooth Fano varieties with Picard number one. As previously mentioned Kollár, Miyaoka and Mori [15] proved the boundedness of smooth Fano varieties in every dimension. Recently Clemens and Ran [21] proved the boundedness of a restricted class of Fano varieties with Picard number one.

Most proofs begin with an idea that goes back to Fano. If the degree is very large then by Riemann Roch for any point p belonging to the smooth locus we may find a $\mathbb{Q}$-divisor B $\mathbb{Q}$-linearly equivalent to $-K_X$ with very high multiplicity at p. The question is how to use the existence of B to obtain a contradiction.

[15] and [6] use reduction modulo p to produce chains of rational curves of small degree connecting any two points. This easily bounds the multiplicity of B and hence the degree.

Borisov [4] uses a similar argument. Unfortunately, due to the presence of singularities, this is extremely delicate and it seems hard to generalise this method to higher dimensions.

Nadel [19] uses the fact that through any point of X there is a rational curve of low degree. If this curve intersects B at p then it must lie in B. He then produces a covering family of varieties V_t with the same property, namely if a rational curve of low degree meets V_t then in fact it must lie in V_t. It is easy to see that this cannot happen when the Picard number is one. The locus V_t is essentially the locus where the divisor B has the same high multiplicity.

Clemens and Ran [21] show that certain sheaves of differential operators are semi-positive. Note that the condition that B has at least a given multiplicity at a point is equivalent to saying that any derivative, of a local defining equation, of order up to the multiplicity vanishes. B then corresponds to a quotient of negative degree of some sheaf of differential operators of high degree on a general curve passing through p. This clearly contradicts semi-positivity.

We use entirely different ideas. The main ideas of our proof are derived from the X-method, which was developed to prove the Cone Theorem. We make use of some of the sophisticated ideas introduced, especially by Kawamata, in an attempt to solve Fujita's conjecture. Another key feature of our proof is to use the connectedness of the locus of log canonical singularities. Connectedness of the locus of log canonical singularities was first observed by Shokurov in the case of surfaces. Kollár proved that connectedness holds in general as a slick application of Kawamata-Viehweg vanishing. As such, just as with the proof of Clemens and Ran [21], we do not use reduction modulo p. In particular we give the first proof of boundedness of smooth Fano varieties without using reduction modulo p.

Instead of producing chains of rational curves of low degree, we produce covering chains of positive dimensional Fano varieties V_t of low degree. Unfortunately we are unable to smooth these varieties, as one can smooth chains of rational curves, see for example [12], and these chains only exist if the degree is sufficiently large. Moreover it is far more problematic to show directly that the relevant intersection numbers are bounded, due to problems of excess intersection which do not arise when one intersects a divisor and a curve. On the other hand these subvarieties are log canonical centres of appropriate log divisors and as such they have a very rich geometry. More precisely

1.7 Theorem. *Let (X, Δ) be a log pair where X is projective of dimension $n > 1$, Δ is an effective divisor and $-(K_X + \Delta)$ is big and nef.*

(1) *If $d > (n!)^n$ then there is a subvariety of the Hilbert scheme of X such that if $f \colon Y \longrightarrow B$ is the normalisation of the universal family and $\pi \colon Y \longrightarrow X$ is the natural morphism, then π is birational, f is a contraction morphism and the fibres of f are Fano varieties and so rationally connected.*

(2) *If further* $d \geq (2^n n!)^n$ *then the fibres of* f *have degree at most* $(2^{k+1}k!)^k$.

(3) *If* $r(K_X + \Delta)$ *is Cartier for some positive integer* r *and*

$$d \geq (2n)^n \left(2^n r(n-1)!\right)^{n(n-1)}$$

then the log pullback Γ *of* Δ *is effective in a neighbourhood of the generic fibre of* f.

(4) *If* $K_X + \Delta$ *is kawamata log terminal of log discrepancy at least* ϵ *and*

$$d \geq \left(\frac{2n}{\epsilon}\right)^n \left(2^n r(n-1)!\right)^{n(n-1)}$$

then π *is small in a neighbourhood of the generic fibre of* f.

A key part of the proof of (1.7) is that we place almost no restrictions on the type of singularities of the pair (X, Δ). Furthermore, even if one were to start with a smooth variety X, with empty divisor Δ, then inductively one needs to deal with the case of an arbitrary log pair, so working at this level of generality is crucial to the proof.

It is not so hard to prove (1.3) using (1.7). For example it is easy to use (1.7) to obtain an explicit bound on the degree when the Picard number is one.

1.8 Corollary. *Let* (X, Δ) *be a kawamata log terminal log pair. If* X *is* $\mathbb{Q}$*-factorial of Picard number one and* $-r(K_X + \Delta)$ *is Cartier and ample, then the degree is at most*

$$(2nr)^n \left(2^n r(n-1)!\right)^{n(n-1)}.$$

I also hope that (1.2) will be the first (and very small) step towards an eventual proof of a very interesting conjecture, due independently to Alexeev and Borisov:

1.9 Conjecture. *Fix an integer* n *and a positive real number* ϵ.

Then the family of all varieties X *such that*

(1) *the log discrepancy of* X *is at least* ϵ *and*

(2) $-K_X$ *is ample,*

is bounded.

Indeed by (1.2) it suffices to bound the index of K_X. In fact more is true; the proof of (1.3) shows that if (1.9) is true in dimension $n-1$ then the degree of $-K_X$ in dimension n is bounded in terms of the log discrepancy. To the best of my knowledge, this is in fact the first concrete evidence in dimension n, for (1.9) beyond that of the evidence of toric varieties.

Alexeev [1] proved (1.9) in dimension two and Borisov and Borisov [5] proved (1.9) for toric varieties. Kawamata [7] proved (1.9) for Fano threefolds with Picard number one and terminal singularities and Kollár, Miyaoka, Mori and Takagi [16] proved that all Fano threefolds with canonical singularities are bounded. Despite this (1.3) has not even been resolved in dimension three. It is also interesting to note that recently Lin has proved that one cannot drop the reference to the log discrepancy ϵ in (1.9) even in the case where one replaces bounded by birationally bounded, see [18].

Note that if a threefold X is uniruled (that is X is covered by rational curves) then, as a consequence of the MMP in dimension three, X is birational to a Mori fibre space, $\pi\colon Y \longrightarrow S$. Quite surprisingly it turns out that many Fano varieties are birationally rigid in the sense that there is only one Mori fibre structure π, up to birational equivalence. REFERENCES? In higher dimensions the existence of the MMP has not yet been established. For this reason, we adopt the following ad hoc definition of birational rigidity in the general case.

1.10 Definition. We will say that a variety X is **birationally rigid** if for any birational model Y of X and any two covering families of curves C_i on Y, such that $-K_X \cdot C_i < 0$, then C_1 is numerically equivalent to a multiple of C_2.

Note that this definition does indeed imply the more usual one, at least in dimension three, by a result of Batryev [3] . With this said, (1.7) has the following unexpected consequence

1.11 Corollary. *Let (X, Δ) be a log pair where X is projective of dimension $n > 1$, Δ is an effective divisor and $-(K_X + \Delta)$ is big and nef.*

If $d > (n!)^n$ then X is not birationally rigid.

One wonders if there is an even stronger result than (1.11)

1.12 Conjecture. *Fix an integer n. Then the family of log pairs (X, Δ) such that*

(1) *$K_X + \Delta$ is kawamata log terminal,*

(2) *$-(K_X + \Delta)$ is ample, and*

(3) *X is birationally rigid*

is bounded.

(1.12) provides an obvious strategy to prove (1.9). Note also that (1.12) is only of interest when the Picard number is one. It might also be interesting to study covering families of tigers in other contexts especially when the degree is not necessarily large. Indeed there are many analogies between the geometry of covering families of tigers and the geometry of covering families of rational curves, some of which we draw out below. Finally it might be of interest to try to connect questions of boundedness with the phenomena of Mirror symmetry. Typically Mirror symmetry associates to one moduli space of Calabi-Yaus another such space. However in some degenerate cases it switches Calabi-Yaus and Fano varieties. Moreover many of the constructions of Mirror symmetry implicitly involve Fano varieties.

Sketch of Proof of (1.3): To navigate through the proof (1.3), let us use the example of cones over a rational normal curve of degree d. The first step is to pick up a covering family of tigers, see §3. As pointed out above this corresponds to finding a divisor Δ_p passing through p of very large multiplicity. The whole point of covering families of tigers, is that we only really need to consider what happens at a general point p of X (see for example (3.2)) and so in practice when we need to apply adjunction or inversion of adjunction, (5.2) and (6.1) we can treat the pair (X, Δ) as though it were log smooth. Roughly speaking the locus V_t is where Δ_t has very large multiplicity. In this sense a covering family of tigers V_t is somewhat similar to the covering family constructed by Nadel.

In the case of a cone there are three obvious candidates,

(1) The family V_t of lines, where $\Delta_t = V_t$.

(2) A pencil of hyperplane sections W_s, where $\Gamma_s = W_s$.

(3) The family of points p_r on S, where $\Theta_r = \Delta_t + \Gamma_s$.

As $-K_S = (d+2)\Delta_t$, the weight of the first family is $d+2$ and as $-K_S = \frac{d+2}{d}\Gamma_s$ the weight of the second family is $\frac{d+2}{d}$. The third family has weight $\frac{d+2}{d+1}$, the harmonic sum of the first two.

The dimension of the first two families is obviously one and of the last zero. In practice then we never choose the second or third family as their weights are not large enough.

The next step is to refine the covering family V_t so that it forms a birational family, see §4. In the case of a cone this is automatic. In the general case it means passing to a covering family of maximal dimension. The idea is to average over all divisors Δ_t passing through p, see (4.2). This is somewhat analogous to growing chains of rational curves. The eats up some of the weight and we end up with a birational family of much smaller weight.

The third step is to decrease the degree of V_t, see §5. Again in the case of a cone this is automatic. In the general case it means passing to a birational family of minimal dimension. The idea is that if the degree of V_t is very large then we can lift a birational family from V_t to the whole variety, see (5.3). This is quite similar to bend and break for curves. This eats up more of the weight and we end up with a birational family of smaller weight still.

Comparing the first family with the second it becomes clearer why we prefer the first; the degree of V_t is bounded whilst the degree of W_t is arbitrarily large.

A key observation is that in the Fano case the family must be non-trivial (that is V_t is not of dimension zero) provided the weight is greater than two, see (3.4); this is a simple consequence of connectedness. This fact and the fact that we have to refine our covering family twice is the reason for the rather large numbers that appear in (1.7).

In the final step we want to connect the log discrepancy to the degree of the fibres, see §6. To do this consider the natural diagram determined by a covering family of tigers (3.1.1). In the case of a cone we get

$$\begin{tikzcd} V_t \arrow[d] & \subset & T \arrow[r, "\pi"] \arrow[d, "f"'] & S, \\ t & \in & \mathbb{P}^1 \end{tikzcd}$$

where T is isomorphic to $\mathbb{F}_d$. The fibres of f are isomorphic to V_t. Thus the fibres of f are of bounded degree. On the other hand if we pick two fibres of f, V_s and V_t then they are not connected but their images in X are connected. Again by connectedness it follows that there is a log canonical centre which connects them. In our case this is the exceptional divisor E of π. As the log discrepancy of S is $2/d$ it follows that the coefficient of E with respect to $\Delta_t + \Delta_s$ is at least $2/d$, that is, by symmetry, the coefficient of E with respect to Δ_t is at least $1/d$. (In fact it is exactly $1/d$ corresponding to the fact that $K_S + \Delta_s + \Delta_t$ is log canonical.) The general case, when π is not small, is not much harder, see (6.2).

By definition of the weight, we can find a $\mathbb{Q}$-divisor $B \sim_{\mathbb{Q}} -(K_X + \Delta)$ such that the coefficient of E with respect to B is at least w/d. Now E is a section of f and $\pi^*(K_X + B)$ restricts to a trivial divisor on the general fibre of f. Thus $w/d \leq 2$ and so $w \leq 2d$. In turn this bounds the degree of $-(K_X + \Delta)$. This gives a bound on the degree of S only in terms of the log discrepancy. In general if π is not small, a similar proof applies, see (6.4). This case is similar but more subtle as in general the log canonical centre connecting V_s and V_t in Y need not be a divisor, see (6.5). This is the only place where the argument is not effective.

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§2 Calculus of log canonical centres

We refer the reader to [14] for the various notions of log terminal and log canonical. We first recall some useful definitions due to Kawamata [8] and collect together some known results that we require for the proof of (1.2) and (1.7).

2.1 Definition. Let (X, Δ) be a log pair. A subvariety V of X is called a **log canonical centre** if it is the image of a divisor of log discrepancy at most zero. A **log canonical place** is a valuation corresponding to a divisor of log discrepancy at most zero. The **log canonical locus** $\mathrm{LLC}(X, \Delta)$ of the pair (X, Δ) is the union of the log canonical centres. We will say that a log canonical centre is **pure** if $K_X + \Delta$ is log canonical at the generic point of V. If in addition there is a unique log canonical place lying over the generic point of V we will say that V is **exceptional**.

Log canonical centres enjoy many special properties. We collect together six of the most important for us.

2.2 Lemma. *(Calculus of Log canonical centres) Let (X, Δ) be a log pair.*

(1) *(Connectedness) If $f \colon X \longrightarrow Z$ is a morphism with connected fibres such that the image of every component of Δ with negative coefficient is of codimension at least two in Z and $-(K_X + \Delta)$ is big and nef over Z then the intersection of $\mathrm{LLC}(X, \Delta)$ with every fibre is connected.*

(2) $\mathrm{LLC}(X, \Delta)$ *is a finite union of log canonical centres.*

(3) *(Convexity) If V is a log canonical centre of $K_X + \Delta + \sum_{i=1}^{d} \Delta_i$, then there is an i such that V is a log canonical centre of $K_X + \Delta + d\Delta_i$.*

Now suppose that in addition Δ is effective and that there is a boundary $\Delta' \leq \Delta$ such that $K_X + \Delta'$ is kawamata log terminal (this always holds for example if X is log terminal and $\mathbb{Q}$-factorial).

(4) *The intersection of two log canonical centres is a union of log canonical centres.*

(5) *If p is a point of* $\mathrm{LLC}(X,\Delta)$ *then there is a unique minimal log canonical centre V containing p.*

(6) *Suppose V is a pure log canonical centre of (X,Δ) and D is a $\mathbb{Q}$-Cartier divisor of the form $A+E$, where A is ample and E is effective. If V is not contained in the support of E, then for all ϵ sufficiently small, there exists a $\mathbb{Q}$-divisor $\Gamma \sim_{\mathbb{Q}} aD$, such that V is an extremal log canonical centre of $(X, \Delta - \epsilon(\Delta - \Delta') + \Gamma)$, where a approaches zero as ϵ approaches zero.*

Proof. (1) follows immediately from (17.4) of [14]. (2) follows by taking a resolution of the pair (X,Δ) and observing that $\mathrm{LLC}(X,\Delta)$ is the image of the exceptional divisors of log discrepancy less than or equal to zero. Note that the convex linear combination of divisors which are kawamata log terminal at the generic point of V is kawamata log terminal at the generic point of V. As $\sum \Delta_i$ is a convex linear combination of the divisors $d\Delta_i$, (3) follows. (4) is (1.5) of [8] and (5) is immediate from (4). (6) is observed in [9]. $\square$

In a recent preprint of Ambro [2] it is proved that (2.2.4) and (2.2.5) hold, even without the unnatural hypothesis on the existence of Δ'.

§3 Covering families of tigers

3.1 Definition. Let (X,Δ) be a log pair, where X is projective. We say that pairs of the form (Δ_t, V_t) form a **covering family of tigers of dimension k and weight w measured against D** if

(1) There is a projective morphism $f\colon Y \longrightarrow B$ of normal projective varieties and an open subset U of B such that the fibre of π over $t \in U$ is V_t.

(2) There is a morphism of B to the Hilbert scheme of X such that B is the normalisation of its image and f is obtained by taking the normalisation of the universal family.

(3) If $\pi\colon Y \longrightarrow X$ is the natural morphism then $\pi(V_t)$ is a pure log canonical centre of $K_X + \Delta + \Delta_t$.

(4) π is dominant.

(5) π is finite.

(6) $D \sim_{\mathbb{Q}} w\Delta_t$, where Δ_t is effective and D is a $\mathbb{Q}$-Cartier divisor.

(7) the dimension of V_t is k.

We will say that the family is a **birational family of tigers** if

(8) $\pi\colon Y \longrightarrow X$ is birational.

Further if

(9)

$$w > 2^n(n-k)!$$

then we will say that (Δ_t, V_t) forms a **covering family of tigers of large weight**.

Finally if

(10)

$$w > 2^{k+1}$$

and the family is birational then we will say that (Δ_t, V_t) forms a **birational family of tigers of large weight**.

The following commutative diagram is perhaps worth a couple of words:

(3.1.1)

$$\begin{tikzcd}
V_t \arrow[d] & \subset & Y \arrow[r, "\pi"] \arrow[d, "f"'] & X. \\
t & \in & B &
\end{tikzcd}$$

Note that $\pi|_{V_t} : V_t \longrightarrow \pi(V_t)$ is finite and birational. We will often abuse notation and identify V_t with its image. Clearly (3.1.6) may be rewritten as $\Delta_t \sim_{\mathbb{Q}} \frac{1}{w}D$. In particular the weight of a sum of divisors is the harmonic sum (that is the reciprocal of the sum of the reciprocals) of the weights. The whole point of (3.1) is that covering family of tigers enjoy some very special properties. When we apply (3.1), we will always take $D = -(K_X + \Delta)$. As the general case is no harder and with a view towards other possible applications, we take D to be arbitrary.

We first show that covering families of tigers exist under weak conditions and at the same time justify the notation of (3.1).

Given a topological space X, we will say that a subset P is countably dense if P is not contained in the union of countably many closed subsets of X. Note that if we decompose as P a countable union

$$P = \bigcup_{m \in \mathbb{N}} P_m$$

then at least one P_m is countably dense.

3.2 Lemma. *Let (X, Δ) be a log pair, where X is projective and let D be a big $\mathbb{Q}$-Cartier divisor. Let w a positive rational number. Let P be a countably dense subset of X. Suppose that for every point $p \in P$ we may find a pair (Δ_p, V_p) such that $p \in V_p$ and V_p is a pure log canonical centre of $K_X + \Delta + \Delta_p$ where $\Delta_p \sim_{\mathbb{Q}} D/w_p$ for some $w_p > w$.*

Then we may find a covering family of tigers of weight w measured against D, together with a countably dense subset Q of P such that for all $p \in Q$, V_p is a fibre of π.

Proof. As D is big, it is of the form $A + E$ where A is ample and E is effective. By (2.2.6), possibly passing to a subset of P, we may assume that V_p is an exceptional log canonical centre. Adding on a multiple of D we may assume that $w_p = w$.

We may find an integer m so that mD/w is integral. Moreover for each $p \in P$ there is an integer m_p such that $m_p\Delta$ is linearly equivalent to m_pD/w. Hence using the observation above, we may assume there is a countably dense subset Q of P such that for all $p \in Q$, $m\Delta_p \in |mD/w|$. Let B be the closure inside this linear system of the points corresponding to $m\Delta_p$ for p in Q. Replace B by an irreducible component that contains a countably dense set of points Q in X and let $f\colon H \longrightarrow B$ be the universal family. Pick a log resolution of the generic fibre of the universal family and extend this to an embedded resolution over an open subset U of B. By assumption there is a unique exceptional divisor of log discrepancy zero over the generic point corresponding to the log canonical centres V_p, for $p \in Q$.

Thus possibly taking a finite cover of B and passing to an open subset of U we may assume that there is a morphism $f \colon Y \longrightarrow B$ whose fibre V_t over $t \in U$ is a log canonical centre of $K_X + \Delta + \Delta_t$. Possibly passing to an open subset of U we may assume that f is flat, so that U maps to the Hilbert scheme. Replacing B by the normalisation of the closure of the image of U in the Hilbert scheme and Y by the normalisation of the pullback of the universal family we may assume that (3.1.1-2) hold. Cutting by hyperplanes in B we may assume that f is finite. The result is now clear. $\square$

We will use (3.2) repeatedly, often without comment.

3.3 Lemma. *Let (X, Δ) be a log pair, where X is a projective variety of dimension n and suppose that D is big and nef. Let w be a positive rational number and let p and q be two points of X contained in the smooth locus of X but not contained in a component of Δ of negative coefficient. Let d be the degree of D.*

(1) *If $d > (wn)^n$ then we may find an effective divisor $\Gamma \sim_{\mathbb{Q}} D/w$ such that $\mathrm{LLC}(X, \Delta + \Gamma)$ contains p.*

(2) *If $d > (wn)^n$ then we may find a covering family of tigers of weight w.*

(3) *If $d > (2^n n!)^n$ then we may find a covering family of tigers of large weight.*

Now assume further that Δ is effective and that $-(K_X + \Delta + D)$ is big and nef.

(4) *If $d > (2n)^n$ then there is an effective divisor $\Gamma \sim_{\mathbb{Q}} D$ and a chain $V_1, V_2, \ldots, V_k$ of log canonical centres of $K_X + \Delta + \Gamma$ such that $p \in V_1$, $q \in V_k$ and $V_i \cap V_{i+1} \neq \emptyset$, for $i \leq k-1$.*

Proof. (1) is well known, see for example (7.1) of [10] for a proof. (2) follows from (1) and (3.2). (3) follows from (2).

By (1) we may find a divisor $\Gamma_p \sim_{\mathbb{Q}} D/2$ (respectively Γ_q) such that $K_X + \Delta + \Gamma_p$ is not kawamata log terminal at p (respectively q). Let $\Gamma = \Gamma_p + \Gamma_q \sim_{\mathbb{Q}} D$. Then $K_X + \Delta + \Gamma$ is not kawamata log terminal at p and q and $-(K_X + \Delta + \Gamma)$ is big and nef. (4) now follows from (2.2.1). $\square$

3.4 Lemma. *Let (X, Δ) be a log pair, where X is projective and Δ is effective and let D be a $\mathbb{Q}$-Cartier divisor such that $-(K_X + \Delta + D)$ is big and nef. Let (Δ_t, V_t) be a covering family of tigers of weight w greater than two.*

Then $k > 0$.

Proof. Suppose not. Then for general p_1 and p_2 we may find Δ_1 and Δ_2 such that $\Delta_i \sim_{\mathbb{Q}} D/w$ where p_i is an exceptional log canonical centre of $K_X + \Delta + \Delta_i$ and $w > 2$. As p_1 and p_2 are general it follows that Δ_2 does not contain p_1. Thus p_1 is an exceptional log canonical centre of $K_X + \Delta + \Delta_1 + \Delta_2$ and hence $\mathrm{LLC}(X, \Delta + \Delta_1 + \Delta_2)$ is not connected. On the other hand as $w > 2$, $-(K_X + \Delta + \Delta_1 + \Delta_2)$ is big and nef. This contradicts (2.2.1) (or indeed (3.3.4)). $\square$

§4 Covering families of maximal dimension

We need an easy Lemma that bounds, both from below and above, the multiplicity of a log canonical divisor at a smooth point of a variety. Even though we are only interested in the case of effective divisors, for the purposes of induction it is expedient to allow negative coefficients.

4.1 Lemma. *Let X be a smooth variety of dimension n. Let D be a normal crossings divisor, let Δ be a boundary and Γ an effective divisor. Let Z be a pure log canonical centre of $K_X + D + \Delta - \Gamma$ of dimension k. Let μ be the multiplicity of Δ at the generic point of Z and let a be the maximum coefficient of D. Assume that $a < 1$.*

(1) *If Γ is empty then $\mu \leq n - k$.*
(2) *$\mu > 1 - a$.*

Proof. The result is local about the generic point of Z, and so we may as well assume that X is affine. Cutting by hyperplanes in X we may assume that Z is a point p. Let $\pi\colon Y \longrightarrow X$ denote the blow up of X at p with exceptional divisor E. Then the log discrepancy of E with respect to $K_X + \Delta$ is $n - \mu$, which is at least zero as $K_X + \Delta$ is log canonical. Hence (1).

On the other hand, by assumption there is an algebraic valuation ν of log discrepancy zero with respect to $K_X + \Delta - \Gamma$. ν determines a series of blows ups, each with centre ν, such that eventually the centre of ν is a divisor, see for example (2.45) of [17]. Suppose that $\mu \leq 1 - a$. We have

$$K_Y + D' + E = \pi^*(K_X + D) + bE,$$

where D' is the strict transform of D and $b \geq (n - na) = n(1 - a)$. Thus the log discrepancy of E with respect to $K_X + D + \Delta - \Gamma$ is at least

$$n(1 - a) - \mu \geq (n - 1)(1 - a) \geq (1 - a).$$

Moreover the coefficient of E is then at most a, $D' \cup E$ still has normal crossings in a neighbourhood of the generic point of the centre of ν, and the multiplicity of the strict transform of Δ is at most the multiplicity of Δ. By induction on the number of blow ups, it follows that ν has log discrepancy at least $1 - a$, a contradiction. $\square$

4.2 Lemma. *Let (X, Δ) be a log pair, where X is a normal projective variety and let D be a big and nef divisor. Let (Δ_t, V_t) be a covering family of tigers of weight less than w and dimension k.*

If (Δ_t, V_t) is not birational then we may find a covering family of tigers (Γ_s, W_s) of weight $w/(n - k)$ and dimension l, where either

(1) *$l > k$, or*

(2) *$l < k$ and (Γ_s, W_s) is a birational family.*

Proof. Let d be the degree of π. By assumption $d > 1$. Pick an open subset $U \subset X$ contained in the smooth locus of X and the complement of the support of Δ such that for all $p \in U$ there exist $t_1, t_2, \dots, t_d$ points of B such that $p \in V_t$ iff $t = t_i$, $i = 1 \dots d$. Set $\Delta_i = \Delta_{t_i}$ and $V_i = V_{t_i}$. Let $G = \sum \Delta_i$ and pick ϕ such that $K_X + \Delta + \phi G$ is maximally log canonical at p. As p is a smooth point of X, by (4.1) we have

$$n - k \geq \mu_p(\phi G) = \phi \sum_i \mu_p(\Delta_i) \geq d\phi.$$

Let W be the unique minimal log canonical centre of $K_X + \Delta + \phi G$ containing p. There is an open subset of $V_1 \cap V_2 \cap \cdots \cap V_d$ such that as we vary p in this set, W does not change. Thus we may assume $V_1 \cap V_2 \cap \cdots \cap V_d \subseteq W$. If we have equality then by (3.2) there is a small perturbation Γ of ϕG such that (Γ_s, W_s) forms a covering family of weight $w/(n-k)$, which is obviously birational.

Thus we may assume that $W \subsetneq V_1$. By symmetry and (2.2.3) W is a log canonical centre of $K_X + d\phi\Delta_1$. Pick $\lambda \leq d\phi$ so that W is a pure log canonical centre of $K_X + \Delta + \lambda\Delta_1$. Now varying $p \in V_1$ it follows that there is an irreducible component of $\mathrm{LLC}(X, \Delta + \lambda\Delta_1)$ that contains V_1 as a proper subset. By (2.2.2), we may assume that V_1 is a proper subset of a log canonical centre W_1 of $\mathrm{LLC}(X, \Delta + \lambda\Delta_1)$. It follows by (3.2) that there is a small perturbation Γ of $\lambda\Delta_1$, so that (Γ_s, W_s) is a covering family of tigers of weight $w/(n-k)$ and greater dimension. $\square$

4.3 Lemma. *Let (X, Δ) be a log pair, where X is a normal projective variety. Let D be a big and nef divisor. Let (Δ_t, V_t) be a covering family of tigers of large weight w and maximal dimension k.*

Then either

(1) *(Δ_t, V_t) is a birational family of tigers of large weight, or*

(2) *there is a birational family (Γ_s, W_s) of tigers of large weight and smaller dimension.*

Proof. By assumption $w > 2^n(n-k)! \geq 2^{k+1}$. If (Δ_t, V_t) is a birational family of tigers (1) holds and there is nothing to prove. Otherwise by (4.2) there is a covering family of tigers of weight $w' > 2^n(n-k-1)!$. The dimension of this family cannot be larger than k, otherwise (Γ_s, W_s) would be a covering family (Γ_s, W_s) of tigers of large weight and greater dimension, which contradicts the maximality of (Δ_t, V_t). By (4.2) the only other possibility is that (Γ_s, W_s) is a birational family of tigers of large weight and smaller dimension and we have (2). $\square$

§5 Birational Families of minimal dimension

We need a form of adjunction and inversion of adjunction. Conjecturally there ought to be a strong form of adjunction for the normalisation of any log canonical centre. Unfortunately inversion of adjunction is only known for divisors and adjunction is only known for some special centres (for example minimal), see [9] and [2]. Fortunately we only need adjunction and inversion of adjunction in very special cases. We state one form of adjunction to do with log canonical centres here. The other form concerns birational families, see (6.1).

5.1 Lemma. *Let $\pi\colon Y \longrightarrow X$ be a smooth morphism of smooth varieties. Let Δ_1 be a $\mathbb{Q}$-divisor on X and let Γ_1 be the pullback of Δ_1 to Y. Let Γ_2 be a boundary on Y, such that the support B of Γ_2 dominates X and $\pi|_B$ is smooth.*

Then (X, Δ) is log canonical (respectively kawamata log terminal, etc.) iff $(Y, \Gamma = \Gamma_1 + \Gamma_2)$ is log canonical (respectively kawamata log terminal, etc.).

Proof. The property of being log canonical is local in the analytic topology. On the other hand, locally in the analytic topology, Y is a product $X_1 \times X_2$, where X_1 is isomorphic to X and Γ_2 is the pullback of a divisor Δ_2 whose support is smooth, so that $\Gamma = \Gamma_1 \times X_2 + X_1 \times \Gamma_2$. But the result is easy in this case, see for example (8.21) of [13]. $\square$

5.2 Lemma. *Let (X, Δ) be a log pair where X is projective and Δ is effective. Suppose that V is an exceptional log canonical centre of $K_X + \Delta$. Then there is an open subset U of the smooth locus of V with the following property:*

For all divisors Θ on X, which do not contain the generic point of V and subvarieties W of V such that $W|_U$ is a pure log canonical centre of $K_U + \Theta|_U$ then W is a pure log canonical centre of $K_X + \Delta + \Theta$.

Proof. This result is local about the generic point of V so we are free to replace X by any open set that contains the generic point of V. The idea is to reduce to the case of a divisor, when the result becomes an easy consequence of inversion of adjunction. Pick a log resolution $\pi\colon Y \longrightarrow X$ of the pair (X, Δ) and let Γ be the log pullback of Δ. By assumption there is a unique divisor E lying over V of log discrepancy zero. Let $f\colon E \longrightarrow V$ be the restriction of π to E. Replacing

X by an open subset, we may assume that f and V are both smooth, and that K_V and $K_X+\Delta$ are $\mathbb{Q}$-linearly equivalent to zero. By adjunction we may write

$$(K_Y+\Gamma)|_E = K_E + B,$$

for some effective divisor B, where both sides are $\mathbb{Q}$-linearly equivalent to zero. Possibly passing to a smaller open set we may assume that every component of B dominates V and that $f|_B$ is smooth. Then

$$K_E + B = (K_Y+\Gamma)|_E = \pi^*(K_X+\Delta)|_E = f^*((K_X+\Delta)|_V) \sim_{\mathbb{Q}} f^*(K_V).$$

Suppose that W is a pure log canonical centre of $K_V+\Theta|_V$. Set $\Theta' = \pi^*\Theta$. As Θ does not contain the generic point of V, E is not a component of Θ', so that $f^*(\Theta|_V) = \Theta'|_E$. It follows by (5.1) that $f^{-1}(W)$ is a pure log canonical centre of $K_E+\Theta'|_E$. The result now follows as in the proof of (17.1.1), (17.6) and (17.7) of [14]. $\square$

5.3 Lemma. *Let (X,Δ) be a log pair and let D be a divisor of the form $A+E$ where A is ample and E is effective. Let (Δ_t, V_t) be a covering family of tigers of weight greater than w and dimension k. Let A_t be the restriction of A to V_t. Suppose that there is an open set of B such that for all $t \in B$ we may find a covering family of tigers $(\Gamma_{t,s}, W_{t,s})$ on V_t of weight, with respect to A_t, greater than w'.*

Then we may find a covering family of tigers (Γ_s, W_s) of dimension less than k and weight

$$w'' = \frac{1}{w} + \frac{1}{w'} = \frac{ww'}{w+w'}.$$

Further if both (Δ_t, V_t) and $(\Gamma_{t,s}, W_{t,s})$ are birational families then so is (Γ_s, W_s).

Proof. Pick r so that rA is Cartier and let $L = \mathcal{O}_X(rA)$ be the corresponding line bundle. Note that by Serre vanishing $H^1(X, \mathcal{I}_V \otimes L^{\otimes m}) = 0$ for m large enough. Hence we may lift $\Gamma_{t,s}$ to a $\mathbb{Q}$-divisor $\Theta_{t,s}$ on X $\mathbb{Q}$-linearly equivalent to the same multiple of A. Adding on a multiple of E we may assume that $\Theta_{t,s}$ is in fact a multiple of D. Thus by (5.2) and (3.2) applied to $(\Delta_t + \Theta_{t,s}, W_{t,s})$ we may find a covering family of tigers (Γ_s, W_s) of weight w''. The rest is clear. $\square$

5.4 Lemma. *Let (X, Δ) be a log pair and let D be a big and nef divisor. Let (Δ_t, V_t) be a birational family of tigers of large weight with minimal dimension k.*

Then $D^k \cdot V_t$ is at most $(2^{k+1}k!)^k$.

Proof. Suppose that $D^k \cdot V_t > (2^{k+1}k!)^k$. Since D is big we may write $D \sim_{\mathbb{Q}} A + E$, where A is ample and E is effective. By Kleiman's criteria for ampleness, as D is nef, $D - \epsilon E$ is ample for any $0 < \epsilon \leq 1$. Thus we may assume that $A^k \cdot V_t > (2^{k+1}k!)^k$. Set $A_t = A|_{V_t}$. Then $(1/2A_t)^k > (2^k k!)^k$. By (4.2) there is a birational family of tigers $(\Gamma_{t,s}, W_{t,s})$ of weight greater than 2^k for $K_{V_t} + \Gamma_t$ and $1/2A_t$, that is of weight greater than 2^{k+1} for A_t. By (5.3) it follows that there is a birational family of tigers (Γ_s, W_s) of dimension less than k and weight greater than 2^k, that is of large weight, which contradicts the minimality of (Δ_t, V_t). $\square$

§6 A bound for the degree of a birational family in terms of the log discrepancy

Here is the other version of adjunction we shall need, mentioned in §5.

6.1 Lemma. *Let X be a projective variety and suppose we are given a subvariety of the Hilbert scheme of X such that if $f : Y \longrightarrow B$ is the normalisation of the universal family and $\pi : Y \longrightarrow X$ is the natural morphism, then π is birational. Let Δ be a $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier.*

Then there is an open subset $U \subset B$ such that for all $t \in U$, the fibre V_t of f over t is smooth in codimension one and we may find Γ_t such that

$$(K_X + \Delta)|_{V_t} = K_{V_t} + \Gamma_t.$$

Moreover the pair (V_t, Γ_t) has the following properties.

(1) *Γ_t is effective iff the log pullback Γ of Δ is effective in a neighbourhood of the generic fibre of f.*

(2) *If Γ is effective in a neighbourhood of the generic fibre of f and $K_X + \Delta$ is kawamata log terminal then V_t is normal.*

(3) *The log discrepancy of* (V_t, Γ_t) *is at least the log discrepancy of* (X, Δ).

Proof. It suffices to define Γ_t and to prove (1-3) on Y, in a neighbourhood of the generic fibre of f.

The generic fibre of f is certainly smooth in codimension one and so the fibres of f are certainly smooth in codimension one over an open subset. Thus over the same open subset $K_X|_{V_t} = K_{V_t}$ and so we can define Γ_t to simply be the restriction of Γ to V_t. (1) is then clear.

If Γ_t and $K_X + \Delta$ is kawamata log terminal it follows that $K_Y + \Gamma$ is kawamata log terminal of log discrepancy at least the log discrepancy of $K_X + \Delta$ in a neighbourhood of the generic fibre of f. In particular it follows that Y is Cohen Macaulay in a neighbourhood of the generic fibre of f and so the fibres of f are Cohen Macaulay over an open subset. In particular V_t is normal. Hence (2).

Picking a resolution of the generic fibre and possibly passing to a smaller open set, (3) follows easily. $\square$

6.2 Lemma. *Let* (X, Δ) *be a log pair, where* X *is projective. Let* D *be any* $\mathbb{Q}$*-Cartier divisor and* (Δ_t, V_t) *a birational family of tigers. Set* $\Gamma_t = \pi^*\Delta_t$.

Then there is an open subset U *of* B *such that if* $t \in U$ *then every component* Z *of the* π*-exceptional locus which dominates* B *is a log canonical centre of* $K_X + \Gamma + 2\Gamma_t$*. In particular every* π*-exceptional divisor has log discrepancy at most zero with respect to* $K_X + \Gamma + 2\Gamma_t$.

Proof. Pick $s \neq t \in B$. Then V_s and V_t are log canonical centres of $K_Y + \Gamma + \Gamma_s + \Gamma_t$. Thus (2.2.1) implies there is some log canonical centre $Z_{s,t}$ contained in Z that connects V_s and V_t. By symmetry and (2.2.3) we may find an open subset $U \subset B$ such that $Z_{s,t}$ is a log canonical centre of $K_Y + \Gamma + 2\Gamma_t$. Varying s, $Z_{s,t}$ sweeps out Z and so by (2.2.2) it follows that Z is a log canonical centre of $K_Y + \Gamma + 2\Gamma_t$. $\square$

6.3 Lemma. *Let* X *be a scheme of pure dimension* n *and let* D *be an ample* $\mathbb{Q}$*-divisor such that* $L = rD$ *is Cartier. Suppose that there are effective* $\mathbb{Q}$*-divisors* B *and* R*, where* $B \neq 0$ *is* *integral and* $D/w \sim_{\mathbb{Q}} aB + R$ *for some constant* a.

Then

$$D^n \geq \frac{wa}{r^{n-1}}.$$

Proof. Indeed

$$r^{n-1}D^n = L^{n-1} \cdot D = w(L^{n-1} \cdot aB + L^{n-1} \cdot R) \geq wa,$$

where we used the fact that $L^{n-1} \cdot B \geq 1$. $\square$

6.4 Lemma. *Suppose that* $K_X + \Delta$ *is kawamata log terminal of log discrepancy at least* $\epsilon > 0$, $-(K_X + \Delta)$ *is big and nef, where* Δ *is an effective* $\mathbb{Q}$*-divisor. Suppose that the dimension of* X *is* n *and that* $r(K_X + \Delta)$ *is Cartier for some integer* r. *Assume that the degree is at least*

$$(2n)^n \left(2^n r(n-1)!\right)^{n(n-1)}$$

and let $\pi \colon Y \longrightarrow X$ *be the birational morphism determined by a birational family of tigers of large weight and minimal dimension, whose existence is guaranteed by (3.3) (here we take* $D = -(K_X + \Delta)$*). Define* Γ *to be the log pullback of* Δ *and define* Γ_t *as in (6.1).*

(1) Γ *is effective in a neighbourhood of the generic fibre of* f.

(2) *There is an open subset* $U \subset B$ *such that* $K_{V_t} + \Gamma_t$ *is kawamata log terminal of log discrepancy at least* ϵ, *for all* $t \in U$.

(3) *If the degree is at least*

$$\left(\frac{2n}{\epsilon}\right)^n \left(2^n r(n-1)!\right)^{n(n-1)}$$

then π *is small in a neighbourhood of the generic fibre of* f.

Proof. (2) follows from (1) and (6.1). Thus it suffices to prove (1) and (3).

Let $E_1, E_2, \dots, E_k$ be the exceptional divisors of π which dominate B. By (6.2) E_i has log discrepancy less than or equal to zero with respect to $K_X + \Delta + 2\Delta_t$. Thus if a_i is the log discrepancy of E_i with respect to $K_X + \Delta$ then the coefficient of E_i with respect to Δ_t is at least $a_i/2$. By (6.3) the degree of $-(K_{V_t} + \Theta_t)$ is at least

$$\frac{wa_i}{2r^{n-1}}$$

On the other hand by (5.4) the degree of $-(K_{V_t} + \Theta_t)$ is at most $(2^{k+1}k!)^k$. Thus

$$w \leq \frac{2r^{n-1}(2^{k+1}k!)^k}{a_i}.$$

By (3.3.2) it follows that

$$d \leq (wn)^n \leq \left(\frac{2nr^{n-1}(2^{k+1}k!)^k}{a_i}\right)^n \leq \left(\frac{2n}{a_i}\right)^n (2^n r(n-1)!)^{n(n-1)}.$$

Note that if Γ is not effective then there exists i such that $a_i \geq 1$ and for all i we have $a_i > \epsilon$. (1) and (3) follow immediately. $\square$

Proof of (1.7). (1.7.1) follows from (3.3.2), (4.2) and (3.4). (1.7.2) follows from (4.3) and (5.4). (1.7.3) and (1.7.4) follow from (6.4). $\square$

Proof of (1.11). Immediate from (1.7.1). $\square$

Proof of (1.8). Suppose not. Then by (1.7) there is a birational morphism $\pi\colon Y \longrightarrow X$ and a contraction morphism $f\colon Y \longrightarrow B$ such that B and the generic fibre have positive dimension. Moreover π is small in a neighbourhood of the generic fibre of f. Let H be the pullback of an ample divisor on B and let G be the pushforward of H. Then G is $\mathbb{Q}$-Cartier as X is $\mathbb{Q}$-factorial and so $\pi^*G = H + E$ where E is exceptional and does not dominate B. Thus G is not big, which contradicts the fact that X has Picard number one. $\square$

6.5 Lemma. *Let $\pi\colon Y \longrightarrow B$ be a bounded family, which parametrises triples (X, L, E), where X is a projective variety, L is a line bundle on X and E is a $\mathbb{Q}$-divisor. Fix an integer r and a positive real number ϵ.*

Then there is a constant w_0 such that for every triple (X, Δ, Γ) where

(1) *Δ is a subboundary such that $\Delta + E$ is effective.*

(2) *$K_X + \Delta$ is kawamata log terminal of log discrepancy at least $\epsilon > 0$.*

(3) *$\mathcal{O}_X(r(K_X + \Delta)) = L$.*

(4) *$K_X + \Delta + \Gamma$ is not kawamata log terminal.*

(5) *$\Gamma \sim_{\mathbb{Q}} L/w$*

Then $w \leq w_0$.

Proof. Note that divisors Δ satisfying (1)-(3) form a bounded family. Thus possibly enlarging B we may assume that B parametrises triples (X, L, Δ). Note that we are free to add a base point free line bundle to L. Adding on a sufficiently ample line bundle to L and passing to a power of L we may therefore assume that L is very ample.

By Noetherian induction we may assume that B is irreducible and it suffices to exhibit a non-empty open subset of B with the given property. Picking a resolution of the geometric generic fibre and passing to an open subset we may therefore assume that π is smooth and that the support of Δ has global normal crossings. As we are assuming that the log discrepancy of $K_X + \Delta$ is at least ϵ it follows that every component of Δ has coefficient at most $1 - \epsilon$.

As $K_X + \Delta + \Gamma$ is not kawamata log terminal (4.1.2) implies that there is a centre Z of X with multiplicity at least ϵ in Γ. Thus we can find divisors $H_1, H_2, \dots, H_k$ belonging to $|L|$ whose intersection with Z is a finite set of points, where k is the dimension of Z. Now pick a point p of this finite set and choose a further series $H_{k+1}, \dots H_{n-1}$ of elements of L that contain p whose intersection with Γ is a finite set of points.

Then

$$L^n = w(H_1 \cdot H_2 \cdot \dots H_{n-1} \cdot \Gamma) \geq w\epsilon,$$

so that

$$w_0 = \frac{L^n}{\epsilon}$$

will do. $\square$

Proof of (1.3). We may as well suppose that the degree of $-(K_X+\Delta)$ is at least

$$(2n)^n\,(2^n r(n-1)!)^{n(n-1)}\,.$$

Thus by (1.7) there is a birational morphism $\pi\colon Y\longrightarrow X$ and a contraction morphism $f\colon Y\longrightarrow B$ where the fibres of f have degree at most $(2^{k+1}k!)^k$. A collection of varieties of bounded degree forms a bounded family and so the fibres of f belong to a bounded family. Moreover by (6.4) if Γ is the log pullback of Δ and Γ_t the restriction of Γ to V_t then we have that $K_{V_t}+\Gamma_t$ is kawamata log terminal of log discrepancy at least ϵ for $t\in U$ an open subset of B.

Set $\Gamma_{t,s}=\pi^*\Delta_s|_{V_t}$. Then by (6.1.3) $K_{V_t}+\Gamma_t+\Gamma_{t,s}$ is not kawamata log terminal, for generic choice of s and t. By (6.5) it follows that there is a constant w_0 such that $w\leq w_0$. Thus by (3.3) the degree of $-(K_X+\Delta)$ is at most $(w_0 n)^n$. □

Proof of (1.2). By the main Theorem of [11] given n and r there is a fixed integer M such that $L=-M(K_X+\Delta)$ is very ample. Thus to prove (1.2) it suffices to bound the degree of $-(K_X+\Delta)$ and we may apply (1.3). □

Proof of (1.6). Suppose that $K_X+\Delta$ is kawamata log terminal and that $-(K_X+\Delta)$ is big and nef. Let r be any positive integer such that $r(K_X+\Delta)$ is Cartier. By (1.3) there is a constant M such that $d<M$ for all pairs (X,Δ) satisfying (1.3.1-4).

Suppose that the orbifold algebraic fundamental group has cardinality at least k. Then there is a finite cover $\pi\colon Y\longrightarrow X$ of X of order at least k such that the log pullback Γ of Δ is effective. Thus the pair (Y,Γ) satisfies (1.3.1-4). Moreover the degree of $-(K_Y+\Gamma)$ is equal to k^n times the degree of $-(K_X+\Delta)$. Thus k is no more than the nth root of M divided by the degree of $-(K_X+\Delta)$. This gives the first statement.

For the second statement, note that if π is étale in codimension one, then it is automatic that the log pullback of Δ is effective. □

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