Flips and flops

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Smooth curves of genus $g \geq 2$ form a moduli space $\mathcal{M}_g$, a quasi-projective variety of dimension $3g - 3$.

It is more natural to work with the projective variety $\overline{\mathcal{M}}_g \supset \mathcal{M}_g$, which parametrises connected nodal curves, with only finitely many automorphisms.
Every smooth plane curve of degree 4 is a curve of genus 3 and vice-versa the general point of $M_3$ is a smooth plane curve of degree 4.
Plane curves of degree four

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Let $I$ be the closure of the graph of $\pi$ and let $p: I \rightarrow \mathbb{P}^{14}, q: I \rightarrow \overline{\mathcal{M}}_3$ be the natural maps.
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**Definition:** $C$ is hyperelliptic if $C$ is a double cover of $\mathbb{P}^1$. Let $\mathcal{H} \subset \overline{\mathcal{M}}_3$ be the closure of the hyperelliptic locus.
Analysis of $\pi$

- $\mathcal{H}$ has dimension 5, since a hyperelliptic curve is determined by its 8 branch points, $0, 1, \infty, \ldots$. 
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- The geometry of the birational map $\pi$ is surprisingly rich, even though $(d, g, r, n) = (4, 3, 2, 1)$ are all relatively small.
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A curve $C$ is of general type if and only if $g \geq 2$. 
Theorem: (BCHM, Siu) The canonical ring $R(X, K_X)$ is f.g.
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As a consequence every variety $Y$ of general type is birational $\phi_{mK_Y}: Y \dashrightarrow X \subset \mathbb{P}^r$ to a variety $X$ naturally embedded in $\mathbb{P}^r$, where $K_X$ is ample and $X$ has canonical singularities.
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- So if we fix the degree there is a family $\pi : \mathcal{X} \to \mathcal{H}$ parametrising all projective varieties such that $K_X$ is ample and $X$ has canonical singularities.
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Plan: Modify the quotient $\mathcal{H}/\text{PGL}(r+1)$ birationally to construct a geometrically meaningful projective moduli space $\overline{\mathcal{M}}$. 

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- $\dim Z < \dim X$. The fibres $F$ of $\pi$ are Fano varieties, $-K_F$ is ample. **STOP:** Mori fibre space.
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- $\dim Z = \dim X$, the locus of curves contracted by $\pi$ is a divisor. Return to (2).
- $\pi$ is small. The intersection number $K_Z \cdot C$ does not even make sense. Instead replace $X$ by $X^+$, the flip.
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**Question:** How do we know that this process terminates? It is clear that we cannot keep contracting divisors, but why could there not be an infinite sequence of flips?
Theorem: (BCHM) Let $X$ be a smooth projective variety, $A$ and an ample $\mathbb{Q}$-divisor, and let $D_1 + D_2 + \cdots + D_k$ be a normal crossings divisor. Then there are only finitely many minimal models, for

$$K_X + A + \sum a_i D_i, \ (a_1, a_2, \ldots, a_k) \in [0, 1]^k.$$
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Example: \(X = \overline{\mathcal{M}}_g, D_i\) the boundary divisors. Conjecturally the minimal models are moduli spaces.

- If we run a MMP whose intermediary steps are all minimal models, then termination is clear.
- Unfortunately this trick does not seem to work to construct the moduli space of varieties of general type. We run into a brick wall called \textit{abundance}.
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- The supremum of $t$ for which $\frac{1}{|f|^{2t}}$ is locally integrable, is called the log canonical threshold.
Examples

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- More generally still, the log canonical threshold of \( (z_1, z_2, \ldots, z_n) \to z_1^{a_1} z_2^{a_2} \ldots z_n^{a_n} \) is \( \min(1/a_1, 1/a_2, \ldots, 1/a_n) \), by Fubini.
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Question: How about $(x, y) \rightarrow y^2 + x^3$?
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$$\Omega_r = \{ (x, y) \in \mathbb{C}^2 \mid 2^{-2(r+1)} < |x| < 2^{-2r}, 2^{-3(r+1)} < |y| < 2^{-3r} \}.$$
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Substituting $x' = 2^2 x$ and $y' = 2^3 y$ we see that $I_{r+1} = 2^{12t-10} I_r$, and summing over $\Omega_r$ the integral is convergent if $t < 5/6$. 
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So the log canonical threshold $a$ is $5/6$. 
Another way

- Clearly the log canonical threshold $\alpha$ only depends on the cuspidal curve $C$, $(y^2 + x^3 = 0) \subset \mathbb{C}^2$. 
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- We already know the log canonical threshold $a$ is the largest $t \leq 1$ such that $\max(a_1, a_2, a_3) \leq 1$. 
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- So the log canonical threshold \(a = 5/6.\)
Yet another way

- \(a_1 = 2t - 1\), \(a_2 = 3t - 3\), and \(a_3 = 6t - 5\). The log canonical threshold \(a\) is \(5/6\).

- But if we contract, \(f : S \rightarrow T\), \(E_1\) and \(E_2\), then \(T\) has two singular points along \(\tilde{C}\), of index 2 and 3.

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- The map \(\psi\) is in fact a weighted blow up.
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If $C$ is given as $y^a + x^b$, then the log canonical threshold is $\min(1/a + 1/b, 1)$, using either integrals or weighted blow ups.
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We hope to prove the full version of Shokurov’s conjecture using birational boundedness.
Termination via ACC

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- This argument is due to Birkar.
Birkar’s local-global argument

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- However, there is some reason to hope that we might prove a version of termination strong enough to construct projective moduli spaces.
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Question: Perhaps one can prove termination of flips for $K_X + \Delta$ Kawamata log terminal and $\Delta$ big?