ACC FOR LOG CANONICAL THRESHOLDS

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To Vyacheslav Shokurov on the occasion of his sixtieth birthday

Abstract. We show that log canonical thresholds satisfy the ACC.

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1. Introduction

We work over an algebraically closed field of characteristic zero. ACC stands for the ascending chain condition whilst DCC stands for the descending chain condition.

Suppose that (X, Δ) is a log canonical pair and $M \geq 0$ is $\mathbb{R}$-Cartier. The **log canonical threshold** of M with respect to (X, Δ) is

$$\operatorname{lct}(X, \Delta; M) = \sup\{\, t \in \mathbb{R} \,|\, (X, \Delta + tM) \text{ is log canonical} \,\}.$$

Let $\mathfrak{T} = \mathfrak{T}_n(I)$ denote the set of log canonical pairs (X, Δ), where X is a variety of dimension n and the coefficients of Δ belong to a set $I \subset [0, 1]$. Set

$$\mathrm{LCT}_n(I, J) = \{\, \operatorname{lct}(X, \Delta; M) \,|\, (X, \Delta) \in \mathfrak{T}_n(I) \,\},$$

where the coefficients of M belong to a subset J of the positive real numbers.

Theorem 1.1 (ACC for the log canonical threshold)**.** *Fix a positive integer n, $I \subset [0, 1]$ and a subset J of the positive real numbers.*

If I and J satisfy the DCC then $\mathrm{LCT}_n(I, J)$ satisfies the ACC.

(1.1) was conjectured by Shokurov [33], see also [22] and [24]. When the dimension is three, [22] proves that 1 is not an accumulation point from below and (1.1) follows from the results of [3]. More recently (1.1) was proved for complete intersections [10] and even when X belongs to a bounded family, [11].

The log canonical threshold is an interesting invariant of the pair (X, Δ) and the divisor M which is a measure of the complexity of the singularities of the triple $(X, \Delta; M)$. It has made many appearances in many different forms, especially in the case of hypersurfaces, see [24], [25] and [34]. The ACC for the log canonical threshold plays a role in inductive approaches to higher dimensional geometry. For example, after [6], we have the following application of (1.1):

Corollary 1.2. *Assume termination of flips for $\mathbb{Q}$-factorial kawamata log terminal pairs in dimension $n - 1$.*

Let (X, Δ) be a kawamata log terminal pair where X is a $\mathbb{Q}$-factorial projective variety of dimension n. If $K_X + \Delta$ is numerically equivalent to a divisor $D \geq 0$ then any sequence of $(K_X + \Delta)$-flips terminates.

(1.1) is a consequence of the following theorem, which was conjectured by Alexeev [3] and Kollár [22]:

Theorem 1.3. *Fix a positive integer n and a set $I \subset [0,1]$ which satisfies the DCC. Let $\mathfrak{D}$ be the set of log canonical pairs (X, Δ) such that the dimension of X is n and the coefficients of Δ belong to I.*

Then there is a constant $\delta > 0$ and a positive integer m with the following properties:

(1) the set

$$\{ \operatorname{vol}(X, K_X + \Delta) \,|\, (X, \Delta) \in \mathfrak{D} \},$$

also satisfies the DCC.

Further, if $(X, \Delta) \in \mathfrak{D}$ and $K_X + \Delta$ is big, then

(2) $\operatorname{vol}(X, K_X + \Delta) \geq \delta$*, and*

(3) $\phi_{m(K_X+\Delta)}$ *is birational.*

Note that, by convention, $\phi_{m(K_X+\Delta)} = \phi_{\lfloor m(K_X+\Delta) \rfloor}$. (1.3) was proved for surfaces in [3]. (1.3) is a generalisation of [15, 1.3], which deals with the case that (X, Δ) is the quotient of a smooth projective variety Y of general type by its automorphism group.

One of the original motivations for (1.3) is to prove the boundedness of the moduli functor for canonically polarised varieties, see [26]. We plan to pursue this application of (1.3) in a forthcoming paper.

To state more results it is convenient to give a simple reformulation of (1.1):

Theorem 1.4. *Fix a positive integer n and a set $I \subset [0,1]$, which satisfies the DCC.*

Then there is a finite subset $I_0 \subset I$ with the following properties:
If (X, Δ) is a log pair such that

(1) X is a variety of dimension n,
(2) (X, Δ) is log canonical,
(3) the coefficients of Δ belong to I, and
(4) there is a non kawamata log terminal centre $Z \subset X$ which is contained in every component of Δ,

then the coefficients of Δ belong to I_0.

(1.4) follows, cf. [33], [27, §18], almost immediately from the existence of divisorial log terminal modifications and from:

Theorem 1.5. *Fix a positive integer n and a set $I \subset [0,1]$, which satisfies the DCC.*

Then there is a finite subset $I_0 \subset I$ with the following properties:
If (X, Δ) is a log pair such that

(1) X is a projective variety of dimension n,
(2) (X, Δ) is log canonical,
(3) the coefficients of Δ belong to I, and
(4) $K_X + \Delta$ is numerically trivial,

then the coefficients of Δ belong to I_0.

We use finiteness of log canonical models to prove a boundedness result for log pairs:

Theorem 1.6. *Fix a positive integer n and two real numbers δ and $\epsilon > 0$.*

Let $\mathfrak{D}$ be a set of log pairs (X, Δ) such that

- *X is a projective variety of dimension n,*
- *$K_X + \Delta$ is ample,*
- *the coefficients of Δ are at least δ, and*
- *the total log discrepancy of (X, Δ) is greater than ϵ.*

If $\mathfrak{D}$ is log birationally bounded then $\mathfrak{D}$ is a bounded family.

Log birationally bounded is defined in (3.5.1). We use (1.5) and (1.6) to prove some boundedness results about Fano varieties.

Corollary 1.7. *Fix a positive integer n, a real number $\epsilon > 0$ and a set $I \subset [0, 1]$ which satisfies the DCC.*

Let $\mathfrak{D}$ be the set of all log pairs (X, Δ), where

- *X is a projective variety of dimension n,*
- *the coefficients of Δ belong to I,*
- *the total log discrepancy of (X, Δ) is greater than ϵ,*
- *$K_X + \Delta$ is numerically trivial, and*
- *$-K_X$ is ample.*

Then $\mathfrak{D}$ forms a bounded family.

As a consequence we are able to prove a result on the boundedness of Fano varieties which was conjectured by Batyrev (cf. [9]):

Corollary 1.8. *Fix two positive integers n and r.*

Let $\mathfrak{D}$ be the set of all kawamata log terminal pairs (X, Δ), where X is a projective variety of dimension n and $-r(K_X + \Delta)$ is an ample Cartier divisor.

Then $\mathfrak{D}$ forms a bounded family.

Definition 1.9. *Let (X, Δ) be a log canonical pair, where X is projective of dimension n and $-(K_X + \Delta)$ is ample. The* ***Fano index*** *of (X, Δ) is the largest real number r such that we can write*

$$-(K_X + \Delta) \sim_{\mathbb{R}} rH,$$

where H is a Cartier divisor.

Fix a set $I \subset [0,1]$ and a positive integer n. Let $\mathfrak{D}$ be the set of log canonical pairs (X,Δ), where X is projective of dimension n, $-(K_X+\Delta)$ is ample and the coefficients of Δ belong to I.

The set

$$R = R_n(I) = \{\, r \in \mathbb{R} \,|\, r \text{ is the Fano index of } (X,\Delta) \in \mathfrak{D} \,\},$$

is called the ***Fano spectrum*** *of $\mathfrak{D}$.*

Corollary 1.10. *Fix a set $I \subset [0,1]$ and a positive integer n.*

If I satisfies the DCC then the Fano spectrum satisfies the ACC.

(1.10) was proved in dimension 2 in [1] and for $R \cap [n-2,\infty)$ in [2].

Now given any set which satisfies the ACC it is natural to try to identify the accumulation points. (1.1) implies that $\mathrm{LCT}_n(I) = \mathrm{LCT}_n(I,\mathbb{N})$ satisfies the ACC. Kollár, cf. [24], [32], [20], conjectured that the accumulation points in dimension n are log canonical thresholds in dimension $n-1$:

Theorem 1.11. *If 1 is the only accumulation point of $I \subset [0,1]$ and $I = I_+$ then the accumulation points of $\mathrm{LCT}_n(I)$ are $\mathrm{LCT}_{n-1}(I) - \{1\}$. In particular, if $I \subset \mathbb{Q}$ then the accumulation points of $\mathrm{LCT}_n(I)$ are rational numbers.*

See §3.4 for the definition of I_+. (1.11) was proved if X is smooth in [20]. Note that in terms of inductive arguments it is quite useful to identify the accumulation points, especially to know that they are rational.

Finally, recall:

Conjecture 1.12 (Borisov-Alexeev-Borisov)**.** *Fix a positive integer n and a positive real number $\epsilon > 0$.*

Let $\mathfrak{D}$ be the set of all projective varieties X of dimension n such that there is a divisor Δ where (X,Δ) has log discrepancy at least ϵ and $-(K_X+\Delta)$ is ample.

Then $\mathfrak{D}$ forms a bounded family.

Note that (1.1), (1.4), (1.5), (1.2) and (1.11) are known to follow from (1.12), (cf. [32]). Instead we use birational boundedness of log pairs of general type cf. (1.3) to prove these results.

2. Description of the proof

Theorem A (ACC for the log canonical threshold)**.** *Fix a positive integer n and a set $I \subset [0,1]$, which satisfies the DCC.*

Then there is a finite subset $I_0 \subset I$ with the following property:

If (X,Δ) is a log pair such that

(1) X is a variety of dimension n,
(2) (X,Δ) is log canonical,
(3) the coefficients of Δ belong to I, and
(4) there is a non kawamata log terminal centre $Z \subset X$ which is contained in every component of Δ,

then the coefficients of Δ belong to I_0.

Theorem B (Upper bounds for the volume)**.** *Let $n \in \mathbb{N}$ and let $I \subset [0,1)$ be a set which satisfies the DCC. Let $\mathfrak{D}$ be the set of kawamata log terminal pairs (X,Δ), where X is projective of dimension n, $K_X+\Delta$ is numerically trivial and the coefficients of Δ belong to I.*

Then the set

$$\{\, \mathrm{vol}(X,\Delta) \,|\, (X,\Delta) \in \mathfrak{D} \,\},$$

is bounded from above.

Theorem C (Birational boundedness)**.** *Fix a positive integer n and a set $I \subset [0,1]$, which satisfies the DCC. Let $\mathfrak{B}$ be the set of log canonical pairs (X,Δ), where X is projective of dimension n, $K_X+\Delta$ is big and the coefficients of Δ belong to I.*

Then there is a positive integer m such that $\phi_{m(K_X+\Delta)}$ is birational, for every $(X,\Delta) \in \mathfrak{B}$.

Theorem D (ACC for numerically trivial pairs)**.** *Fix a positive integer n and a set $I \subset [0,1]$, which satisfies the DCC.*

Then there is a finite subset $I_0 \subset I$ with the following property:

If (X,Δ) is a log pair such that

(1) X is projective of dimension n,
(2) the coefficients of Δ belong to I,
(3) (X,Δ) is log canonical, and
(4) $K_X+\Delta$ is numerically trivial,

then the coefficients of Δ belong to I_0.

The proof of Theorem A, Theorem B, Theorem C, and Theorem D proceeds by induction:

- Theorem D_{n-1} implies Theorem A_n, cf. (5.3).
- Theorem D_{n-1} and Theorem A_{n-1} imply Theorem B_n, cf. (6.2).
- Theorem C_{n-1}, Theorem A_{n-1}, and Theorem B_n imply Theorem C_n, cf. (7.4).
- Theorem D_{n-1} and Theorem C_n implies Theorem D_n, cf. (8.1).

2.1. **Sketch of the proof.** The basic idea of the proof of (1.1) goes back to Shokurov and we start by explaining this.

Consider the following simple family of plane curve singularities,

$$C = (y^a + x^b = 0) \subset \mathbb{C}^2,$$

where a and b are two positive integers. A priori, to calculate the log discrepancy c, one should take a log resolution of the pair $(X = \mathbb{C}^2, C)$, write down the log discrepancy of every exceptional divisor E_i with respect to the pair (X, tC) as a function of t and then find out the largest value c of t for which all of these log discrepancies are non-negative. However there is an easier way. We know that when $t = c$ there is at least one divisor of log discrepancy zero (and every other divisor has non-negative log discrepancy). Let $\pi \colon Y \longrightarrow X$ extract just this divisor. To construct π we simply contract all other divisors on the log resolution.

Almost by definition we can write

$$K_Y + E + cD = \pi^*(K_X + cC),$$

where E is the exceptional divisor and D is the strict transform of C. Restrict both sides of this equation to E. As the RHS is a pullback, we get a numerically trivial divisor.

To compute the LHS we apply adjunction. E is a copy of $\mathbb{P}^1$. One slightly delicate issue is that Y is singular along E and the adjunction formula has to take account of this. In fact $Y \longrightarrow X$ is precisely the weighted blow up of $X = \mathbb{C}^2$, with weights (a, b), in the given coordinates x, y. There are two singular points p and q of Y along C, of index a and b, and D intersects C transversally at another point r. If we apply adjunction we get

$$(K_Y + E + cD)|_E = K_E + \left(\frac{a-1}{a}\right) p + \left(\frac{b-1}{b}\right) q + cr.$$

As $(K_Y + E + cD)|_E$ is numerically trivial we have $(K_Y + E + cD) \cdot E = 0$ so that

$$-2 + \frac{a-1}{a} + \frac{b-1}{b} + c = 0,$$

and so

$$c = \frac{1}{a} + \frac{1}{b}.$$

Now let us consider the general case. As with the example above the first step is to extract divisors of log discrepancy zero, $\pi \colon Y \longrightarrow X$. To construct π we mimic the argument above; pick a log resolution for the pair $(X, \Delta + C)$ and contract every divisor whose log discrepancy is not zero. The fact that we can do this in all dimensions follows from the MMP (minimal model program), see (3.3.1) and π is called a divisorially log terminal modification.

The next step is the same, restrict to the general fibre of some divisor of log discrepancy zero, see (5.1). There are similar formulae for the coefficients of the restricted divisor, see (4.1). In this way, we reduce the problem from a local one in dimension n to a global problem in dimension $n-1$, see §5. This explains how to go from Theorem D_{n-1} to Theorem A_n, see the proof of (5.3).

The global problem involves log canonical pairs (X, Δ), where X is projective and $K_X + \Delta$ is numerically trivial. One reason that the dimension one case is easy is that there is only one possibility for X, X must be isomorphic to $\mathbb{P}^1$. In higher dimensions it is not hard, running the MMP again, to reduce to the case where X has Picard number one, so that at least X is a Fano variety and Δ is ample. In this case we perturb Δ by increasing one of its coefficients to get a kawamata log terminal pair (X, Λ) such that $K_X + \Lambda$ is ample. We then exploit the fact that some fixed multiple $m(K_X + \Lambda)$ of $K_X + \Lambda$ gives a birational map $\phi_{m(K_X+\Lambda)}$. By definition this means that $\phi_{\lfloor m(K_X+\Lambda)\rfloor}$ is a birational map, which in particular means that $K_X + \Lambda_{\lfloor m \rfloor}$ (see (3.1) for the definition of $\Lambda_{\lfloor m \rfloor}$) is big . This forces $\Delta \leq \Lambda_{\lfloor m \rfloor}$ which implies that there are lots of gaps. This explains how to go from Theorem C_n to Theorem D_n, see the proof of (8.1).

It is clear then that the main thing to prove is that if (X, Δ) is a kawamata log terminal pair, $K_X + \Delta$ is big and the coefficients of Δ belong to a DCC set then some fixed multiple of $K_X + \Delta$, gives a birational map $\phi_{m(K_X+\Delta)}$. Following some ideas of Tsuji, we developed a fairly general method to prove such a result in [15], see (3.5.2) and (3.5.5). We use the technique of cutting non kawamata log terminal centres as developed in [5], see [24]. The main issue is to find a boundary on the non kawamata log terminal centre so that we can run an induction.

There are two key hypotheses to apply (3.5.5). One of them requires that the volume of $K_X + \Delta$ restricted to appropriate non kawamata log terminal centres is bounded from below. The other places a requirement on the coefficients of Δ which is stronger than the DCC.

The first condition follows by induction on the dimension and a strong version of Kawamata's subadjunction formula, (4.2), which we now explain. If (X, Λ) is a log pair and V is a non kawamata log terminal centre such that (X, Λ) is log canonical at the generic point of V, then one can write

$$(K_X + \Lambda)|_W = K_W + \Theta_b + J,$$

where W is the normalisation of V, Θ_b is the discriminant divisor and J is the moduli part. Not much is known about the moduli part J

beyond the fact that it is pseudo-effective. On the other hand $\Theta_b \geq 0$ behaves very well. If (X, Λ) is log canonical at the generic point of a prime divisor B on W then the coefficient of B in Θ_b is at most one. In fact there is a simple way to compute the coefficient of B involving the log canonical threshold. By assumption there is a log canonical place, that is, a valuation with centre V of log discrepancy zero. Then we can find a divisorially log terminal modification $g\colon Y \longrightarrow X$ such that the centre of this log canonical place is a divisor S on Y. Note that there is a commutative diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ W \arrow[r] & X. \end{tikzcd}$$

If we pullback $K_X + \Delta$ to Y and restrict to S we get a divisor Φ' on S. Let

$$\begin{aligned} \lambda = \sup\{\, t \in \mathbb{R} \,|\, (S, \Phi' + tf^*B) \text{ is log canonical over a} \\ \text{neighbourhood of the generic point of } B \,\}, \end{aligned}$$

be the log canonical threshold. Then the coefficient of B in Θ_b is $1 - \lambda$.

In practice we start with a divisor Δ whose coefficients belong to I such that (X, Δ) is kawamata log terminal. We then find a divisor Δ_0, whose coefficients we have no control on, and V is a non kawamata log terminal centre of $(X, \Lambda = \Delta + \Delta_0)$. It follows that the coefficients of Φ' do not behave well and we have no control on the coefficients of Θ_b.

To circumvent this we simply mimic the same construction for (X, Δ) rather than (X, Λ). First we construct a divisor Φ on S whose coefficients of Φ belong to $D(I)$, see (4.1). Then we construct a divisor Θ whose coefficients automatically belong to the set

$$\{\, a \,|\, 1 - a \in \mathrm{LCT}_{n-1}(D(I)) \,\} \cup \{\, 1 \,\}.$$

It is clear from the construction that $\Theta_b \geq \Theta$, so that if we bound the volume of $K_W + \Theta$ from below we bound the volume of $(K_X + \Delta + \Delta_0)|_W$ from below.

On the other hand, as part of the induction we assume that Theorem A_{n-1} holds. Hence $\mathrm{LCT}_{n-1}(D(I))$ satisfies the ACC and the coefficients of Θ belong to a set which satisfies the DCC. The final step is to observe that if we choose V to pass through a general point then it belongs to a family which covers X. If we assume that V is a general member of such a family then we can pullback $K_X + \Delta$ to this family and restrict to V. It is straightforward to check that the difference between $K_W + \Theta$ and $(K_X + \Delta)|_W$ on a log resolution of the family is pseudo-effective (for example, if X and V are smooth then this follows from the fact that the first chern class of the normal bundle is pseudo-effective), so that if $K_X + \Delta$ is big then so is $K_W + \Theta$. In this case we know the volume is bounded from below by induction.

We now explain the condition on the coefficients. To apply (3.5.5) we require that either I is a finite set or

$$I = \{ \frac{r-1}{r} \,|\, r \in \mathbb{N} \}.$$

The first lemma, (7.2), simply assumes this condition on I and we deduce the result in this case.

The key is then to reduce to the case when I is finite. Given any positive integer p and a log pair (X, Δ), let $\Delta_{\lfloor p \rfloor}$ denote the largest divisor less than Δ such that $p\Delta_{\lfloor p \rfloor}$ is integral. Given I it suffices to find a fixed positive integer p such that if we start with (X, Δ) such that $K_X + \Delta$ is big and the coefficients belong to I then $K_X + \Delta_{\lfloor p \rfloor}$ is big, since the coefficients of $\Delta_{\lfloor p \rfloor}$ belong to the finite set

$$\{ \frac{i}{p} \,|\, 1 \leq i \leq p \}.$$

Let

$$\lambda = \inf\{ t \in \mathbb{R} \,|\, K_X + t\Delta \text{ is big} \},$$

be the pseudo-effective threshold. A simple computation, (7.4), shows that it suffices to bound λ away from one. Running the MMP we reduce to the case when X has Picard number one. Since $K_X + \lambda\Delta$ is numerically trivial and kawamata log terminal, Theorem B implies that the volume of Δ is bounded away from one. Passing to a log resolution we may assume that (X, D) has simple normal crossings where D is the sum of the components of Δ. As $K_X + D$ is big then so is $K_X + \frac{r-1}{r}D$ for any positive integer r which is sufficiently large. It follows that some fixed multiple of $K_X + \frac{r-1}{r}D$ gives a birational map, and (3.5.2) implies that (X, D) belongs to log birationally bounded family. In this case, it is easy to bound the pseudo-effective threshold λ away from one, see (7.3). This explains how to go from Theorem B_n to Theorem C_n, cf. (7.4).

We now explain the last implication. Suppose that (X, Δ) is kawamata log terminal and $K_X + \Delta$ is numerically trivial. If the volume of Δ is large then we may find a divisor Π numerically equivalent to a small multiple of Δ with large multiplicity at a general point, so that (X, Π) is not kawamata log terminal. In particular we may find Φ arbitrarily close to Δ such that (X, Φ) is not kawamata log terminal. The key lemma is to show that this is impossible, (6.1). By assumption we may extract a divisor S of log discrepancy zero with respect to (X, Φ). After we run the MMP we get down a log pair $(Y, S+\Gamma)$ where Γ is the strict transform of Δ and both $K_Y + S + \Gamma$ and $-(K_Y + S + (1-\epsilon)\Gamma)$ are ample. Here $\epsilon > 0$ is arbitrarily close to zero. If we restrict to S and apply adjunction, it is easy to see that this contradicts either ACC for the log canonical threshold or ACC for numerically trivial pairs. This explains how to go from Theorem D_{n-1} and Theorem A_{n-1} to Theorem B_n, cf. (6.2).

It is interesting to note that if (X, Δ) is log canonical then there is no bound on the volume of Δ:

Example 2.1.1. *Let X be the weighted projective surface $\mathbb{P}(p, q, r)$, where p, q and r are three positive integers and let Δ be the sum of the three coordinate lines. Then $K_X + \Delta \sim_{\mathbb{Q}} 0$ and*

$$\operatorname{vol}(X, \Delta) = \frac{(p+q+r)^2}{pqr}.$$

But the set

$$\{ \frac{(p+q+r)^2}{pqr} \mid (p, q, r) \in \mathbb{N}^3 \},$$

is dense in the positive real numbers, cf. [19, 22.5].

We now explain the proof of (1.11) which mirrors the proof of (1.1). We are given a sequence of log pairs $(X, \Delta) = (X_i, \Delta_i)$ and we want to identify the limit points of the log canonical thresholds. The first step is to show that the set of log canonical thresholds is essentially the same as the set of pseudo-effective thresholds. In §5 we showed that every log canonical threshold in dimension $n+1$ is a numerically trivial threshold in dimension n. To show the reverse inclusion, one takes the cone (Y, Γ) over a log canonical pair (X, Δ) where $K_X + \Delta$ is numerically trivial, (11.5).

In this way we are reduced to looking at log canonical pairs (X, Δ) such that $K_X + \Delta$ is numerically trivial. The basic idea is to generate a component of coefficient one and apply adjunction. To this end, we need to deal with the case where some coefficients of Δ don't necessarily belong to I but instead they are increasing towards one, (11.7).

Running the MMP we reduce to the case of Picard number one, Case A, Step 1 and Case B, Steps 3 and 5. We may also assume that the non kawamata log terminal locus is a divisor. In particular $-K_X$ is ample, any two components of Δ intersect and we may assume that the number of components of Δ is constant, (11.6). If (X, Δ) is not kawamata log terminal then there is a component of coefficient one and we are done, Case B, Step 2.

The argument now splits into two cases. Case A deals with the case that the coefficients of Δ are bounded away from one. In this case if the volume of Δ is arbitrarily large then we can create a component of coefficient one and we reduce to the other case, Case B. Otherwise (1.6) implies that (X, Δ) belongs to a bounded family, which contradicts the fact that the coefficients of Δ are not constant.

So we may assume we are in Case B, namely that some of the coefficients of the components of Δ are approaching one. We decompose Δ as $A + B + C$ where the coefficients of A are approaching one, the coefficients of B are fixed, and we are trying to identify the limit of the coefficients of C. Using the fact that the Picard number of X is one, we may increase the coefficients of A to one and decrease the coefficients of C, without changing the limit of the coefficients of C. At this point we apply adjunction and induction, Case B, Step 6.

3. Preliminaries

3.1. **Notation and Conventions.** If $D = \sum d_i D_i$ is an $\mathbb{R}$-divisor on a normal variety X, then the *round down* of D is $\lfloor D \rfloor = \sum \lfloor d_i \rfloor D_i$, where $\lfloor d \rfloor$ denotes the largest integer which is at most d, the *fractional part* of D is $\{D\} = D - \lfloor D \rfloor$, and the *round up* of D is $\lceil D \rceil = -\lfloor -D \rfloor$. If m is a positive integer, then let

$$D_{\lfloor m \rfloor} = \frac{\lfloor mD \rfloor}{m}.$$

Note that $D_{\lfloor m \rfloor}$ is the largest divisor less than or equal to D such that $mD_{\lfloor m \rfloor}$ is integral.

The sheaf $\mathcal{O}_X(D)$ is defined by

$$\mathcal{O}_X(D)(U) = \{\, f \in K(X) \mid (f)|_U + D|_U \geq 0 \,\},$$

so that $\mathcal{O}_X(D) = \mathcal{O}_X(\lfloor D \rfloor)$. Similarly we define $|D| = |\lfloor D \rfloor|$. If X is normal, and D is an $\mathbb{R}$-divisor on X, the *rational map ϕ_D associated to D* is the rational map determined by the restriction of $\lfloor D \rfloor$ to the smooth locus of X.

We say that D is $\mathbb{R}$-*Cartier* if it is a real linear combination of Cartier divisors. If $f \colon Y \longrightarrow X$ is a morphism then $D|_Y$ denotes the pullback of D to Y, f^*D. In general $D|_Y$ is only well-defined up to $\mathbb{R}$-linear equivalence. However if $f(Y)$ is not contained in the support of D then $D|_Y$ is a well-defined $\mathbb{R}$-Cartier divisor. An $\mathbb{R}$-Cartier divisor D on a normal variety X is *nef* if $D \cdot C \geq 0$ for any curve $C \subset X$. We say that two $\mathbb{R}$-divisors D_1 and D_2 are $\mathbb{R}$-*linearly equivalent*, denoted $D_1 \sim_{\mathbb{R}} D_2$, if the difference is an $\mathbb{R}$-linear combination of principal divisors.

A *log pair* (X, Δ) consists of a normal variety X and a $\mathbb{R}$-Weil divisor $\Delta \geq 0$ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. The *support* of $\Delta = \sum_{i \in I} d_i D_i$ (where $d_i \neq 0$) is the sum $D = \sum_{i \in I} D_i$. If (X, Δ) has simple normal crossings, a *stratum* of (X, Δ) is an irreducible component of the intersection $\cap_{j \in J} D_j$, where J is a non-empty subset of I (in particular, a stratum of (X, Δ) is always a proper closed subset of X). If we are given a morphism $X \longrightarrow T$, then we say that (X, Δ) has *simple normal crossings over* T if (X, Δ) has simple normal crossings and both X and every stratum of (X, D) is smooth over T. We say that the birational morphism $f \colon Y \longrightarrow X$ only *blows up strata* of (X, Δ), if f is the composition of birational morphisms $f_i \colon X_{i+1} \longrightarrow X_i$, $1 \leq i \leq k$, with $X = X_0$, $Y = X_{k+1}$, and f_i is the blow up of a stratum of (X_i, Δ_i), where Δ_i is the sum of the strict transform of Δ and the exceptional locus.

A *log resolution* of the pair (X, Δ) is a projective birational morphism $\mu \colon Y \longrightarrow X$ such that the exceptional locus is the support of a μ-ample divisor and (Y, G) has simple normal crossings, where G is the support of the strict transform of Δ and the exceptional divisors. If we write

$$K_Y + \Gamma + \sum b_i E_i = \mu^*(K_X + \Delta)$$

where Γ is the strict transform of Δ, then b_i is called the *coefficient of* E_i *with respect to* (X, Δ). The *log discrepancy* of E_i is $a(E_i, X, \Delta) = 1 - b_i$. The *log discrepancy* of (X, Δ) is the infimum over all log resolutions of the log discrepancy of any exceptional divisor. The *total log discrepancy* of (X, Δ) is the minimum of the log discrepancy of (X, Δ) and $1 - a$ where a ranges over the coefficients of the components of Δ. The pair (X, Δ) is *kawamata log terminal* (respectively *log canonical*; *purely log terminal*; *divisorially log terminal*) if $b_i < 1$ for all i and $\lfloor \Delta \rfloor = 0$ (respectively $b_i \leq 1$ for all i and for all log resolutions; $b_i < 1$ for all i and for all log resolutions; the coefficients of Δ belong to $[0, 1]$ and there exists a log resolution such that $b_i < 1$ for all i). If we drop the condition that $\Delta \geq 0$ but all of the coefficients of Γ are at most one then we say that (X, Δ) is *sub log canonical.* A *non kawamata log terminal centre* is the centre of any valuation associated to a divisor E_i with $b_i \geq 1$. In this paper, we only consider valuations ν of X whose centre on some birational model Y of X is a divisor.

We now introduce some results some of which are well known to experts but which are included for the convenience of the reader.

3.2. **The volume.**

Definition 3.2.1. *Let X be an irreducible projective variety of dimension n and let D be an $\mathbb{R}$-divisor. The* ***volume*** *of D is*

$$\operatorname{vol}(X, D) = \limsup_{m \to \infty} \frac{n! h^0(X, \mathcal{O}_X(mD))}{m^n}.$$

We say that D is ***big*** *if $\operatorname{vol}(X, D) > 0$.*

For more background, see [31].

Lemma 3.2.2. *Let X be a projective variety and let (X, Δ) be a log pair.*

If D is an $\mathbb{R}$-divisor and $\operatorname{vol}(X, D) > n^n$ then for every point $x \in X$ we may find $\Pi \sim_{\mathbb{R}} D$ such that $(X, \Delta + \Pi)$ is not kawamata log terminal at $x \in X$.

Proof. Arguing as in the proof of [24, 6.7.1] we may assume that $x \in X$ is a general point, so that in particular x is a smooth point of X. As the volume is a continuous function of D we may assume that D is a $\mathbb{Q}$-divisor, [30, 2.2.44]. The result then follows as in the proof of [24, 6.1]. $\square$

Lemma 3.2.3. *Let X be a quasi-projective $\mathbb{Q}$-factorial variety and let (X, Δ) be a kawamata log terminal pair.*

If $(X, \Delta + D)$ is not log canonical, where $D \geq 0$ is big, then we may find $0 \leq D' \sim_{\mathbb{R}} tD$, for some $0 < t < 1$ such that $(X, \Delta + D')$ has exactly one log canonical place.

Proof. As $(X, \Delta + D)$ is not log canonical we may find $\delta > 0$ such that $(X, \Delta + (1-\delta)D)$ is not log canonical. As D is big we may find divisors $A \geq 0$ and $B \geq 0$ such that $D \sim_{\mathbb{R}} A + B$ and A is ample. Replacing D by $(1 - \delta)D + \delta A + \delta B$ we may assume that there is an ample divisor $A \geq 0$ such that $D \geq A$.

Let

$$\pi \colon Y \longrightarrow X$$

be a log resolution. We may write

$$K_Y + \Gamma + \sum a_i E_i = \pi^*(K_X + \Delta + tD),$$

where Γ is the strict transform of Δ and a_i are linear functions of t. By assumption $a_i < 1$ when $t = 0$ and there is an index i such that $a_i > 1$ when $t = 1$. It follows that we may find $t \in (0, 1)$ such that $a_i \leq 1$ for all indices with equality for at least one index i. Possibly using A to tie-break, see [24], we may assume that there is at most one index i such that $a_i = 1$. $\square$

3.3. **Divisorially log terminal modifications.** If (X, Δ) is not kawamata log terminal then we may find a modification which is divisorially log terminal, so that the non kawamata log terminal locus is a divisor:

Proposition 3.3.1. *Let (X, Δ) be a log pair where X is a variety and the coefficients of Δ belong to $[0, 1]$.*

Then there is a projective birational morphism $\pi \colon Y \longrightarrow X$ such that

(1) Y is $\mathbb{Q}$-factorial,
(2) π only extracts divisors of log discrepancy at most zero,
(3) if $E = \sum E_i$ is the sum of the π-exceptional divisors and Γ is the strict transform of Δ, then $(Y, \Gamma + E)$ is divisorially log terminal and

$$K_Y + E + \Gamma = \pi^*(K_X + \Delta) + \sum_{a(E_i, X, B) < 0} a(E_i, X, B) E_i.$$

(4) Further, if (X, Δ) is log canonical and S is a component of Δ then there is a nef divisor of the form $-T - F$, where T is the strict transform of S and $F \geq 0$ is a sum of exceptional divisor whose centres are contained in S.

Any birational morphism $\pi \colon Y \longrightarrow X$ satisfying (1–3) is called a ***divisorially log terminal modification****.*

Proof. The proof of (1–3) is due to the first author and can be found in [13], [28, 3.1], and also [4].

Now suppose that (X, Δ) is log canonical and S is a component of Δ. In this case

$$K_Y + E + \Gamma = \pi^*(K_X + \Delta).$$

Pick $\epsilon > 0$ so that $\Gamma - \epsilon T \geq 0$. Note that $(Y, E + \Gamma - \epsilon T)$ is divisorially log terminal, as Y is $\mathbb{Q}$-factorial and $(Y, E + \Gamma)$ is divisorially log terminal. By Theorem 1.1 of [7] or by Theorem 1.6 of [16], we may replace Y by a log terminal model of $(Y, E + \Gamma - \epsilon T)$ over X, gaining the fact that $-T$ is nef over X, at the expense of temporarily losing the property that $(Y, \Gamma + E)$ is divisorially log terminal, whilst preserving the condition that $K_Y + E + \Gamma$ is log canonical and numerically trivial over X. If $g \colon W \longrightarrow Y$ is a divisorially log terminal modification of $(Y, \Gamma + E)$ and we replace Y by W then $g^*(-T)$ is a nef divisor over X of the correct form. □

3.4. **DCC sets.** We say that a set I of real numbers satisfies the *descending chain condition* or DCC, if it does not contain any infinite strictly decreasing sequence. For example,

$$I = \{ \frac{r-1}{r} \,|\, r \in \mathbb{N} \},$$

satisfies the DCC. Let $I \subset [0,1]$. We define

$$I_+ := \{\, 0 \,\} \cup \{\, j \in [0,1] \,|\, j = \sum_{p=1}^{l} i_p, \text{ for some } i_1, i_2, \dots, i_l \in I \,\},$$

and

$$D(I) := \{\, a \leq 1 \,|\, a = \frac{m-1+f}{m}, m \in \mathbb{N}, f \in I_+ \,\}.$$

As usual, $\overline{I}$ denotes the closure of I. Note that the set $D(I)$ appears when we apply adjunction, (4.1).

Proposition 3.4.1. *Let $I \subset [0,1]$.*

(1) $D(D(I)) = D(I) \cup \{\, 1 \,\}$.

(2) I satisfies the DCC if and only if $\overline{I}$ satisfies the DCC.

(3) I satisfies the DCC if and only if $D(I)$ satisfies the DCC.

Proof. Straightforward, see for example [32, 4.4]. □

3.5. **Bounded pairs.** We recall some results and definitions from [15], stated in a convenient form.

Definition 3.5.1. *We say that a set $\mathfrak{X}$ of varieties is **birationally bounded** if there is a projective morphism $Z \longrightarrow T$, where T is of finite type, such that for every $X \in \mathfrak{X}$, there is a closed point $t \in T$ and a birational map $f \colon Z_t \dashrightarrow X$.*

*We say that a set $\mathfrak{D}$ of log pairs is **log birationally bounded** (respectively **bounded**) if there is a log pair (Z,B), where the coefficients of B are all one, and a projective morphism $Z \longrightarrow T$, where T is of finite type, such that for every $(X,\Delta) \in \mathfrak{D}$, there is a closed point $t \in T$ and a birational map $f \colon Z_t \dashrightarrow X$ (respectively isomorphism of varieties) such that the support of B_t is not the whole of Z_t and yet B_t contains the support of the strict transform of Δ and any f-exceptional divisor (respectively $f(B_t) = \Delta$).*

Theorem 3.5.2. *Fix a positive integer n and a set $I \subset [0,1] \cap \mathbb{Q}$, which satisfies the DCC. Let $\mathfrak{B}_0$ be a set of log canonical pairs (X,Δ), where X is projective of dimension n, $K_X + \Delta$ is big and the coefficients of Δ belong to I.*

Suppose that there is a constant M such that for every $(X,\Delta) \in \mathfrak{B}_0$ there is a positive integer k such that $\phi_{k(K_X+\Delta)}$ is birational and

$$\operatorname{vol}(X, k(K_X+\Delta)) \leq M.$$

Then the set

$$\{\, \operatorname{vol}(X, K_X+\Delta) \,|\, (X,\Delta) \in \mathfrak{B}_0 \,\},$$

satisfies the DCC.

Proof. Follows from (2.3.4), (3.1) and (1.9) of [15]. □

Recall:

Definition 3.5.3. *Let X be a normal projective variety and let D be a big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X.*

If x and y are two general points of X then, possibly switching x and y, we may find $0 \leq \Delta \sim_{\mathbb{Q}} (1-\epsilon)D$, for some $0 < \epsilon < 1$, where (X, Δ) is not kawamata log terminal at y, (X, Δ) is log canonical at x and $\{x\}$ is a non kawamata log terminal centre, then we say that D is ***potentially birational****.*

Note that this is a slight variation on the definition which appears in [15], where general is replaced by very general.

Theorem 3.5.4. *Let (X, Δ) be a kawamata log terminal pair, where X is projective of dimension n and let H be an ample $\mathbb{Q}$-divisor. Suppose there is a constant $\gamma \geq 1$ and a family of subvarieties $V \longrightarrow B$ with the following property.*

If x and y are two general points of X then, possibly switching x and y, we can find $b \in B$ and $0 \leq \Delta_b \sim_{\mathbb{Q}} (1-\delta)H$, for some $\delta > 0$, such that $(X, \Delta + \Delta_b)$ is not kawamata log terminal at y and there is a unique non kawamata log terminal place of $(X, \Delta + \Delta_b)$ whose centre V_b contains x. Further there is a divisor D on W, the normalisation of V_b, such that ϕ_D is birational and $\gamma H|_W - D$ is pseudo-effective.

Then mH is potentially birational, where $m = 2p^2\gamma + 1$ and $p = \dim V_b$.

Proof. Let x and y be two general points of X. Possibly switching x and y, we will prove by descending induction on k that there is a $\mathbb{Q}$-divisor $\Delta_0 \geq 0$ such that:
$(\flat)_k$ $\Delta_0 \sim_{\mathbb{Q}} \lambda H$, for some $\lambda < 2(p-k)p\gamma + 1$, where $(X, \Delta + \Delta_0)$ is log canonical at x, not kawamata log terminal at y and there is a non kawamata log terminal centre $Z \subset V_b$ of dimension at most k containing x.

Suppose $k = p$. $(X, \Delta + \Delta_b)$ is not kawamata log terminal but log canonical at x since there is a unique non kawamata log terminal place whose centre contains x. Thus $\Delta_0 = \Delta_b \sim_{\mathbb{Q}} \lambda H$, where $\lambda = 1 - \delta < 1$, satisfies $(\flat)_k$ and so this is the start of the induction.

Now suppose that we may find a $\mathbb{Q}$-divisor Δ_0 satisfying $(\flat)_k$. We may assume that Z is the minimal non kawamata log terminal centre containing x and that Z has dimension k. Let $Y \subset W$ be the inverse image of Z. As x is a general point of X it is also a general point of W, Y and Z. In particular the restriction of $\gamma H|_W - D$ to Y is pseudo-effective, $Y \longrightarrow Z$ is birational, and as ϕ_D is birational and x is general, the restriction of ϕ_D to Y is birational. Thus

$$\operatorname{vol}(Y, \gamma H|_Y) \geq \operatorname{vol}(Y, D|_Y) \geq 1,$$

where the last inequality is proved, for example, in [14, 2.2]. Note that

$$\operatorname{vol}(Z, \gamma H|_Z) = \operatorname{vol}(Y, \gamma H|_Y),$$

as H is nef, see for example [23, VI.2.15]. Thus

$$\operatorname{vol}(Z, 2p\gamma H|_V) > \operatorname{vol}(Z, 2k\gamma H|_V) \geq 2k^k,$$

so that by [15, 2.3.5], we may find $\Delta_1 \sim_{\mathbb{Q}} \mu H$, where $\mu < 2p\gamma$ and constants $0 < a_i \leq 1$ such that $(X, \Delta + a_0\Delta_0 + a_1\Delta_1)$ is log canonical at x, not kawamata log terminal at y and there is a non kawamata log terminal centre Z' containing x, whose dimension is less than k. As

$$a_0\Delta_0 + a_1\Delta_1 \sim_{\mathbb{Q}} (a_0\lambda + a_1\mu)H,$$

and

$$\lambda' = a_0\lambda + a_1\mu < 2(p-k)p\gamma + 1 + 2p\gamma = 2(p-(k-1))p\gamma + 1,$$

$a_0\Delta_0 + a_1\Delta_1$ satisfies $(\flat)_{k-1}$. This completes the induction and the proof. $\square$

Theorem 3.5.5. *Fix a positive integer n. Let $\mathfrak{B}_0$ be a set of kawamata log terminal pairs (X, Δ), where X is projective of dimension n and $K_X + \Delta$ is ample.*

Suppose that there are positive integers p, k and l such that for every $(X, \Delta) \in \mathfrak{B}_0$ we have:

1. *There is a family of subvarieties $V \longrightarrow B$ such that if x and y are two general points of X then, possibly switching x and y, we can find $b \in B$ and $0 \leq \Delta_b \sim_{\mathbb{Q}} (1-\delta)H$, for some $\delta > 0$, such that $(X, \Delta + \Delta_b)$ is not kawamata log terminal at y and there is a unique non kawamata log terminal place of $(X, \Delta + \Delta_b)$ whose centre V_b contains x, where $H = k(K_X + \Delta)$. Further there is a divisor D on W, the normalisation of V_b, such that ϕ_D is birational and $lH|_W - D$ is pseudo-effective.*
2. *Either $p\Delta$ is integral or the coefficients of Δ belong to*

$$\{\, \frac{r-1}{r} \mid r \in \mathbb{N} \,\}.$$

Then there is a positive integer m such that $\phi_{mk(K_X+\Delta)}$ is birational, for every $(X, \Delta) \in \mathfrak{B}_0$.

Proof. Let $m_0 = 2(n-1)^2 l + 1$. (3.5.4) implies that $m_0 H$ is potentially birational. But then [15, 2.3.4.1] implies that $\phi_{K_X + \lceil m_0 j H \rceil}$ is birational for all positive integers j.

If $p\Delta$ is integral then

$$K_X + \lceil m_0 kp(K_X + \Delta) \rceil = \lfloor (m_0 kp + 1)(K_X + \Delta) \rfloor,$$

and if the coefficients of Δ belong to

$$\{ \frac{r-1}{r} \mid r \in \mathbb{N} \},$$

then

$$K_X + \lceil m_0 kp(K_X + \Delta) \rceil = \lfloor (m_0 kp + 1)(K_X + \Delta) \rfloor.$$

Let $m = (m_0 + 1)p$. $\square$

4. Adjunction

We will need the following basic result about adjunction (see for example §16 in [27]).

Lemma 4.1. *Let $(X, \Delta = S' + B)$ be a log canonical pair, where S has coefficient one in Δ. If S is the normalisation of S' then there is a divisor $\Theta = \operatorname{Diff}_S(B)$ on S such that*

$$(K_X + \Delta)|_S = K_S + \Theta.$$

1. *If (X, Δ) is purely log terminal then (S, Θ) is kawamata log terminal.*
2. *If (X, Δ) is divisorially log terminal then (S, Θ) is divisorially log terminal.*
3. *If $B = \sum b_i B_i$ then the coefficients of Θ belong to the set $D(\{ b_1, b_2, \dots, b_m \})$.*

In particular, if (X, Δ) is divisorially log terminal and the coefficients of B belong to the set I then the coefficients of Θ belong to the set $D(I)$.

Theorem 4.2. *Let I be a subset of $[0, 1]$ which contains 1. Let X be a projective variety of dimension n and let V be an irreducible closed subvariety, with normalisation W. Suppose we are given a log pair (X, Δ) and an $\mathbb{R}$-Cartier divisor $\Delta' \geq 0$, with the following properties:*

1. *the coefficients of Δ belong to I,*
2. *(X, Δ) is kawamata log terminal, and*
3. *there is a unique non kawamata log terminal place ν for $(X, \Delta + \Delta')$, with centre V.*

Then there is a divisor Θ on W whose coefficients belong to

$$\{\, a \,|\, 1-a \in \mathrm{LCT}_{n-1}(D(I)) \,\} \cup \{\, 1 \,\},$$

such that the difference

$$(K_X + \Delta + \Delta')|_W - (K_W + \Theta),$$

is pseudo-effective.

Now suppose that V is the general member of a covering family of subvarieties of X. Let $\psi\colon U \longrightarrow W$ be a log resolution of W and let Ψ be the sum of the strict transform of Θ and the exceptional divisors. Then

$$K_U + \Psi \geq (K_X + \Delta)|_U.$$

Proof. Since there is a unique non kawamata log terminal place with centre V, it follows that $(X, \Delta+\Delta')$ is log canonical but not kawamata log terminal at the generic point of V, see (2.31) of [29]. Let $g\colon Y \longrightarrow X$ be a divisorially log terminal modification of $(X, \Delta+\Delta')$, (3.3.1), so that the centre of ν is a divisor S on Y and this is the only exceptional divisor with centre V. As $(X, \Delta+\Delta')$ is divisorially log terminal, S is normal and so there is a commutative diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ W \arrow[r] & X. \end{tikzcd}$$

We may write

$$K_Y+S+\Gamma = g^*(K_X+\Delta)+E \qquad \text{and} \qquad K_Y+S+\Gamma+\Gamma' = g^*(K_X+\Delta+\Delta'),$$

where Γ is the sum of the strict transform of Δ and the exceptional divisors, apart from S. In particular the coefficients of Γ belong to I. As (X, Δ) is kawamata log terminal, $E \geq 0$. As g is a divisorially log terminal modification of $(X, \Delta+\Delta')$, $\Gamma' \geq 0$ and $(Y, S+\Gamma)$ is divisorially log terminal. We may write

$$(K_Y + S + \Gamma)|_S = K_S + \Phi \qquad \text{and} \qquad (K_Y + S + \Gamma + \Gamma')|_S = K_S + \Phi'.$$

Note that the coefficients of Φ belong to $D(I)$. Let B be a prime divisor on W. Let

$$\mu = \sup\{\, t \in \mathbb{R} \,|\, (S, \Phi + tf^*B) \text{ is log canonical over a neighbourhood of the generic point of } B \,\},$$

be the log canonical threshold over a neighbourhood of the generic point of B. We define Θ by

$$\mathrm{mult}_B(\Theta) = 1 - \mu.$$

It is clear that the coefficients of Θ belong to

$$\{\, a \mid 1-a \in \mathrm{LCT}_{n-1}(D(I)) \,\} \cup \{\, 1 \,\}.$$

Let

$$\lambda = \sup\{\, t \in \mathbb{R} \mid (S, \Phi' + tf^*B) \text{ is log canonical over a} \\ \text{neighbourhood of the generic point of } B \,\},$$

be the log canonical threshold over a neighbourhood of the generic point of B. We define a divisor Θ_b on W by

$$\mathrm{mult}_B(\Theta_b) = 1 - \lambda.$$

As $\Gamma' \geq 0$ we have $\Phi \leq \Phi'$, so that $\lambda \leq \mu$. But then

$$\Theta \leq \Theta_b.$$

Note that Θ_b is precisely the divisor defined in Kawamata's subadjunction formula, see Theorem 1 and 2 of [18] and also (8.5.1) and (8.6.1) of [25]. It follows that the difference

$$(K_X + \Delta + \Delta')|_W - (K_W + \Theta_b),$$

is pseudo-effective, so that the difference

$$(K_X + \Delta + \Delta')|_W - (K_W + \Theta),$$

is certainly pseudo-effective.

Now suppose that V is the general member of a covering family of subvarieties of X. We first relate the definition of Θ, which uses the log canonical threshold on S, to a log canonical threshold on X. Let B be a prime divisor on W and let A be its image on V. Pick any $\mathbb{Q}$-divisor $H \geq 0$ on X which is $\mathbb{Q}$-Cartier in a neighbourhood of the generic point of A and which does not contain V such that

$$\mathrm{mult}_B(H|_W) = 1.$$

We have

$$K_Y + S + \Gamma + tg^*H = g^*(K_X + \Delta + tH) + E,$$

and so

$$(K_Y + S + \Gamma + tg^*H)|_S = K_S + \Phi + tf^*B,$$

over a neighbourhood of the generic point of B. Now if $(X, \Delta + tH)$ is not log canonical in a neighbourhood of the generic point of A then $K_Y + S + \Gamma + tg^*H$ is not log canonical over a neighbourhood of the generic point of B. Inversion of adjunction on Y, cf [17], implies that $K_Y + S + \Gamma + tg^*H$ is log canonical over a neighbourhood of the generic point of B if and only if $K_S + \Phi + tf^*B$ is log canonical over a neighbourhood of the generic point of B. It follows that if

$$\mu = \sup\{\, t \in \mathbb{R} \mid (S, \Phi + tf^*B) \text{ is log canonical over a } \text{neighbourhood of the generic point of } B \,\},$$

the log canonical threshold of f^*B over a neighbourhood of the generic point of B, and

$$\xi = \sup\{\, t \in \mathbb{R} \mid (X, \Delta + tH) \text{ is log canonical at the generic point of } A \,\},$$

the log canonical threshold of H at the generic point of A, then $\mu \leq \xi$.

By assumption we may pick a component R_0 of the Hilbert scheme $\mathcal{H}$ whose universal family dominates X, such that V is the general member of R_0. Let $\pi_0 \colon Z_0 \longrightarrow R_0$ be the restriction of the normalisation of the universal family. Cutting by hyperplanes in R_0 we may find $R \subset R_0$ with $V \in R$ such that if $\pi \colon Z \longrightarrow R$ is the restriction of π_0 then the natural morphism $h \colon Z \longrightarrow X$ is generically finite (note that if we take the hyperplanes successively from a fixed sequence of general pencils of hyperplanes then we won't lose the fact that V is a general element). We may write

$$K_Z + \Xi = h^*(K_X + \Delta).$$

Possibly blowing up, we may assume that (Z, Ξ) has simple normal crossings over a dense open subset R_1 of R. Let U be the fibre of π corresponding to W. As V is a general member of R_0, we may assume that $r = \pi(U) \in R_1$ and so $(U, \Xi|_U)$ has simple normal crossings. As the coefficients of $\Xi|_U$ are at most one it follows that $(U, \Xi|_U)$ is sub log canonical. Therefore it is enough to check that

$$K_U + \Psi \geq (K_X + \Delta)|_U = K_U + \Xi|_U,$$

on the given model and in fact we just have to check that $\Psi \geq \Xi|_U$.

Let C be a prime divisor on U. If $\operatorname{mult}_C \Xi|_U \leq 0$ there is nothing to prove as $\Psi \geq 0$. If C is an exceptional divisor of $U \longrightarrow V$ then $\operatorname{mult}_C \Psi = 1$ and there is again nothing to prove as $\operatorname{mult}_C \Xi|_U \leq 1$.

Otherwise pick a prime component G of Ξ such that $\operatorname{mult}_C(G|_U) = 1$. If $h(G)$ is a divisor then let $H = h(G)/e$ where e is the ramification index at G. Note that the pullback of H to W is $\mathbb{Q}$-Cartier in a neighbourhood of the generic point of $B = \psi(C)$. Otherwise, pick a $\mathbb{Q}$-Cartier divisor $H \geq 0$, which does not contain V, such that $\operatorname{mult}_G(h^*H) = 1$. Either way, as $r \in R$ is general it follows that $\operatorname{mult}_C(h^*H|_U) = 1$. But then

$$\operatorname{mult}_B(H|_W) = \operatorname{mult}_C(h^*H|_U) = 1.$$

We may write

$$K_Z + \Xi + \xi h^* H = h^*(K_X + \Delta + \xi H).$$

As $(X, \Delta + \xi H)$ is log canonical in a neighbourhood of the generic point of B, $K_Z + \Xi + \xi h^* H$ is sub log canonical in a neighbourhood of the generic point of C. Note that in a neighbourhood of the generic point of C,

$$(K_Z + \Xi + \xi h^* H)|_U = K_U + \Xi|_U + \xi C + J,$$

where $J \geq 0$. As r is a general point of R, $(U, \Xi|_U + \xi C + J)$ is sub log canonical in a neighbourhood of the generic point of C. It follows that

$$\operatorname{mult}_C \Xi|_U + \xi \leq 1,$$

so that

$$\operatorname{mult}_C \Psi = \operatorname{mult}_B \Theta = 1 - \mu \geq 1 - \xi \geq \operatorname{mult}_C \Xi|_U.$$

Thus $\Psi \geq \Xi|_U$. □

5. Global to local

Lemma 5.1. *Fix a positive integer n and a set $1 \in I \subset [0, 1]$.*

Suppose (X, Δ) is a log canonical pair where X is a variety of dimension $n + 1$, the coefficients of Δ belong to I and there is a non kawamata log terminal centre $V \subset X$. Suppose that $c \in I$ is the coefficient of some component M of Δ which contains V.

Then we may find a log canonical pair (S, Θ) where S is a projective variety of dimension at most n, the coefficients of Θ belong to $D(I)$, $K_S + \Theta$ is numerically trivial and some component of Θ has coefficient

$$\frac{m - 1 + f + kc}{m},$$

where m, $k \in \mathbb{N}$ and $f \in D(I)$.

Proof. Possibly passing to an open subset of X and replacing V by a maximal (with respect to inclusion) non kawamata log terminal centre, we may assume that X is quasi-projective. If V is a divisor then $M = V$ is a component of Δ with coefficient one so that $c = 1$. As $1 \in I$ we may take $(S, \Theta) = (\mathbb{P}^1, p + q)$, where p and q are two points of $\mathbb{P}^1$.

Otherwise, let $\pi\colon Y \longrightarrow X$ be a divisorially log terminal modification of (X, Δ). Then Y is $\mathbb{Q}$-factorial and we may write

$$K_Y + E + \Gamma = \pi^*(K_X + \Delta),$$

where Γ is the strict transform of Δ, E is the sum of the exceptional divisors and the pair $(Y, E + \Gamma)$ is divisorially log terminal. By (4) of (3.3.1) we may choose π so that there is a nef divisor of the form

$-N - F$, where N is the strict transform of M and $F \geq 0$ is a sum of exceptional divisors whose centres are contained in M.

By assumption π is not an isomorphism over the generic point of V. It follows that N must intersect an exceptional divisor S of π whose centre is V. We may write

$$(K_Y + E + \Gamma)|_S = K_S + \Theta,$$

by adjunction, where (S, Θ) is divisorially log terminal, the coefficients of Θ belong to $D(I)$ and some component of Θ has a coefficient of the form

$$\frac{m - 1 + f + kc}{m},$$

where m, $k \in \mathbb{N}$ and $f \in D(I)$. Note that $N \cap S$ dominates V. If $v \in V$ is a general point then (S_v, Θ_v) is divisorially log terminal, S_v is projective of dimension at most n, the coefficients of Θ_v belong to $D(I)$, some component of Θ_v has a coefficient of the form

$$\frac{m - 1 + f + kc}{m},$$

and $K_{S_v} + \Theta_v$ is numerically trivial. □

Lemma 5.2. *Let $I \subset [0, 1]$ be a set which satisfies the DCC.*

If $J_0 \subset [0, 1]$ is a finite set then

$$I_0 = \{\, c \in I \mid \frac{m - 1 + f + kc}{m} \in J_0, \text{ for some } k,\ m \in \mathbb{N} \text{ and } f \in D(I) \,\}$$

is a finite set.

Proof. We may assume that $c \neq 0$. Suppose that

$$l = \frac{m - 1 + f + kc}{m} \in J_0.$$

Then $kc \leq 1$. As I satisfies the DCC, we may find $\delta > 0$ such that $c > \delta$. It follows that $k < 1/\delta$ so that k can take on only finitely many values. As J_0 is finite, we may find $\epsilon > 0$ such that if $l < 1$ then $l < 1 - \epsilon$. But then $m < \frac{1}{\epsilon}$. If $l = 1$ then $f + kc = 1$, in which case we may take $m = 1$. Either way, we may assume that m takes on only finitely many values.

Fix k, m and l. Then

$$c = \frac{(ml - m + 1) - f}{k}.$$

The LHS belongs to I, a set which satisfies the DCC. The RHS belongs to a set which satisfies the ACC. But the only set which satisfies both the DCC and the ACC is a finite set. □

Lemma 5.3. *Theorem D_{n-1} implies Theorem A_n.*

Proof. As I satisfies the DCC so does $J = D(I)$. As we are assuming Theorem D_{n-1}, there is a finite set $J_0 \subset J$ such that if (S, Θ) is a log canonical pair where S is projective of dimension at most $n-1$, the coefficients of Θ belong to J and $K_S + \Theta$ is numerically trivial, then the coefficients of Θ belong to J_0. Let

$$I_0 = \{\, c \in I \mid \frac{m-1+f+kc}{m} \in J_0 \text{ for some } k \text{ and } m \in \mathbb{N} \text{ and } f \in I_+ \,\}.$$

As J_0 is a finite set, (5.2) implies that I_0 is also a finite set.

Suppose that (X, Δ) is a log canonical pair where X is a quasi-projective variety of dimension n, the coefficients of Δ belong to I, and there is a non kawamata log terminal centre $Z \subset X$ which is contained in every component of Δ. (5.1) implies that the coefficients of Δ belong to I_0. $\square$

6. Upper bounds on the volume

Lemma 6.1. *Using the notation of Theorem B_n, Theorem D_{n-1} and Theorem A_{n-1} imply that there is a constant $\epsilon > 0$ with the following property:*

If $(X, \Delta) \in \mathfrak{D}$, where X has dimension n, Δ is big and $K_X + \Phi$ is numerically trivial, where

$$\Phi \geq (1-\delta)\Delta,$$

for some $\delta < \epsilon$, then (X, Φ) is kawamata log terminal.

Proof. Theorem D_{n-1} and Theorem A_{n-1} imply that we may find $\epsilon > 0$ with the following property: if S is a projective variety of dimension $n-1$, (S, Θ) and (S, Θ') are two log pairs, the coefficients of Θ belong to $D(I)$, and

$$(1-\epsilon)\Theta \leq \Theta' \leq \Theta,$$

then (S, Θ) is log canonical if (S, Θ') is log canonical, and moreover $\Theta = \Theta'$ if in addition $K_S + \Theta'$ is numerically trivial.

Suppose that (X, Φ) is not kawamata log terminal, where

$$\Phi \geq (1-\delta)\Delta,$$

for some $\delta < \epsilon$ and $K_X + \Phi$ is numerically trivial. As $\delta < \epsilon$ and Φ is big we may assume that $K_X + \Phi$ is not log canonical. Pick $\lambda \in (0, 1]$ such that $(X, (1-\lambda)\Delta + \lambda\Phi)$ is log canonical but not kawamata log terminal. As Φ is big, $\delta < \epsilon$ and (X, Δ) is kawamata log terminal, (3.2.3) implies that, perturbing Φ, we may assume $(X, (1-\lambda)\Delta + \lambda\Phi)$ has only one non kawamata log terminal place.

Replacing Φ by $(1-\lambda)\Delta + \lambda\Phi$ we may assume that (X, Φ) is purely log terminal and the non kawamata log terminal locus is irreducible. Let $\phi\colon Y \longrightarrow X$ be a divisorially log terminal modification of (X, Φ). We may write

$$K_Y + \Psi = \phi^*(K_X + \Phi) \qquad \text{and} \qquad K_Y + \Gamma + aS = \phi^*(K_X + \Delta),$$

where $S = \lfloor \Psi \rfloor$ is a prime divisor, Γ is the strict transform of Δ and $a < 1$, as (X, Δ) is kawamata log terminal.

As $K_Y + \Psi$ is numerically trivial, $K_Y + \Psi - S$ is not pseudo-effective. By [8, 1.3.3], we may run $f\colon Y \dashrightarrow W$ the $(K_Y + \Psi - S)$-MMP until we end with a Mori fibre space $\pi\colon W \longrightarrow Z$. As $K_Y + \Psi$ is numerically trivial, every step of this MMP is S-positive, so that the strict transform T of S dominates Z. Let F be the general fibre of π. Replacing Y, Γ and Ψ by F and the restriction of $\pi_*\Gamma$ and $\pi_*\Psi$ to F, we may assume that S, Ψ and Γ are $\mathbb{Q}$-linearly equivalent to multiples of the same ample divisor.

In particular $K_Y + \Gamma + S$ is ample. As $\Psi \geq (1-\epsilon)\Gamma + S$, it follows that $K_Y + (1-\eta)\Gamma + S$ is numerically trivial, for some $0 < \eta < \epsilon$, and $K_Y + (1-\epsilon)\Gamma + S$ is log canonical. We may write

$$\begin{aligned} (K_Y + (1-\epsilon)\Gamma + S)|_S &= K_S + \Theta_1, \\ (K_Y + (1-\eta)\Gamma + S)|_S &= K_S + \Theta_2, \quad \text{and} \\ (K_Y + \Gamma + S)|_S &= K_S + \Theta, \end{aligned}$$

where the coefficients of Θ belong to $D(I)$. Note that

$$(1-\epsilon)\Theta \leq \Theta_1 \leq \Theta_2 \leq \Theta,$$

where by (4.1) the first inequality follows from the inequality

$$t\left(\frac{m-1+f}{m}\right) \leq \frac{m-1+tf}{m} \qquad \text{for any} \qquad t \leq 1.$$

As (S, Θ_1) is log canonical, it follows that (S, Θ) is log canonical. In particular (S, Θ_2) is also log canonical. As $K_S + \Theta_2$ is numerically trivial, $\Theta = \Theta_2$, a contradiction. $\square$

Lemma 6.2. *Theorem D_{n-1} and Theorem A_{n-1} imply Theorem B_n.*

Proof. Let $\epsilon > 0$ be the constant given by (6.1). If $(X, \Delta) \in \mathfrak{D}$, Δ is big, $\Pi \sim_{\mathbb{R}} \eta\Delta$ and $(X, \Pi + (1-\eta)\Delta)$ is not kawamata log terminal, then (6.1) implies that $\eta \geq \epsilon$. But then (3.2.2) implies that

$$\operatorname{vol}(X, \Delta) \leq \left(\frac{n}{\epsilon}\right)^n. \qquad \square$$

7. Birational boundedness

Lemma 7.1. *Let (X, Δ) be a log pair, where X is a projective variety of dimension n and let D be a big $\mathbb{R}$-divisor.*

If $\operatorname{vol}(X, D) > (2n)^n$ then there is a family $V \longrightarrow B$ of subvarieties of X such that if x and y are two general points of X then we may find $b \in B$ and $0 \leq \Delta_b \sim_{\mathbb{R}} D$ such that $(X, \Delta + \Delta_b)$ is not kawamata log terminal at y and there is a unique non kawamata log terminal place of $(X, \Delta + \Delta_b)$ whose centre V_b contains x. Further, if $B_1, B_2, \dots, B_k$ are the irreducible components of B and $V_i \longrightarrow B_i$ is the corresponding family then the natural map $V_i \longrightarrow X$ is dominant.

Proof. Let K be the algebraic closure of the function field of X. There is a fibre square

$$\begin{tikzcd} X_K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} K \arrow[r] & \operatorname{Spec} k. \end{tikzcd}$$

Let ξ be the closed point of X_K corresponding to the generic point of X, and let Δ_K and D_K be the pullbacks of Δ and D to X_K. (3.2.2) implies that we may find $0 \leq D_\xi \sim_{\mathbb{R}} D_K/2$ such that $(X_K, \Delta_K + D_\xi)$ is not log canonical at ξ. By standard arguments we may spread out D_ξ to a family of divisors D_t, $t \in T$, where there is dominant morphism $g\colon T \longrightarrow X$ such that $(X, \Delta + D_t)$ is not log canonical at $x = g(t)$ and where $D_t \sim_{\mathbb{R}} D/2$.

Let y be a general point of X. Pick s such that $(X, \Delta + D_s)$ is not log canonical at $y = g(s)$, where $D_s \sim_{\mathbb{R}} D/2$. Let

$$\beta = \beta_{s,t} = \sup\{\, \lambda \in \mathbb{R} \,|\, (X, \Delta + \lambda(D_t + D_s)) \text{ is log canonical at } x \,\},$$

be the log canonical threshold. Thus $(X, \Delta + \beta(D_s + D_t))$ is log canonical but not kawamata log terminal at x. Possibly switching s and t, we may assume that $(X, \Delta + \beta(D_s + D_t))$ is not kawamata log terminal at y. Perturbing, by (3.2.3) we may assume that there is a unique non kawamata log terminal place of $(X, \Delta + \beta(D_t + D_s))$ whose centre $V_{(s,t)}$ contains x (as y is general, we will not lose the property that $(X, \Delta + \beta(D_t + D_s))$ is not kawamata log terminal at y). Decomposing $B = T \times T$ into finitely many locally closed subsets, we may assume that the log canonical threshold is constant on each irreducible component of B, and moreover that $V_{s,t}$ forms a family $V \longrightarrow B$. Possibly discarding components of B, we may assume that every component of V dominates X. Then the image of B in $X \times X$ contains an open subset of the form $U \times U$. $\square$

Lemma 7.2. *Assume Theorem C_{n-1} and Theorem A_{n-1}. Fix a positive integer p.*

Let $\mathfrak{B}_1$ be the set of kawamata log terminal pairs (X, Δ), where X is projective of dimension n, $K_X + \Delta$ is big and either $p\Delta$ is integral or the coefficients of Δ belong to

$$\{ \frac{r-1}{r} \,|\, r \in \mathbb{N} \}.$$

Then there is a positive integer m such that $\phi_{m(K_X+\Delta)}$ is birational, for every $(X, \Delta) \in \mathfrak{B}_1$.

Proof. Passing to a log canonical model of (X, Δ) we may assume that $K_X + \Delta$ is ample.

Pick a positive integer k such that $\mathrm{vol}(X, k(K_X + \Delta)) > (2n)^n$. We will apply (3.5.5) to $k(K_X + \Delta)$. (2) holds by hypothesis.

Let

$$J = \{ 1 - a \,|\, a \in \mathrm{LCT}_{n-1}(D(I)) \} \cup \{ 1 \}.$$

Theorem A_{n-1} implies that J satisfies the DCC.

Theorem C_{n-1} implies that there is a positive integer l such that if (U, Ψ) is a log canonical pair, where U is projective of dimension at most $n-1$, the coefficients of Ψ belong to J and $K_U + \Psi$ is big, then $\phi_{l(K_U+\Psi)}$ is birational.

Apply (7.1) to $k(K_X + \Delta)$ to get a family $V \longrightarrow B$. Let $b \in B$ be a general point. Let $\nu \colon W \longrightarrow V_b$ be the normalisation of V_b and let $0 \leq \Delta_b \sim_{\mathbb{R}} k(K_X + \Delta)$ be the divisor given by (7.1), so that V_b is the unique non kawamata log terminal place of $(X, \Delta + \Delta_b)$ containing x. $(4.2)_n$ implies that we may find Θ on W such that

$$(K_X + \Delta + \Delta_b)|_W - (K_W + \Theta),$$

is pseudo-effective, where the coefficients of Θ belong to J.

Let $\psi \colon U \longrightarrow W$ be a log resolution of (W, Θ) and let Ψ be the sum of the strict transform of Θ and the exceptional divisors. $(4.2)_n$ implies that

$$(K_U + \Psi) \geq (K_X + \Delta)|_U,$$

so that $K_U + \Psi$ is big. As the coefficients of Θ belong to J, it follows that the coefficients of Ψ belong to J. But then $\phi_{l(K_U+\Psi)}$ is birational. It is easy to see (1) of (3.5.5) holds.

As the hypotheses of (3.5.5) hold, there is a positive integer m_0 such that $\phi_{m_0 k(K_X+\Delta)}$ is birational. If $\mathrm{vol}(X, K_X + \Delta) \geq 1$ then $\mathrm{vol}(X, 2(n+1)(K_X + \Delta)) > (2n)^n$ and $\phi_{2m_0(n+1)(K_X+\Delta)}$ is birational.

Otherwise, if $\mathrm{vol}(X, K_X + \Delta) < 1$, then we may find k such that

$$(2n)^n < \mathrm{vol}(X, k(K_X + \Delta)) \leq (4n)^n.$$

It follows that

$$\operatorname{vol}(X, m_0k(K_X + \Delta)) \leq (4m_0n)^n.$$

(3.5.2) implies that there is a constant $0 < \delta < 1$ such that if $(X, \Delta) \in \mathfrak{B}$, then

$$\operatorname{vol}(X, K_X + \Delta) > \delta.$$

In this case,

$$\operatorname{vol}(X, \alpha(K_X + \Delta)) > (2n)^n,$$

where

$$\alpha = \frac{2n}{\delta},$$

and we may take $m = \max(m_0 \ulcorner \alpha \urcorner, 2m_0(n + 1))$. $\square$

Lemma 7.3. *Using the notation of Theorem C_n, assume Theorem C_{n-1}, Theorem A_{n-1}, and Theorem B_n.*

Then there is a constant $\beta < 1$ such that if $(X, \Delta) \in \mathfrak{B}$ then the pseudo-effective threshold

$$\lambda = \inf\{\, t \in \mathbb{R} \,|\, K_X + t\Delta \textit{ is big}\,\},$$

is at most β.

Proof. We may assume that $1 \in I$. Suppose that $(X, \Delta) \in \mathfrak{B}$. Let $\pi \colon W \longrightarrow X$ be a log resolution of (X, Δ). We may write

$$K_W + \Xi = \pi^*(K_X + \Delta) + F,$$

where Ξ is the strict transform of Δ plus the sum of the exceptional divisors and $F \geq 0$ is exceptional as (X, Δ) is log canonical. Let

$$\mu = \inf\{\, t \in \mathbb{R} \,|\, K_W + t\Xi \text{ is big}\,\}.$$

be the pseudo-effective threshold. As $\pi_*(K_W + \mu\Xi) = K_X + \mu\Delta$ is pseudo-effective it follows that $\lambda \leq \mu$ and so it suffices to bound μ away from one. Replacing (X, Δ) by (W, Ξ) we may assume that (X, Δ) has simple normal crossings.

We may assume that $\lambda > 1/2$, so that K_X is not pseudo-effective. As $K_X + \Delta$ is big we may find $0 \leq D \sim_{\mathbb{R}} (K_X + \Delta)$. If $\epsilon > 0$ then

$$(1 + \epsilon)(K_X + \lambda\Delta) \sim_{\mathbb{R}} K_X + \mu\Delta + \epsilon D,$$

where $\mu = (1 + \epsilon)\lambda - \epsilon < \lambda$. It follows that if ϵ is sufficiently small then $K_X + \mu\Delta + \epsilon D$ is kawamata log terminal. By [8, 1.4.2], we may run $f \colon X \dashrightarrow Y$ the $(K_X + \lambda\Delta)$-MMP with scaling until $K_Y + \Gamma$ is kawamata log terminal and nef, where $\Gamma = f_*(\lambda\Delta)$. Now we may run the $(K_Y + \mu f_*\Delta)$-MMP with scaling of f_*D until we get to a Mori fibre space $\pi \colon Y \longrightarrow Z$; all steps of this MMP are $(K_Y + \Gamma)$-trivial, as all steps of this MMP are $(K_Y + \mu f_*\Delta + \epsilon f_*D)$-trivial, so that (Y, Γ)

remains kawamata log terminal and nef. Replacing (X, Δ) by a log resolution, we may assume that f is a morphism. Replacing X by the general fibre of the composition of f and π, we may assume that Z is a point, so that $K_Y + \Gamma$ is numerically trivial.

Suppose that we have a sequence of such log pairs $(X_l, \Delta_l) \in \mathfrak{B}$. We may assume that the pseudo-effective threshold is an increasing sequence,

$$\lambda_1 < \lambda_2 < \lambda_3 < \dots,$$

and it suffices to bound this sequence away from one. Let

$$J = \{\, \lambda_l i \,|\, i \in I, l \in \mathbb{N} \,\}.$$

Then J satisfies the DCC, as λ_l are an increasing sequence.

Theorem B_n implies that there is a constant C such that $\operatorname{vol}(Y, \Gamma) < C$ for any Γ whose coefficients belong to J. Let α be the smallest non-zero element of J and let $G = G_l$ be the sum of the components of $\Gamma = \Gamma_l$. Let $Y = Y_l$. Then

$$\begin{aligned}
\operatorname{vol}(Y, K_Y + G) &= \operatorname{vol}(Y, G - \Gamma) \\
&\leq \operatorname{vol}(Y, G) \\
&\leq \operatorname{vol}(Y, \frac{1}{\alpha}\Gamma) \\
&\leq \frac{C}{\alpha^n}.
\end{aligned}$$

Let D be the sum of the components of Δ. Certainly $K_X + D$ is big. We may write

$$K_X + D = f^*(K_Y + G) + F,$$

where F is supported on the exceptional locus. It follows that

$$\operatorname{vol}(X, K_X + D) \leq \operatorname{vol}(Y, K_Y + G) \leq \frac{C}{\alpha^n}.$$

Given (X_l, D_l) we may pick $r \in \mathbb{N}$ such that

$$K_{X_l} + \Theta_l = K_{X_l} + \frac{r-1}{r} D_l$$

is big. As the coefficients of Θ_l belong to

$$\{\, \frac{r-1}{r} \,|\, r \in \mathbb{N} \,\},$$

(7.2) implies that

$$\{\, (X_l, \Theta_l) \,|\, l \in \mathbb{N} \,\},$$

is log birationally bounded. But then

$$\{\, (X_l, \Delta_l) \,|\, l \in \mathbb{N} \,\},$$

is log birationally bounded. In particular, [15, 1.9] implies that there is a constant $\delta > 0$ such that

$$\operatorname{vol}(X_l, K_{X_l} + \Delta_l) \geq \delta,$$

for every $l \in \mathbb{N}$. In this case

$$\delta \leq \operatorname{vol}(X, K_X+\Delta) \leq \operatorname{vol}(Y, K_Y+\frac{1}{\lambda}\Gamma) = (\frac{1}{\lambda}-1)^n \operatorname{vol}(Y, \Gamma) \leq (\frac{1}{\lambda}-1)^n C,$$

so that we may take

$$\beta = \frac{1}{1 + \left(\frac{\delta}{C}\right)^{1/n}}. \qquad \square$$

Lemma 7.4. *Theorem C_{n-1}, Theorem A_{n-1}, and Theorem B_n imply Theorem C_n.*

Proof. Replacing I by

$$I \cup \{\, \frac{r-1}{r} \,|\, r \in \mathbb{N} \,\} \cup \{1\},$$

we may assume that 1 is both an accumulation point of I and an element of I. Let α be the smallest non-zero element of I. By (7.3) there is a constant $\beta < 1$ such that if $(X, \Delta) \in \mathfrak{B}$ then the pseudo-effective threshold

$$\lambda = \inf\{\, t \in \mathbb{R} \,|\, K_X + t\Delta \text{ is big} \,\},$$

is at most β.

Pick $(X, \Delta) \in \mathfrak{B}$. Let $\pi \colon Y \longrightarrow X$ be a log resolution of (X, Δ). Then we may write

$$K_Y + \Gamma = \pi^*(K_X + \Delta) + E,$$

where Γ is the strict transform of Δ plus the sum of the exceptional divisors. Replacing (X, Δ) by (Y, Γ) we may assume that (X, Δ) is log smooth. If $S = \lfloor \Delta \rfloor$, then we may pick $r \in \mathbb{N}$ such that

$$K_X + \Delta' = K_X + \frac{r-1}{r} S + \{\Delta\},$$

is big. Replacing (X, Δ) by (X, Δ'), we may assume that (X, Δ) is kawamata log terminal.

Pick p such that

$$p > \frac{2}{\alpha(1-\beta)}.$$

If a is the coefficient of a component of Δ then

$$\begin{aligned}\frac{\lfloor pa \rfloor}{p} &> a - \frac{1}{p} \\ &> a - \frac{\alpha(1-\beta)}{2} \\ &\geq a - \frac{a(1-\beta)}{2} \\ &= \frac{a(1+\beta)}{2}.\end{aligned}$$

It follows that

$$\frac{\beta+1}{2}\Delta \leq \Delta_{\lfloor p \rfloor} \leq \Delta,$$

so that $K_X + \Delta_{\lfloor p \rfloor}$ is big. Since the coefficients of $\Delta_{\lfloor p \rfloor}$ belong to

$$I_0 = \{\, \frac{i}{p} \,|\, 1 \leq i \leq p-1 \,\},$$

(7.2) implies that there is a positive integer m such that $\phi_{m(K_X+\Delta_{\lfloor p \rfloor})}$ is birational. But then $\phi_{m(K_X+\Delta)}$ is birational as well. □

8. Numerically trivial log pairs

Lemma 8.1. *Theorem D_{n-1} and Theorem C_n imply Theorem D_n.*

Proof. We may assume that $1 \in I$ and $n > 1$.

As we are assuming Theorem D_{n-1} there is a finite set $J_0 \subset J = D(I)$ with the following property. If (S, Θ) is a log pair such that S is projective of dimension $n-1$, the coefficients of Θ belong to J, (S, Θ) is log canonical, and $K_S + \Theta$ is numerically trivial, then the coefficients of Θ belong to J_0. Let I_1 be the largest subset of I such that $D(I_1) \subset J_0$. (5.2) implies that I_1 is finite.

Theorem C_n implies that there is a constant m with the following property: if (Y, Γ) is log canonical, Y is a projective variety of dimension n, $K_Y + \Gamma$ is big and the coefficients of Γ belong to I, then $\phi_{m(K_Y+\Gamma)}$ is birational.

For every $1 \leq l \leq m$, let

$$A_l = [(l-1)/m, l/m),$$

and $A_{m+1} = \{\, 1 \,\}$ so that

$$[0, 1] = \bigcup_{l=1}^{m+1} A_l.$$

Let I_2 be the union of the largest elements of $A_l \cap I$ (if $A_l \cap I$ does not have a largest element, either because it is empty or because it has infinitely many elements, then we ignore the elements of $A_l \cap I$). Then I_2 has at most $m+1$ elements, so that I_2 is certainly finite. Let I_0 be the union of I_1 and I_2.

Suppose that (X, Δ) satisfies (1–4) of Theorem D_n. Let $\pi\colon Y \longrightarrow X$ be a divisorially log terminal modification, so that Y is $\mathbb{Q}$-factorial. As (X, Δ) is log canonical if we write

$$K_Y + \Gamma = \pi^*(K_X + \Delta),$$

then Γ is the strict transform of Δ plus the exceptional divisors, so that (Y, Γ) is numerically trivial and divisorially log terminal. Replacing (X, Δ) by (Y, Γ), we may assume that X is $\mathbb{Q}$-factorial. Further (X, Δ) is kawamata log terminal if and only if $\lfloor \Delta \rfloor = 0$. Suppose that B is a prime component of Δ with coefficient i. It suffices to prove that $i \in I_0$. We may assume that $i \neq 1$. Suppose that B intersects a component of $\lfloor \Delta \rfloor$. If S is the normalisation of this component then by adjunction we may write

$$(K_X + \Delta)|_S = K_S + \Theta,$$

where the coefficients of Θ belong to $J = D(I)$ by (4.1). As S is projective of dimension $n-1$, (S, Θ) is log canonical, and $K_S + \Theta$ is numerically trivial, the coefficients of Θ belong to J_0. But then $i \in I_1$.

As $K_X + \Delta$ is numerically trivial, $K_X + \Delta - iB$ is not pseudo-effective. By [8, 1.3.3] we may run $f\colon X \dashrightarrow Y$ the $(K_X + \Delta - iB)$-MMP until we reach a Mori fibre space. As $K_X + \Delta$ is numerically trivial, it follows that every step of this MMP is B-positive. If at some step of this MMP we contract a component S of $\lfloor \Delta \rfloor$ then this component intersects B and $i \in I_1$ by the argument above. Otherwise, it follows that $(Y, f_*\Delta)$ is kawamata log terminal if and only if $\lfloor f_*\Delta \rfloor = 0$. Further B is not contracted and so replacing (X, Δ) by $(Y, f_*\Delta)$, we may assume that X is a Mori fibre space $\pi\colon X \longrightarrow Z$, where B dominates Z.

If Z is not a point, then replacing X by the general fibre of π we are done by induction. So we may assume that X has Picard number one. If $\lfloor \Delta \rfloor \neq 0$ then any component S of $\lfloor \Delta \rfloor$ intersects B and so $i \in I_1$. Otherwise $\lfloor \Delta \rfloor = 0$ and we may assume that (X, Δ) is kawamata log terminal.

Suppose that $j \in I$ and $j > i$. Let $\pi\colon Y \longrightarrow X$ be a log resolution of (X, Δ). Let Γ_0 be the strict transform of Δ, let E by the sum of the exceptional divisors, and let C be the strict transform of B. Set

$$\Gamma = \Gamma_0 + E + (j - i)C.$$

Then (Y, Γ) is log canonical and the coefficients of Γ belong to I. We may write

$$K_Y + \Gamma_0 + E = \pi^*(K_X + \Delta) + F,$$

where $F \geq 0$ contains the full exceptional locus. Pick $\epsilon > 0$ such that $F \geq \epsilon E$. Note that $(j - i)C + \epsilon E > \delta\pi^* B$ for any $\delta > 0$ sufficiently small, so that

$$K_Y + \Gamma = (K_Y + \Gamma_0 + (1 - \epsilon)E) + (j - i)C + \epsilon E,$$

is big. Hence $\phi_{m(K_Y+\Gamma)}$ is birational, so that $K_Y + \Gamma_{\lfloor m \rfloor}$ is big. But then $K_X + \Lambda_{\lfloor m \rfloor}$ is big, where

$$\Lambda = \pi_*\Gamma = \Delta + (j - i)B.$$

It follows that if $i \in A_l$, then $j \geq l/m$, so that i is the largest element of the interval A_l which also belongs to I. Hence $i \in I_2$. $\square$

9. Proofs of Theorems

Proof of (1.5) *and* (1.4). This is Theorem A and Theorem D. $\square$

Proof of (1.1). Suppose that $c_1, c_2, \dots \in \mathrm{LCT}_n(I, J)$, where $c_i \leq c_{i+1}$. It suffices to show that $c_i = c_{i+1}$ for i sufficiently large. By assumption we may find log canonical pairs (X_i, Δ_i) and $\mathbb{R}$-Cartier divisors M_i, where X_i is a variety of dimension n, the coefficients of Δ_i belong to I, the coefficients of M_i belong to J and c_i is the log canonical threshold,

$$c_i = \sup\{\, t \in \mathbb{R} \,|\, (X_i, \Delta_i + c_i M_i) \text{ is log canonical} \,\}.$$

Let $\Theta_i = \Delta_i + c_i M_i$ and

$$K = I \cup \{\, c_i j \,|\, i \in \mathbb{N}, j \in J \,\}.$$

Then (X_i, Θ_i) is log canonical, X_i is a variety of dimension n, the coefficients of Θ_i belong to K and there is a non kawamata log terminal centre V contained in the support of M_i. Possibly throwing away components of Θ_i which don't contain V and passing to an open subset which contains the generic point of V, we may assume that every component of Θ_i contains V.

As K satisfies the DCC, (1.5) implies that the coefficients of Θ_i belong to a finite subset K_0 of K. It follows $c_i = c_{i+1}$ for i sufficiently large. $\square$

Proof of (1.3). (3) is Theorem C.

Fix a constant $V > 0$ and let

$$\mathfrak{D}_V = \{\, (X, \Delta) \in \mathfrak{D} \,|\, 0 < \mathrm{vol}(X, K_X + \Delta) \leq V \,\}.$$

(3) implies that $\phi_{m(K_X+\Delta)}$ is birational. (3.5.2) implies that the set

$$\{\, \mathrm{vol}(X, K_X+\Delta) \,|\, (X,\Delta) \in \mathfrak{D}_V \,\},$$

satisfies the DCC, which implies that (1) and (2) of (1.3) hold in dimension n. $\square$

Lemma 9.1. *Let $Z \longrightarrow T$ be a projective morphism to a variety and suppose that (Z,Φ) has simple normal crossings over T. Suppose that the restriction of any irreducible component of Φ to any fibre is irreducible. Suppose that (Z,Φ) is kawamata log terminal and there is a closed point $0 \in T$ such that $K_{Z_0}+\Phi_0$ is big. Let $0 \leq \Theta \leq \Phi$ be any divisor with the same support as Φ.*

Then we may find finitely many birational contractions $f_i \colon Z \dashrightarrow X_i$ over T such that if $f\colon Z_t \dashrightarrow Y$ is the log canonical model of (Z_t,Ψ) for some $t \in T$ and $\Theta_t \leq \Psi \leq \Phi_t$ then $f = f_{it}$ for some index i.

Proof. [15, 1.7] implies that $K_Z+\Phi$ is big over T. Pick

$$0 \leq D \sim_{\mathbb{R},T} (K_Z+\Phi).$$

Let

$$B = \frac{\epsilon}{1-\epsilon} D.$$

If we pick $\epsilon > 0$ sufficiently small then $K_Z+B+\Phi$ is kawamata log terminal and we may find a divisor $0 \leq \Theta' \leq \Theta$ with

$$K_Z+\Theta = \epsilon(K_Z+\Phi)+(1-\epsilon)(K_Z+\Theta').$$

If $\Theta \leq \Xi \leq \Phi$ then

$$K_Z+\Xi \sim_{\mathbb{R},T} (1-\epsilon)(K_Z+B+\Xi'),$$

where $\Theta' \leq \Xi' \leq \Xi$. It is proved in [8, 1.1.5] that there are finitely many $f_1, f_2, \dots, f_k$ birational contractions $f_i\colon Z \dashrightarrow X_i$ over T such that if $g\colon Z \dashrightarrow X$ is the log canonical model of $K_Z+\Xi$ over T then $g = f_i$ for some index $1 \leq i \leq k$.

It suffices to show that if $\Xi|_{Z_t} = \Psi$ and g is the log canonical model of $K_Z+\Xi$ then $f = g_t$. For this we may assume that T is affine.

In this case the (relative) log canonical model is given by taking Proj

$$X_i = \mathrm{Proj}(Z, R(Z, k(K_Z+\Xi))),$$

of the (truncation of the) canonical ring

$$R(Z, k(K_Z+\Xi)) = \bigoplus_{m\in\mathbb{N}} H^0(Z, \mathcal{O}_Z(mk(K_Z+\Xi))).$$

On the other hand [15, 1.7] implies that if k is sufficiently divisible then

$$R(Z, k(K_Z+\Xi)) \longrightarrow R(Z_t, k(K_{Z_t}+\Psi)),$$

is surjective and so $f = g_t$. $\square$

Proof of (1.6). By definition there is a log pair (Z, B) and a projective morphism $Z \longrightarrow T$, where T is of finite type with the following property. If $(X, \Delta) \in \mathfrak{D}$ then there is a closed point $t \in T$ and a birational map $f \colon X \dashrightarrow Z_t$ such that the support of B_t is a divisor on Z_t which contains the support of the strict transform of Δ and any f^{-1}-exceptional divisor.

We may assume that T is reduced. Decomposing T into a finite union of locally closed subsets and throwing away some components, we may assume that every fibre Z_t is a variety and that B does not contain Z_t; blowing up and decomposing T into a finite union of locally closed subsets, we may assume that (Z, B) has simple normal crossings; passing to an open subset of T, we may assume that the fibres of $Z \longrightarrow T$ are log pairs, so that (Z, B) has simple normal crossings over T; passing to a finite cover of T, we may assume that every stratum of (Z, B) has irreducible fibres over T; decomposing T into a finite union of locally closed subsets, we may assume that T is smooth; finally passing to a connected component of T, we may assume that T is integral.

Let $a = 1 - \epsilon < 1$. By assumption $\delta \leq a \leq 1$. Let $\Phi = aB$ and $\Theta = \delta B$, so that Φ, Θ and B have the same support but the coefficients of Φ are all a, the coefficients of Θ are all δ and the coefficients of B are all one. As (Z, Φ) is kawamata log terminal it follows that there are only finitely many valuations of log discrepancy at most one with respect to (Z, Φ). As (Z, Φ) has simple normal crossings there is a sequence of blow ups $Y \longrightarrow Z$ of strata, which extracts every divisor of log discrepancy at most one. Note that as (Z, Φ) has simple normal crossings over T, it follows that if $t \in T$ is a closed point then every valuation of log discrepancy at most one with respect to (Z_t, Φ_t) has centre a divisor on Y_t.

Suppose that $(X, \Delta) \in \mathfrak{D}$. Then there is a closed point $t \in T$ and a birational map $f \colon X \dashrightarrow Z_t$ such that the support of B_t contains the support of the strict transform of Δ_t and any f^{-1}-exceptional divisor. Let $p \colon W \longrightarrow X$ and $q \colon W \longrightarrow Z_t$ resolve f. Let S be the sum of the p-exceptional divisors and let Ξ be the sum of the strict transform of Δ and aS, so that S and Ξ are divisors on W. We may write

$$K_W + \Xi = p^*(K_X + \Delta) + E,$$

where E is a sum of p-exceptional divisors and $E \geq 0$ as the log discrepancy of (X, Δ) is greater than ϵ.

Let $\Psi = q_*\Xi$. We may write

$$p^*(K_X + \Delta) + E + F = q^*(K_{Z_t} + \Psi),$$

where F is q-exceptional. As $p^*(K_X + \Delta)$ is nef, it is q-nef so that $E + F \geq 0$ by negativity of contraction. If ν is any valuation whose centre is a divisor on X then

$$\begin{aligned} a(Z_t, \Phi_t, \nu) &\leq a(Z_t, \Psi, \nu) && \text{as } \Phi_t \geq \Psi \\ &\leq a(X, \Delta, \nu) && \text{as } E + F \geq 0 \\ &\leq 1 && \text{as the centre of } \nu \text{ is a divisor on } X. \end{aligned}$$

Therefore the induced birational map $Y_t \dashrightarrow X$ is a birational contraction. Thus replacing Z by Y and B by its strict transform union the exceptional divisor, we may assume that $g = f^{-1} \colon Z_t \dashrightarrow X$ is a birational contraction. In this case F is p-exceptional and so g is the log canonical model of (Z_t, Θ_t).

Since there are only finitely integral divisors $0 \leq B' \leq B$, replacing B we may assume that Ψ has the same support as B_t. $K_{Z_t} + \Phi_t$ is big as $K_{Z_t} + \Psi$ is big and $\Phi_t \geq \Psi$. Finally $\Theta_t \leq \Psi \leq \Phi_t$ and so we are done by (9.1). □

10. Proofs of Corollaries

Proof of (1.2). This follows from (1.1) and the main result of [6]. □

Proof of (1.7). (1.5) implies that there is a finite subset $I_0 \subset I$ such that the coefficients of Δ belong to I_0. Thus there is a positive integer r such that $r\Delta$ is integral.

On the other hand, Theorem B implies that there is a constant C such that $\operatorname{vol}(X, \Delta) < C$. Let D be the sum of the components of Δ. Then $K_X + D$ is big and

$$\begin{aligned} \operatorname{vol}(X, K_X + D) &= \operatorname{vol}(X, D - \Delta) \\ &\leq \operatorname{vol}(X, D) \\ &\leq \operatorname{vol}(X, r\Delta) \\ &\leq Cr^n. \end{aligned}$$

Let $\pi \colon Y \longrightarrow X$ be a log resolution of (X, Δ). Let G be the sum of the strict transform of the components of Δ and the exceptional divisors. Then (Y, G) has simple normal crossings. Pick $\eta > 0$ such that $(X, (1+\eta)\Delta)$ is kawamata log terminal and the log discrepancy is greater than ϵ. Then $K_X + (1 + \eta)\Delta$ is ample and we may write

$$K_Y + \Gamma = \pi^*(K_X + (1 + \eta)\Delta),$$

where $\Gamma \leq G$. As $K_Y + \Gamma$ is big it follows that $K_Y + G$ is big. (1.3) implies that there is a positive integer m such that $\phi_{m(K_Y+G)}$ is birational, for every $(X, \Delta) \in \mathfrak{D}$. But then $\mathfrak{D}$ is log birationally bounded by [15, 2.4.2.3-4]. Now apply (1.6). $\square$

Proof of (1.8). Let $D = -r(K_X + \Delta)$. Then D is an ample Cartier divisor and $D - (K_X + \Delta)$ is ample. By Kollár's effective base point free theorem (cf. [21]), there is a fixed positive integer m such that the linear system $|mD|$ is base point free. Pick a general divisor $H \in |mD|$. Then $(X, \Lambda = \Delta + \frac{1}{mr}H)$ is kawamata log terminal and

$$K_X + \Lambda \sim_{\mathbb{Q}} 0.$$

Note the coefficients of Λ belong to the finite set

$$I = \{\, \frac{i}{r} \,|\, 1 \leq i \leq r-1 \,\} \cup \{\, \frac{1}{mr} \,\}.$$

There are two ways to proceed. On the one hand we may apply (1.7).

Here is a more direct approach. Theorem B implies that

$$\operatorname{vol}(X, \Lambda),$$

is bounded from above. But then

$$\operatorname{vol}(X, mD) \leq (mr)^n \operatorname{vol}(X, \Lambda),$$

is bounded from above. $\square$

Proof of (1.10). Suppose that $r_1 \leq r_2 \leq \dots$ is a non-decreasing sequence in R. For each i we may find $(X, \Delta) = (X_i, \Delta_i) \in \mathfrak{D}$ and a Cartier divisor H such that $-(K_X + \Delta) \sim_{\mathbb{R}} rH$. By the cone theorem we may find a curve C such that $-(K_X + \Delta) \cdot C \leq 2n$, cf. Theorem 18.2 of [13]. In particular $r \leq 2n$, as $H \cdot C \geq 1$. By Fujino's extension, [12], of Kollár's effective base point free theorem, [21], to the case of log canonical pairs, there is a fixed positive integer m such that the linear system $|mH|$ is base point free. Possibly replacing m by a multiple we may assume that $m > 2n$. Pick a general divisor $D \in |mH|$.

Then $(X, \Lambda = \Delta + \frac{r}{m}D)$ is log canonical and

$$K_X + \Lambda \sim_{\mathbb{R}} 0.$$

Then the coefficients of $\Lambda_i = \Lambda$ belong to the set

$$I \cup \{\, \frac{r_i}{m} \,|\, i \in \mathbb{N} \,\},$$

which satisfies the DCC. (1.4) implies that the coefficients of Λ belong to a finite subset. But then $r_i = r_{i+1}$ is eventually constant and so R satisfies the ACC. $\square$

11. Accumulation points

Definition 11.1. *Given $I \subset [0,1]$ and $c \in [0,1]$ let*

$$D_c(I) = \{\, a \leq 1 \mid a = \frac{m-1+f+kc}{m}, k, m \in \mathbb{N}, f \in I_+ \,\} \subset D(I \cup \{\, c \,\}).$$

Let $\mathfrak{N}_n(I,c)$ be the set of log canonical pairs (X,Δ) such that X is a projective variety of dimension n, $K_X + \Delta$ is numerically trivial and we may write $\Delta = B + C$, where the coefficients of B belong to $D(I)$ and the coefficients of $C \neq 0$ belong to $D_c(I)$.

Let

$$N_n(I) = \{\, c \in [0,1] \mid \mathfrak{N}_n(I,c) \text{ is non-empty} \,\}.$$

Lemma 11.2. *Let $n \in \mathbb{N}$ and $I \subset [0,1]$.*

(1) $\mathrm{LCT}_n(I) \subset \mathrm{LCT}_{n+1}(I)$.
(2) $N_n(I) \subset N_{n+1}(I)$.
(3) If $f \in I_+$ and $k \in \mathbb{N}$ then

$$c = \frac{1-f}{k} \in N_n(I).$$

Proof. Let E be an elliptic curve. If $(X, \Delta = \sum d_i \Delta_i)$ is a log pair then (Y, Γ) is a log pair, where $Y = X \times E$ and $\Gamma = \sum d_i(\Delta_i \times E)$. By construction Γ has the same coefficients as Δ.

Note that (X,Δ) is log canonical if and only if (Y,Γ) is log canonical. This gives (1). Further if $c \in [0,1]$ and $(X,\Delta) \in \mathfrak{N}_n(I,c)$ then $(Y,\Gamma) \in \mathfrak{N}_{n+1}(I,c)$. This is (2).

Using (2), it suffices to prove (3) when $n = 1$. Let $X = \mathbb{P}^1$ and $\Delta = B + C$, where $B = fp + fq$, $C = 2kcr$, and p, q and r are three points of $\mathbb{P}^1$. Then $(X,\Delta) \in \mathfrak{N}_1(I,c)$ (take $m = 1$) so that $c \in N_1(I)$. This is (3). □

For technical reasons, it is convenient to introduce a smaller set than $\mathfrak{N}_n(I,c)$:

Definition 11.3. *Given $I \subset [0,1]$ and $c \in [0,1]$ let $\mathfrak{K}_n(I,c) \subset \mathfrak{N}_n(I,c)$ be the subset consisting of kawamata log terminal pairs (X,Δ), where X is $\mathbb{Q}$-factorial of Picard number one.*

Let

$$K_n(I) = \{\, c \in [0,1] \mid \mathfrak{K}_m(I,c) \text{ is non-empty, for some } m \leq n \,\}.$$

Lemma 11.4. *If $n \in \mathbb{N}$ and $I \subset [0,1]$ then*

$$N_n(I \cup \{\, 1 \,\}) = K_n(I).$$

In particular, $N_n(I \cup \{\, 1 \,\}) = N_n(I)$.

Proof. By (2) of (11.2), it suffices to show that

$$N_n(I \cup \{1\}) \subset K_n(I).$$

Suppose that $c \in N_n(I \cup \{1\})$. Then we may find $(X, \Delta) \in \mathfrak{N}_n(I \cup \{1\}, c)$. By assumption we may write $\Delta = A + B + C$, where the coefficients of A are one, the coefficients of B belong to $D(I)$ and the coefficients of $C \neq 0$ belong to $D_c(I)$.

Let $\pi\colon X' \longrightarrow X$ be a divisorially log terminal modification of (X, Δ). If we write

$$K_{X'} + \Delta' = \pi^*(K_X + \Delta),$$

then X' is projective of dimension n, X' is $\mathbb{Q}$-factorial, (X', Δ') is divisorially log terminal and $K_{X'} + \Delta'$ is numerically trivial. Let B' and C' be the strict transforms of B and C and let $A' = \Delta' - B' - C'$. Then the coefficients of A' are one, the coefficients of B' belong to $D(I)$ and the coefficients of $C' \neq 0$ belong to $D_c(I)$. Thus $(X', \Delta') \in \mathfrak{N}_n(I \cup \{1\}, c)$. Replacing (X, Δ) by (X', Δ') we may assume that X is $\mathbb{Q}$-factorial and $(X, A + B)$ is divisorially log terminal. Note that (X, Δ) is kawamata log terminal if and only if $A = 0$.

Suppose that A and C intersect. Let S be an irreducible component of A which intersects C. Then we may write

$$(K_X + \Delta)|_S = K_S + \Theta,$$

by adjunction, where (S, Θ) is divisorially log terminal and moreover we may write $\Theta = A' + B' + C'$, where the coefficients of A' are one, the coefficients of B' belong to $D(I)$ and the coefficients of $C' \neq 0$ belong to $D_c(I)$. Thus $(S, \Theta) \in \mathfrak{N}_{n-1}(I \cup \{1\}, c)$. Hence $c \in N_{n-1}(I \cup \{1\})$ and so $c \in K_{n-1}(I) \subset K_n(I)$, by induction on n.

Let $f\colon X \dashrightarrow X'$ be a step of the $(K_X + A + B)$-MMP. As $K_X + \Delta$ is numerically trivial, f is automatically C-positive. Suppose that f is birational. Let $A' = f_*A$, $B' = f_*B$ and $C' = f_*C$, so that $\Delta' = f_*\Delta = A' + B' + C'$. $C' \neq 0$, as f is C-positive. X' is a projective variety of dimension n, (X', Δ') is log canonical, $K_{X'} + \Delta'$ is numerically trivial, the coefficients of A' are all one, the coefficients of B' belong to $D(I)$ and the coefficients of $C' \neq 0$ belong to $D_c(I)$. Thus $(X', \Delta') \in \mathfrak{N}_n(I \cup \{1\}, c)$. Further X' is $\mathbb{Q}$-factorial and $(X', A' + B')$ is divisorially log terminal. If a component of A is contracted then A and C intersect and we are done. Otherwise (X', Δ') is kawamata log terminal if and only if $A' = 0$.

If we run the $(K_X + A + B)$-MMP with scaling of an ample divisor then we end with a Mori fibre space. Therefore, replacing (X, Δ) by (X', Δ') finitely many times, we may assume that $f\colon X \dashrightarrow Z = X'$ is a Mori fibre space and C dominates Z. If $\dim Z > 0$ then let $z \in Z$ be a general point. Then $(X_z, \Delta_z) \in \mathfrak{N}_{n-k}(I \cup \{1\}, c)$, where $k = \dim Z$, and we are done by induction on the dimension.

So we may assume that Z is a point in which case X has Picard number one. If $A \neq 0$ then A and C intersect and we are done. If $A = 0$ then (X, Δ) is kawamata log terminal and so $(X, \Delta) \in \mathfrak{K}_n(I, c)$. But then $c \in K_n(I)$. $\square$

Proposition 11.5. *If $I \subset [0, 1]$, $I = I_+$ and $n \in \mathbb{N}$ then* $\mathrm{LCT}_{n+1}(I) = N_n(I)$.

Proof. We first show that $\mathrm{LCT}_{n+1}(I) \subset N_n(I)$. Pick $0 \neq c \in \mathrm{LCT}_{n+1}(I)$. By definition we may find a log canonical pair $(X, \Delta + cM)$ where X has dimension $n+1$, the coefficients of Δ belong to I, M is an integral $\mathbb{Q}$-Cartier divisor and there is a non kawamata log terminal centre V contained in the support of M. Possibly passing to an open subset of X and replacing V by a maximal non kawamata log terminal centre, we may assume that V is the only non kawamata log terminal centre of $(X, \Delta + cM)$. In particular, (X, Δ) is kawamata log terminal.

If V is a component of M then V has coefficient one in $\Delta + cM$ and $c = \frac{1-f}{k} \in N_n(I)$ by (3) of (11.2). Otherwise let $f\colon Y \longrightarrow X$ be a divisorially log terminal modification of $(X, \Delta + cM)$. Then Y is $\mathbb{Q}$-factorial and we may write

$$K_Y + T + \Delta' + cM' = f^*(K_X + \Delta + cM)$$

where Δ' and M' are the strict transforms of Δ and M, T is the sum of the exceptional divisors and the pair $(Y, T + \Delta' + cM')$ is divisorially log terminal. By (4) of (3.3.1) we may choose f so that T contains the inverse image of V. Let S be an irreducible component of T which intersects M'. Then we may write

$$(K_Y + T + \Delta' + cM')|_S = K_S + \Theta,$$

by adjunction, where (S, Θ) is divisorially log terminal and moreover we may write $\Theta = A + B + C$, where the coefficients of A are one, the coefficients of B belong to $D(I)$ and the coefficients of $C \neq 0$ belong to $D_c(I)$. As S is a non kawamata log terminal centre, the centre of S on X is V so that there is a morphism $S \longrightarrow V$. If $v \in V$ is a general point then $(S_v, \Theta_v) \in \mathfrak{N}_k(I \cup \{1\}, c)$, for some $k \leq n$. Thus $c \in N_k(I \cup \{1\}) \subset N_n(I)$.

We now show that $\mathrm{LCT}_{n+1}(I) \supset N_n(I)$. Pick $0 \neq c \in N_n(I)$. Then we may find a pair $(X, \Delta) \in \mathfrak{K}_m(I, c)$, some $m \leq n$. If $m < n$ then we are done by induction on the dimension. Otherwise X has dimension

n. As $-K_X$ is ample, we may pick d such that $-dK_X$ is very ample and embed X into projective space by the linear system $|-dK_X|$.

Let Y be the cone over X and let Γ_j be the cone over Δ_j. Then Y is a quasi-projective variety of dimension $n+1$. Y is $\mathbb{Q}$-factorial as X has Picard number one. $(Y, \Gamma = \sum d_i\Gamma_i)$ is log canonical but not kawamata log terminal at the vertex p of the cone. By assumption we may write

$$d_i = \frac{m_i - 1 + f_i + k_i c}{m_i},$$

for each i, where m_i is a positive integer, k_i is a non-negative integer ($k_i = 0$ if Γ_i is a component of B_i and $k_i > 0$ if Γ_i is a component of C_i) and $f_i \in I_+$. Since we are working locally around p, the vertex of Y, we may find a cover of $\pi\colon \tilde{Y} \longrightarrow Y$ which ramifies over Γ_i to index m_i for every i and is otherwise unramified at the generic point of any divisor. We may write

$$K_{\tilde{Y}} + \tilde{\Gamma} = \pi^*(K_Y + \Gamma),$$

where the coefficients of $\tilde{\Gamma}$ belong to the set

$$\{\, f_i + k_i c \,|\, i \,\}.$$

$\tilde{Y}$ is a $\mathbb{Q}$-factorial quasi-projective variety of dimension $n+1$ and $(\tilde{Y}, \tilde{\Gamma})$ is log canonical but not kawamata log terminal over any point q lying over p. Let

$$\Theta = \sum f_i \Gamma_i \qquad \text{and} \qquad M_i = \sum k_i \Gamma_i.$$

Then the coefficients of Θ belong to $I_+ = I$, M_i is an integral $\mathbb{Q}$-Cartier divisor and

$$c = \sup\{\, t \in \mathbb{R} \,|\, (X, \Theta + tM) \text{ is log canonical} \,\},$$

is the log canonical threshold. But then $c \in \mathrm{LCT}_{n+1}(I)$. $\square$

Lemma 11.6. *Let (X, Δ) be a log canonical pair, where X is $\mathbb{Q}$-factorial of dimension n and Picard number one and $K_X + \Delta$ is numerically trivial.*

If the coefficients of Δ are at least δ then Δ has at most $\frac{n+1}{\delta}$ components.

Proof. [27, 18.24] implies that the sum of the coefficients of Δ is at most $n+1$. $\square$

Proposition 11.7. *Fix a positive integer n and a set $I \subset [0,1]$ whose only accumulation point is one such that $I = I_+$.*

Let $c_1, c_2, \ldots \in [0,1]$ be a strictly decreasing sequence with limit $c \neq 0$ with the following property. There is a sequence of log canonical pairs

(X_i, Δ_i) such that X_i is a projective variety of dimension n, $K_{X_i} + \Delta_i$ is numerically trivial and we may write $\Delta_i = A_i + B_i + C_i$, where the coefficients of A_i are approaching one, the coefficients of B_i belong to $D(I)$ and the coefficients of $C_i \neq 0$ belong to $D_{c_i}(I)$.

Then $c \in N_{n-1}(I)$.

Proof. We may assume that A_i and $B_i + C_i$ have no common components. Replacing B_i by $B_i - \lfloor B_i \rfloor$ and A_i by $A_i + \lfloor B_i \rfloor$ we may assume that $\lfloor \Delta_i \rfloor = \lfloor A_i \rfloor$. As the coefficients of $A_i + B_i$ belong to a set which satisfies the DCC, (1.5) implies that not all of the coefficients of C_i are increasing. In particular at least one coefficient of C_i is bounded away from one.

Let a_i be the total log discrepancy of (X_i, Δ_i).

Case A: $\lim a_i > 0$.

In this case, we assume that a_i is bounded away from zero.

Case A, Step 1: We reduce to the case X_i is $\mathbb{Q}$-factorial and the Picard number of X_i is one.

As we are assuming that a_i is bounded away from zero, $A_i = 0$ and so $(X_i, \Delta_i) \in \mathfrak{N}_n(I, c_i)$, so that $c_i \in N_n(I) = K_n(I)$, by (11.4). Thus we may assume that $(X_i, \Delta_i) \in \mathfrak{K}_m(I, c_i)$, for some $m \leq n$. If $m < n$ then we are done by induction. Otherwise we may assume that X_i is $\mathbb{Q}$-factorial and the Picard number of X_i is one.

Possibly passing to a subsequence, (11.6) implies that we may assume that the number of components of B_i and C_i is fixed. As the only accumulation point of $D(I)$ is one and the coefficients of B_i are bounded away from one, possibly passing to a subsequence we may assume that the coefficients of B_i are fixed and that the coefficients of C_i have the form

$$\frac{r-1}{r} + \frac{f}{r} + \frac{kc_i}{r},$$

where k, r and f depend on the component but not on i.

Given $t \in [0, 1]$, let $C_i(t)$ be the divisor with the same components as C_i but now with coefficients

$$\frac{r-1}{r} + \frac{f}{r} + \frac{kt}{r},$$

so that $C_i = C_i(c_i)$. Let

$$h_i = \sup\{\, t \mid (X_i, B_i + C_i(t)) \text{ is log canonical} \,\},$$

be the log canonical threshold. Set $h = \lim h_i$.

Case A, Step 2: We reduce to the case $h > c$.

Suppose that $h \leq c$. As $c_i \leq h_i$, it follows that $h = c$. Now

$$h_i \in \mathrm{LCT}_n(D(I)) = N_{n-1}(I),$$

so that we are done by induction in this case.

Case A, Step 3: We reduce to the case $\operatorname{vol}(X_i, C_i)$ is unbounded.

Suppose not, suppose that $\operatorname{vol}(X_i, C_i)$ is bounded from above. Let

$$d_i = \frac{c_i + h_i}{2} \qquad \text{and} \qquad d = \frac{c+h}{2}.$$

Then the coefficients of $(X_i, B_i + C_i(d))$ are fixed. The log discrepancy of $(X_i, B_i + C_i(d_i))$ is at least $a_i/2$ so that the log discrepancy of $(X_i, B_i + C_i(d))$ is bounded away from zero. As $h > c$, possibly passing to a tail of the sequence, we may assume that $d > c_i$ so that $K_{X_i} + B_i + C_i(d)$ is ample. Note that

$$\operatorname{vol}(X_i, K_{X_i} + B_i + C_i(d)) = \operatorname{vol}(X_i, C_i(d) - C_i)$$

is bounded from above by assumption. (1.3) implies that there is a positive integer m such that $\phi_{m(K_{X_i}+B_i+C_i(d))}$ is birational. But then $\{ (X_i, \Delta_i) \,|\, i \in \mathbb{N} \}$ is log birationally bounded by [15, 2.4.2.4]. (1.6) implies that (X_i, Δ_i) belongs to a bounded family. Thus we may find an ample Cartier divisor H_i such that the intersection numbers $T_i \cdot H_i^{n-1}$ and $-K_{X_i} \cdot H_i^{n-1}$ are bounded, where T_i is any component of Δ_i. Possibly passing to a subsequence, we may assume that these intersection numbers are constant. But then

$$(K_{X_i} + \Delta_i) \cdot H_i^{n-1} = 0, \qquad A_i \cdot H_i^{n-1} = 0 \qquad \text{and} \qquad B_i \cdot H_i^{n-1}$$

is independent of i, whilst $C_i \cdot H_i^{n-1}$ is not constant, a contradiction.

Case A, Step 4: We finish case A.

As $\operatorname{vol}(X_i, C_i)$ is unbounded (3.2.2) implies that we may find $\epsilon_i > 0$ and divisors $0 \leq C_i' \sim_{\mathbb{R}} \epsilon_i C_i$ such that $(X_i, \Delta_i + C_i')$ is not log canonical. Passing to a subsequence, and using (3.2.3), we may find $g_i < c_i$ and a divisor

$$0 \leq \Theta_i \sim_{\mathbb{R}} C_i - C_i(g_i) \qquad \text{with} \qquad \lim g_i = c,$$

such that $(X_i, \Phi_i = B_i + C_i(g_i) + \Theta_i)$ has a unique non kawamata log terminal place. If $\phi\colon Y_i \longrightarrow X_i$ is a divisorially log terminal modification then ϕ extracts a unique prime divisor S_i of log discrepancy zero with respect to (X_i, Φ_i). We may write

$$K_{Y_i} + \Psi_i = \phi^*(K_{X_i} + \Phi_i) \qquad \text{and} \qquad K_{Y_i} + B_i' + C_i' + s_i S_i = \phi^*(K_{X_i} + \Delta_i),$$

where $S_i = \lfloor \Psi_i \rfloor$, B_i' and C_i' are the strict transform of B_i and C_i, and $s_i < 1$, as (X_i, Δ_i) is kawamata log terminal.

As $K_{Y_i} + \Psi_i$ is numerically trivial, $K_{Y_i} + \Psi_i - S_i$ is not pseudo-effective. By [8, 1.3.3], we may run $f\colon Y_i \dashrightarrow W_i$ the $(K_{Y_i} + \Psi_i - S_i)$-MMP until we end with a Mori fibre space $\pi_i\colon W_i \longrightarrow Z_i$. As $K_{Y_i} + \Psi_i$ is numerically trivial, every step of this MMP is S_i-positive, so that the strict transform T_i of S_i dominates Z_i. Let F_i be the general fibre of

π_i. Replacing Y_i, B'_i, C'_i and Ψ_i by F_i and the restriction of $f_*B'_i$, $f_*C'_i$ and $f_*\Psi_i$ to F_i, we may assume that S_i, Ψ_i, B'_i and C'_i are multiples of the same ample divisor. In particular $K_{Y_i} + B'_i + C'_i + S_i$ is ample.

We let $C'_i(t)$ denote the strict transform of $C_i(t)$. We may write

$$(K_{Y_i} + S_i + B'_i + C'_i(t))|_{S_i} = K_{S_i} + B''_i + C''_i(t),$$

where the coefficients of B''_i belong to $D(I)$ and the coefficients of $C''_i(t) \neq 0$ belong to $D_t(I)$. We let $C''_i = C''_i(c_i)$.

There are two cases. Suppose that $(S_i, B''_i + C''_i)$ is not log canonical. Let

$$k_i = \sup\{\, t \,|\, (S_i, B''_i + C''_i(t)) \text{ is log canonical}\,\},$$

be the log canonical threshold. Then $k_i \in \mathrm{LCT}_{n-1}(D(I)) = N_{n-2}(I)$. Then $k = \lim k_i \in N_{n-2}(I) \subset N_{n-1}(I)$ by induction on n. As $(S_i, B''_i + C''_i(g_i))$ is kawamata log terminal, $k_i \in (g_i, c_i)$. Thus

$$c = \lim c_i = \lim k_i = k \in N_{n-1}(I).$$

Otherwise we may suppose that $(S_i, B''_i + C''_i)$ is log canonical. Let

$$l_i = \sup\{\, t \,|\, (S_i, B''_i + C''_i(t)) \text{ is pseudo-effective}\,\},$$

be the pseudo-effective threshold. Then $l_i \in N_{n-1}(I)$ and $l = \lim l_i \in N_{n-1}(I)$ by induction on n. On the other hand $l_i \in (g_i, c_i)$. Thus

$$c = \lim c_i = \lim l_i = l \in N_{n-1}(I).$$

Case B: $\lim a_i = 0$.

In this case, we assume that a_i approaches 0.

Case B, Step 1: We reduce to the case $A_i \neq 0$, X_i is $\mathbb{Q}$-factorial and (X_i, Δ_i) is kawamata log terminal if and only if $\lfloor A_i \rfloor = 0$.

Possibly passing to a subsequence we may assume that $a_i \geq a_{i+1}$ and $a_i \leq 1$. If (X_i, Δ_i) is not divisorially log terminal or $A_i \neq 0$ but X_i is not $\mathbb{Q}$-factorial then let $\pi_i \colon X'_i \longrightarrow X_i$ be a divisorially log terminal modification. If $A_i = 0$ then let $\pi_i \colon X'_i \longrightarrow X_i$ extract a divisor of log discrepancy a_i, where X'_i is $\mathbb{Q}$-factorial. Either way, we may write

$$K_{X'_i} + \Delta'_i = \pi_i^*(K_{X_i} + \Delta_i),$$

where Δ'_i is a sum of the strict transform of Δ_i and a divisor which is exceptional. Let B'_i and C'_i be the strict transforms of B_i and C_i and let $A'_i = \Delta'_i - B'_i - C'_i \neq 0$. Then X'_i is a $\mathbb{Q}$-factorial projective variety of dimension n, (X'_i, Δ'_i) is a divisorially log terminal pair, $K_{X'_i} + \Delta'_i$ is numerically trivial, the coefficients of $A'_i \neq 0$ are approaching one, the coefficients of B'_i belong to $D(I)$ and the coefficients of $C'_i \neq 0$ belong to $D_{c_i}(I)$. Replacing (X_i, Δ_i) by (X'_i, Δ'_i), we may assume that $A_i \neq 0$ and X_i is $\mathbb{Q}$-factorial. Moreover (X_i, Δ_i) is kawamata log terminal if and only if $\lfloor A_i \rfloor = 0$.

Case B, Step 2: We are done if the support of C_i and $\lfloor A_i \rfloor$ intersect.

Suppose that a component of C_i intersects the normalisation of a component S_i of $\lfloor A_i \rfloor$. Then we may write

$$(K_{X_i} + \Delta_i)|_{S_i} = K_{S_i} + \Theta_i,$$

by adjunction. S_i is projective of dimension $n-1$, (S_i, Θ_i) is log canonical, $K_{S_i} + \Theta_i$ is numerically trivial, and we may write $\Theta_i = A'_i + B'_i + C'_i$, where the coefficients of A'_i approach one, the coefficients of B'_i belong to $D(I)$ and the coefficients of $C'_i \neq 0$ belong to $D_{c_i}(I)$. In this case, the limit c belongs to $N_{n-2}(I) \subset N_{n-1}(I)$ by induction.

Case B, Step 3: We are done if $f_i \colon X_i \longrightarrow Z_i$ is a Mori fibre space, A_i dominates Z_i and $\dim Z_i > 0$.

Let F_i be the general fibre of f_i. We may write

$$(K_{X_i} + \Delta_i)|_{F_i} = K_{F_i} + \Theta_i,$$

by adjunction. F_i is projective of dimension at most $n-1$, (F_i, Θ_i) is log canonical, $K_{F_i} + \Theta_i$ is numerically trivial, and we may write $\Theta_i = A'_i + B'_i + C'_i$, where the coefficients of A'_i approach one, the coefficients of B'_i belong to $D(I)$ and the coefficients of C'_i belong to $D_{c_i}(I)$.

There are two cases. Suppose that $C'_i = 0$. Then (1.5) implies that the coefficients of A'_i are fixed, so that $\lfloor A'_i \rfloor = A'_i$. But then $\lfloor A_i \rfloor \neq 0$ dominates Z_i. On the other hand, as $C'_i = 0$, C_i does not intersect F_i, that is, C_i does not dominate Z_i. But then C_i must contain a fibre so that A_i and C_i intersect and we are done by Case B, Step 2. Otherwise $C'_i \neq 0$. In this case $c_i \in N_{n-1}(I)$ so that

$$c = \lim c_i \in N_{n-2}(I) \subset N_{n-1}(I),$$

by induction.

Case B, Step 4: We reduce to the case (X_i, Δ_i) is kawamata log terminal.

Suppose not, suppose that (X_i, Δ_i) is not kawamata log terminal. By Case B, Step 1, this implies that $S_i = \lfloor A_i \rfloor$ is not the zero divisor. Let $\Theta_i = \Delta_i - S_i$. We run the $(K_{X_i} + \Theta_i)$-MMP with scaling of some ample divisor. Let $f_i \colon X_i \dashrightarrow X'_i$ be a step of the $(K_{X_i} + \Theta_i)$-MMP. As $K_{X_i} + \Delta_i$ is numerically trivial, f_i is automatically S_i-positive. Let $A'_i = f_{i*}A_i$, $B'_i = f_{i*}B_i$ and $C'_i = f_{i*}C_i$. First suppose that f_i is birational. If $C'_i = 0$ then (1.5) implies that the coefficients of A'_i are all one. As f_i contracts C_i it does not contract a component of A_i and so it follows that the coefficients of A_i are all one, that is, $S_i = A_i$. As f_i contracts C_i and f_i is S_i-positive, C_i intersects S_i and we are done by Case B, Step 2. Therefore we may assume that $C'_i \neq 0$ and we may replace (X_i, Δ_i) by (X'_i, Δ'_i). As the MMP must terminate with a Mori fibre space, replacing (X_i, Δ_i) with (X'_i, Δ'_i) finitely many times, we may assume that $f_i \colon X_i \longrightarrow Z_i = X'_i$ is a Mori fibre space and S_i dominates Z_i. By Case B, Step 3, we may assume that Z_i is a point. But then the support of S_i and C_i intersect and we are done by Case B, Step 2.

Case B, Step 5: We reduce to the case X_i has Picard number one.

We run the $(K_{X_i}+B_i+C_i)$-MMP with scaling of some ample divisor. Let $f_i \colon X_i \dashrightarrow X'_i$ be a step of the $(K_{X_i}+B_i+C_i)$-MMP. As $K_{X_i}+\Delta_i$ is numerically trivial f_i is automatically A_i-positive. Let $A'_i = f_{i*}A_i$, $B'_i = f_{i*}B_i$ and $C'_i = f_{i*}C_i$. First suppose that f_i is birational. Suppose $C'_i = 0$. As f_i contracts only one divisor and A_i and C_i are non-zero by assumption, it follows that $A'_i \neq 0$. (1.5) implies that the coefficients of A'_i are all one, which contradicts the fact that (X_i, Δ_i) is kawamata log terminal. Therefore we may assume that $C'_i \neq 0$ and we may replace (X_i, Δ_i) by (X'_i, Δ'_i). As the MMP must terminate with a Mori fibre space, replacing (X_i, Δ_i) with (X'_i, Δ'_i) finitely many times, we may assume that $f_i \colon X_i \longrightarrow Z_i = X'_i$ is a Mori fibre space and A_i dominates Z_i.

By Case B, Step 3 we may assume that Z_i is a point, so that X_i has Picard number one.

Case B, Step 6: We finish case B and the proof.

Possibly passing to a subsequence, (11.6) implies that we may assume that the number of components of B_i and C_i is fixed. As the only accumulation point of $D(I)$ is one and the coefficients of B_i are bounded away from one, possibly passing to a subsequence we may assume that the coefficients of B_i are fixed and that the coefficients of C_i have the form
$$\frac{r-1}{r} + \frac{f}{r} + \frac{kc_i}{r},$$
where k, r and f depend on the component but not on i.

Given $t \in [0,1]$, let $C_i(t)$ be the divisor with the same components as C_i but now with coefficients
$$\frac{r-1}{r} + \frac{f}{r} + \frac{kt}{r},$$
so that $C_i = C_i(c_i)$.

Let T_i be the sum of the components of A_i, so that T_i has the same components as A_i but now every component has coefficient one. Then $A_i \leq T_i$ and $C_i(c) \leq C_i$. Note that $(X_i, A_i + B_i + C_i(c))$ is kawamata log terminal as $(X_i, A_i + B_i + C_i)$ is kawamata log terminal. Let
$$s_i = \sup\{\, s \in [0,1] \,|\, (X_i, A_i + B_i + C_i(c) + s(T_i - A_i)) \text{ is log canonical}\,\}$$

be the log canonical threshold. Then

$$A_i + B_i + C_i(c) \leq A_i + B_i + C_i(c) + s_i(T_i - A_i) \leq T_i + B_i + C_i(c),$$

As the coefficients of $A_i + B_i + C_i(c)$ belong to a set which satisfies the DCC and the coefficients of $T_i - A_i$ approach zero, the coefficients of $A_i + B_i + C_i(c) + s_i(T_i - A_i)$ belong to a set which satisfies the DCC. Therefore, possibly passing to a tail of the sequence, (1.4) implies that $s_i = 1$, so that $(X_i, T_i + B_i + C_i(c))$ is log canonical.

Suppose that $(X_i, T_i + B_i + C_i)$ is not log canonical. Let

$$d_i = \sup\{\, t \in [c, c_i) \,|\, (X_i, T_i + B_i + C_i(t)) \text{ is log canonical} \,\},$$

be the log canonical threshold. Then $d_i \in \operatorname{LCT}_n(D(I)) = N_{n-1}(I)$ and $c = \lim d_i$ and so we are done by induction on the dimension.

Thus we may assume that $(X_i, T_i + B_i + C_i)$ is log canonical. Let

$$e_i = \sup\{\, t \in \mathbb{R} \,|\, K_{X_i} + T_i + B_i + C_i(t)) \text{ is pseudo-effective} \,\},$$

be the pseudo-effective threshold. Suppose that $e_i < c$. Let

$$f_i = \sup\{\, t \in \mathbb{R} \,|\, K_{X_i} + tT_i + B_i + C_i(c)) \text{ is pseudo-effective} \,\},$$

be the pseudo-effective threshold. As $e_i < c$, $f_i < 1$ and $\lim f_i = 1$, so that the coefficients of $f_iT_i + B_i + C_i(c)$ belong to a set which satisfies the DCC, which contradicts (1.5). Thus $e_i \geq c$. On the other hand $e_i < c_i$ as $K_{X_i} + T_i + B_i + C_i$ is strictly bigger than $K_{X_i} + A_i + B_i + C_i$, which is numerically trivial. Thus $\lim e_i = c$. Possibly passing to a subsequence we may assume that either $e_i > e_{i+1}$ for all i or $e_i = c$. In the former case we might as well replace $C_i = C_i(c_i)$ by $C_i(e_i)$. In this case some component of C_i intersects a component S_i of T_i and we are done by Case B, Step 2. In the latter case we restrict to a component S_i of T_i and apply adjunction to conclude that $c = e_i \in N_{n-1}(I)$. □

Proof of (1.11). By (11.5) it suffices to prove that the accumulation points of $N_n(I)$ belong to $N_{n-1}(I)$. Suppose that $c_1, c_2, \dots \in [0, 1]$ is a strictly decreasing sequence of real numbers such that $\mathfrak{N}(I, c_i)$ is non-empty. Pick $(X_i, \Delta_i) \in \mathfrak{N}(I, c_i)$. By assumption we may write $\Delta_i = B_i + C_i$ where the coefficients of B_i belong to $D(I)$ and the coefficients of $C_i \neq 0$ belong to $D_{c_i}(I)$, and so (11.7) implies that the limit c belongs to $N_{n-1}(I)$. □

References

[1] V. Alexeev, *Fractional indices of log del Pezzo surfaces*, Izv. Akad. Nauk SSSR Ser. Mat. **52** (1988), no. 6, 1288–1304, 1328.

[2] ———, *Theorems about good divisors on log Fano varieties (case of index $r > n-2$)*, Algebraic geometry (Chicago, IL, 1989), Lecture Notes in Math., vol. 1479, Springer, Berlin, 1991, pp. 1–9.

[3] ———, *Boundedness and K^2 for log surfaces*, International J. Math. **5** (1994), 779–810.

[4] V. Alexeev and C. D. Hacon, *Non-rational centers of log canonical singularities*, J. Algebra **369** (2012), 1–15, arXiv:math.AG/1109.4164.

[5] U. Angehrn and Y. Siu, *Effective freeness and point separation for adjoint bundles*, Invent. Math. **122** (1995), no. 2, 291–308.

[6] C. Birkar, *Ascending chain condition for log canonical thresholds and termination of log flips*, Duke Math. J. **136** (2007), no. 1.

[7] ———, *Existence of log canonical flips and a special LMMP*, Publ. Math. Inst. Hautes Études Sci. (2012), 325–368.

[8] C. Birkar, P. Cascini, C. Hacon, and J. M^{c}Kernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. **23** (2010), no. 2, 405–468, arXiv:math.AG/0610203.

[9] A. Borisov, *Boundedness of Fano threefolds with log-terminal singularities of given index*, J. Math. Sci. Univ. Tokyo **8** (2001), no. 2, 329–342.

[10] T. de Fernex, L. Ein, and M. Mustaţă, *Shokurov's ACC Conjecture for log canonical thresholds on smooth varieties*, Duke Math. J. **152** (2010), no. 1, 93–114, arXiv:0905.3775v3.

[11] ———, *Log canonical thresholds on varieties with bounded singularities*, Classification of algebraic varieties, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2011, pp. 221–257.

[12] O. Fujino, *Effective base point free theorem for log canonical pairs—Kollár type theorem*, Tohoku Math. J. (2) **61** (2009), no. 4, 475–481.

[13] ———, *Fundamental theorems for the log minimal model program*, Publ. Res. Inst. Math. Sci. **47** (2011), no. 3, 727–789.

[14] C. Hacon and J. M^{c}Kernan, *Boundedness of pluricanonical maps of varieties of general type*, Invent. Math. **166** (2006), no. 1, 1–25.

[15] C. Hacon, J. M^{c}Kernan, and C. Xu, *On the birational automorphisms of varieties of general type*, Ann. of Math. **177** (2013), no. 3, 1077–1111.

[16] C. Hacon and C. Xu, *Existence of log canonical closures*, Invent. Math. **192** (2013), no. 1, 161–195.

[17] M. Kawakita, *Inversion of adjunction on log canonicity*, Invent. Math. **167** (2007), no. 1, 129–133.

[18] Y. Kawamata, *Subadjunction of log canonical divisors. II*, Amer. J. Math. **120** (1998), no. 5, 893–899.

[19] S. Keel and J. M^{c}Kernan, *Rational curves on quasi-projective surfaces*, Mem. Amer. Math. Soc. **140** (1999), no. 669, viii+153.

[20] J. Kollár, *Which powers of holomorphic functions are integrable?*, arXiv:0805.0756.

[21] ———, *Effective base point freeness*, Math. Ann. **296** (1993), no. 4, 595–605.

[22] ———, *Log surfaces of general type; some conjectures*, Contemp. Math. **162** (1994), 261–275.

[23] ———, *Rational Curves on Algebraic Varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 32, Springer, 1996.

[24] ______, *Singularities of pairs*, Algebraic geometry—Santa Cruz 1995, Amer. Math. Soc., Providence, RI, 1997, pp. 221–287.
[25] ______, *Kodaira's canonical bundle formula and adjunction*, Flips for 3-folds and 4-folds, Oxford Lecture Ser. Math. Appl., vol. 35, Oxford Univ. Press, Oxford, 2007, pp. 134–162.
[26] ______, *Moduli of varieties of general type*, Handbook of Moduli: Volume II, Adv. Lect. Math. (ALM), vol. 24, Int. Press, Somerville, MA, 2013, arXiv:1008.0621v1, pp. 115–130.
[27] J. Kollár et al., *Flips and abundance for algebraic threefolds*, Société Mathématique de France, Paris, 1992, Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, Astérisque No. 211 (1992).
[28] J. Kollár and S. Kovács, *Log canonical singularities are Du Bois*, J. Amer. Math. Soc. **23** (2010), no. 3, 791–813.
[29] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge tracts in mathematics, vol. 134, Cambridge University Press, 1998.
[30] R. Lazarsfeld, *Positivity in algebraic geometry. I*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 48, Springer-Verlag, Berlin, 2004, Classical setting: line bundles and linear series.
[31] ______, *Positivity in algebraic geometry. II*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 49, Springer-Verlag, Berlin, 2004, Positivity for vector bundles, and multiplier ideals.
[32] J. McKernan and Y. Prokhorov, *Threefold Thresholds*, Manuscripta Math. **114** (2004), no. 3, 281–304.
[33] V. V. Shokurov, *Three-dimensional log perestroikas*, Izv. Ross. Akad. Nauk Ser. Mat. **56** (1992), no. 1, 105–203.
[34] B. Totaro, *The ACC conjecture for log canonical thresholds (after de Fernex, Ein, Mustaţă, Kollár)*, Astérisque (2011), no. 339, Exp. No. 1025, ix, 371–385, Séminaire Bourbaki. Vol. 2009/2010. Exposés 1012–1026.

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