Flips and flops

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- Smooth curves of genus $g \geq 2$ form a moduli space $\mathcal{M}_g$, a quasi-projective variety of dimension $3g - 3$.
- It is more natural to work with the projective variety $\overline{\mathcal{M}}_g \supset \mathcal{M}_g$, which parametrises connected nodal curves, with only finitely many automorphisms.

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finition: C is hyperelliptic if C is a double cover of $\mathbb{P}^1$. Let $\mathcal{H} \subset \overline{\mathcal{M}}_3$ be the closure of the hyperelliptic locus.

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- So π cannot possibly be a morphism.
- Similarly p blows up some components of the locus of singular plane curves (for example, a double line union a conic) and q blows up some nodal curves.
- The geometry of the birational map π is surprisingly rich, even though $(d, g, r, n) = (4, 3, 2, 1)$ are all relatively small.

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- We expect the geometry of a projective variety to be reflected by the sign of the canonical divisor.
- X is a variety of general type if K_X is big (maximally positive).
- A curve C is of general type if and only if $g \geq 2$.

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Plan: Modify the quotient $\mathcal{H}/\operatorname{PGL}(r+1)$ birationally to construct a geometrically meaningful projective moduli space $\overline{\mathcal{M}}$.

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- $\dim Z = \dim X$, the locus of curves contracted by π is a divisor. Return to (2).
- π is small. The intersection number $K_Z \cdot C$ does not even make sense. Instead replace X by X^+, the flip.

Flips

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It is clear that we cannot keep contracting divisors, but why could there not be an infinite sequence of flips?

Global approach to termination

heorem: (BCHM) Let X be a smooth projective variety, A and an ample $\mathbb{Q}$-divisor, and let $D_1 + D_2 + \cdots + D_k$ be a normal crossings divisor. Then there are only finitely many minimal models, for $K_X + A + \sum a_i D_i$, $(a_1, a_2, \ldots, a_k) \in [0,1]^k$.

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- If we run a MMP whose intermediary steps are all minimal models, then termination is clear.
- Unfortunately this trick does not seem to work to construct the moduli space of varieties of general type. We run into a brick wall called abundance.

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- As we increase t, $\frac{1}{|f|^{2t}}$ is more singular.
- The supremum of t for which $\frac{1}{|f|^{2t}}$ is locally integrable, is called the log canonical threshold.

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- We write $K_S + t\tilde{C} + a_1 E_1 + a_2 E_2 + a_3 E_3 = \pi^*(K_{\mathbb{C}^2} + tC)$.
- We already know the log canonical threshold a is the largest $t \leq 1$ such that $\max(a_1, a_2, a_3) \leq 1$.

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- $0 = (K_T + a\tilde{C} + E)|_E = -2 + 1/2 + 2/3 + a$, where we applied orbifold adjunction.

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- So the log canonical threshold $a = 5/6$.

Yet another way

- $a_1 = 2t - 1$, $a_2 = 3t - 3$, and $a_3 = 6t - 5$. The log canonical threshold a is $5/6$.
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- The map ψ is in fact a weighted blow up.

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- If C is given as $y^a + x^b$, then the log canonical threshold is $\min(1/a + 1/b, 1)$, using either integrals or weighted blow ups.

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- We hope to prove the full version of Shokurov's conjecture using birational boundedness.

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- This argument is due to Birkar.

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- However, there is some reason to hope that we might prove a version of termination strong enough to construct projective moduli spaces.

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