A new approach to Mori theory

James McC Kernan

UCSB
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The idea is to focus on three extreme cases,

- $K_X$ is ample.
- $K_X$ is trivial.
- $-K_X$ is ample.

The hope is that any variety is constructed, in a sense to be explained, using only these building blocks.
Some basic definitions

- We say that a birational map $f : Y \rightarrow X$ is a contraction if $f^{-1}$ does not contract any divisors.

- A log pair is a pair $(X; P_i a_i)$, where $X$ is normal, $P_i$ and $K_X + P_i$ are $\mathbb{Q}$-Cartier, where $a_i \in [0; 1]$, for all $i$. 
Some basic definitions

- We say that a birational map $f : Y \dasharrow X$ is a **contraction** if $f^{-1}$ does not contract any divisors.

- We say that $f$ is **$D$-negative**, if there is a resolution $p : W \longrightarrow X$ and $q : W \longrightarrow Y$ such that if we write

  $$p^*D = q^*D' + E,$$

  then $E \geq 0$, where both $D$ and $D' = f_*D$ are $\mathbb{Q}$-Cartier.
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then \( E \geq 0 \), where both \( D \) and \( D' = f_* D \) are \( \mathbb{Q} \)-Cartier.

- A \textit{log pair} is a pair \((X, \Delta = \sum_i a_i \Delta_i)\), where \( X \) is normal, \( \Delta \geq 0 \) and \( K_X + \Delta \) is \( \mathbb{Q} \)-Cartier, where \( a_i \in [0, 1] \), for all \( i \).
Conjecture (Minimal Model Conjecture). Let \((Y, \Gamma)\) be a log smooth projective pair. Then there is a \(K_Y + \Gamma\)-negative birational contraction \(f: Y \dashrightarrow X\) and a morphism \(\pi: X \rightarrow S\) such that either

1. \(K_X + \Delta = \pi^* H\), where \(H\) is ample, or
2. \(-(K_X + \Delta)\) is relatively ample, where \(\dim Z < \dim X\),

and \(\Delta = f_* \Gamma\).
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Note that this is the contraction of two conjectures, the minimal model conjecture and the abundance conjecture.
Traditional Approach

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First we try to construct a pair \((X, \Delta)\) such that either \(K_X + \Delta\) is nef or \(- (K_X + \Delta)\) is relatively ample, for some morphism \(\pi\).
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- We try to construct \(X\) from \(Y\) by a sequence of elementary birational modifications.
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- The **abundance conjecture** then states that if \(K_X + \Delta\) is nef, then it is semiample.

- The problem is that both of these parts seem hard.
Mori’s program

- Start with any birational model $Y$. 
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- Start with any birational model $Y$.
- Desingularise $Y$. 

If $K_Y$ is nef, then STOP.

Otherwise there is an extremal contraction, $Y \to X$, which is $(K_Y + \cdot)$-ample.

If $\dim X < \dim Y$, then STOP.

If $\cdot$ is divisorial replace $(Y; \cdot)$ by $(X; \cdot)$.

If $\cdot$ is small, then $K_X + \cdot$ is not $Q$-Cartier, and we need to do something different.
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- If $\dim X < \dim Y$, then STOP.
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Mori’s program

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- Desingularise $Y$.
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- Otherwise there is an extremal contraction, $\pi: Y \rightarrow X$, which is $-(K_Y + \Gamma)$-ample.
- If $\dim X < \dim Y$, then STOP.
- If $\pi$ is divisorial replace $(Y, \Gamma)$ by $(X, \Delta)$.
- If $\pi$ is small, then $K_X + \Delta$ is not $\mathbb{Q}$-Cartier, and we need to do something different.
Instead we try to replace $Y$ by another birational model $Y^+$, $Y \dashrightarrow Y^+$, such that $\pi^+: Y^+ \to X$ is $(K_{Y^+} + \Gamma^+)$-ample.
Three main Conjectures

Conjecture. *(Existence)* Suppose that $K_X + \Delta$ is kawamata log terminal. Let $f : X \to Y$ be a small extremal contraction.
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Conjecture. (Existence) Suppose that $K_X + \Delta$ is kawamata log terminal. Let $f : X \to Y$ be a small extremal contraction. Then the flip of $f$ exists.

Conjecture. (Termination) There is no infinite sequence of kawamata log terminal flips.

Conjecture. (Abundance) Suppose that $K_X + \Delta$ is kawamata log terminal and nef. Then $K_X + \Delta$ is semiample.
The new approach

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- Let $G_1, G_2, \ldots, G_k$ be $k$ $\mathbb{Q}$-divisors and let $Y$ be a projective variety. The Cox ring is the multigraded ring

$$R(Y, G^\bullet) = \bigoplus_{m \in \mathbb{N}^k} H^0(Y, \mathcal{O}_Y(\sum_i m_iG_i)).$$
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$$R(Y, G^*) = \bigoplus_{m \in \mathbb{N}^k} H^0(Y, \mathcal{O}_Y(\sum_i m_i G_i)).$$

If $R(Y, G)$ is finitely generated, where $G = K_Y + \Gamma$ ($k = 1$), then the log canonical model is equal to

$$X = \operatorname{Proj} R(Y, G).$$
One conjecture to rule them all

Conjecture (Finite Generation). Let \((Y, \Gamma_i)\) be a log smooth pair, where \(\Gamma_i\) has rational coefficients, where \(Y\) is projective. Set \(G_i = K_Y + \Gamma_i\). Then \(R(Y, G^\bullet)\) is finitely generated.
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I hope to persuade everyone of two things:

- This conjecture does indeed imply the minimal model conjecture.
- There is some chance that attacking finite generation directly is better than the step by step approach sketched previously.
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- Assume that $K_Y + \Gamma + tD$ is kawamata log terminal and $K_Y + \Gamma$ is big.
The MMP with scaling

Choose $t$ minimal such that $K_Y + \Gamma + tD$ is nef. The trick is to contract only special extremal rays.
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  - If $f$ is divisorial then replace $(Y, \Gamma)$ by $(X, \Delta)$.
  - If $f$ is small, then replace $Y$ by the flip.
- At this point we use existence of flips, which is guaranteed by finite generation.
Termination of the MMP with scaling

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Since we are assuming that $K_Y + \Gamma$ is big, varying the coefficients of $\Gamma$, for a small perturbation $K_Y + \Gamma'$ of $K_Y + \Gamma + tD$, $K_Y + \Gamma'$ is ample.
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- For every step of the MMP, this small perturbation changes.
- So termination follows from finiteness.
We have already seen that if we assume finite generation and $K_Y + \Gamma$ is big and $\mathbb{Q}$-Cartier then we can construct the log canonical model $(X, \Delta)$. Since we are assuming that $K_Y + \Gamma$ is big, varying the coefficients of $K_Y$ for a small perturbation $K_Y + 0$ of $K_Y + 0$, $K_Y + 0$ is big and $\mathbb{Q}$-Cartier. Since there are only finitely many log canonical models in a neighbourhood of $K_Y$, taking the limit as $0$ approaches $K_Y$, we may assume that there is a model such that $K_X + \Delta$ is nef. Now apply the base point free Theorem, to obtain a log canonical model.
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Real log canonical models

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Now apply the base point free Theorem, to obtain a log canonical model.
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- Let \( H \) be an ample divisor and take the infimum such that \( K_Y + \Gamma + tH \) is big.
- Then \( K_Y + \Gamma + tH \) is pseudo-effective but not big and \( t > 0 \).
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- Then \( K_Y + \Gamma + tH \) is pseudo-effective but not big and \( t > 0 \).
- Let \((X, \Delta + tH')\) be the log canonical model, \( f : Y \to X \). Let \( \pi : X \to Z \) be the morphism whose existence is guaranteed by the base point free Theorem.
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- Then \( -(K_X + \Delta) \) is relatively big. We may now modify \( \pi \) so that \( -(K_X + \Delta) \) is ample, using the MMP with scaling.
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- Then \( -(K_X + \Delta) \) is relatively big. We may now modify \( \pi \) so that \( -(K_X + \Delta) \) is ample, using the MMP with scaling.
- Further as \( f \) is \( K_Y + \Gamma + tH \) negative, \( f \) is surely \( K_Y + \Gamma \) negative.
Suppose that $K_X + \Delta$ is kawamata log terminal and nef.
Abundance via finite generation

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\[ \kappa(K_X + \Delta) \geq 0 \]
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- $\kappa(K_X + \Delta) \geq 0$ and
- if $K_X + \Delta$ is not numerically trivial, then $\kappa(K_X + \Delta) = \kappa(K_Y + \Gamma) > 0$. 
An embedding of \((X, \Delta)\)

- Pick any projectively normal embedding of \(X \subset \mathbb{P}^n\), and let \((\bar{X}, \bar{\Delta})\) be the cone over \((X, \Delta)\) with vertex \(p\).
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- Let \(W \rightarrow \bar{X}\) be the blow up of \(p\), with exceptional divisor \(E\). Let \(\Theta = E + \Theta' + H\), where \(\Theta'\) is the strict transform of \(\Theta\), and \(H\) is the strict transform of a sufficiently general and sufficiently ample divisor.
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- Then \(E\) is isomorphic to \(X\), and under this identification,

\[
(K_W + \Theta)|_E = K_X + \Delta.
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An embedding of \((X, \Delta)\)

- Pick any projectively normal embedding of \(X \subset \mathbb{P}^n\), and let \((\tilde{X}, \tilde{\Delta})\) be the cone over \((X, \Delta)\) with vertex \(p\).

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- Then \(E\) is isomorphic to \(X\), and under this identification,

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(K_W + \Theta)|_E = K_X + \Delta.
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- In addition \(K_W + \Theta\) is big and log canonical.
The log canonical model of \((W, \Theta)\)

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The log canonical model of \((W, \Theta)\)

- Since we are assuming finite generation, \((W, \Theta)\) has a log canonical model \((W', \Theta')\), \(f : W \to W'\)
- \(f\) is birational as \(K_W + \Theta\) is big.
- First suppose that \(\kappa(K_X + \Delta) = -\infty\). Then running the MMP with scaling, it follows that \(X\) is covered by curves \(C\) such that \((K_X + \Delta) \cdot C < 0\).
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- Thus finite generation implies abundance.
Finite Generation

- **Baby case** Suppose that $G = K_Y + \Gamma$ is very ample. Then considering

$$0 \longrightarrow \mathcal{O}_Y((k-1)G) \longrightarrow \mathcal{O}_Y(kG) \longrightarrow \mathcal{O}_G(kG) \longrightarrow 0,$$

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- However the theory of multiplier ideals, developed by Siu and Kawamata, gives a way to lift sections (hopefully generators), from log canonical centres, under suitable assumptions.
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Another idea of Shokurov gives a way to check finite generation locally.
Further work

- To prove finite generation following the general line sketched above, we need to extend the theory of multiplier ideal sheaves beyond the big case.

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