Birational geometry and moduli spaces of varieties of general type

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- K_X is a natural polarisation on a curve of $g \geq 2$. In general classification is a birational problem.

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- every pluricanonical form lifts to a resolution (even though $\overline{\mathcal{M}}_g$ is not canonical).
- $K_{\overline{\mathcal{M}}_g} + D$ is log canonical and ample, where D is the sum of the boundary divisors, $\partial\overline{\mathcal{M}}_g = \overline{\mathcal{M}}_g - \mathcal{M}_g$.

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- projective X, K_X is ample (canonically polarised).
- more generally log canonical pairs (X, Δ) such that $K_X + \Delta$ is ample (from $\overline{\mathcal{M}}_g$ to $\overline{\mathcal{M}}_{g,n}$).

First results

- Gieseker showed that the set of smooth projective surfaces S of general type with fixed Hilbert polynomial $h(t) = \chi(S, \mathcal{O}_S(tK_S))$ is the set of closed point of a quasi-projective variety $\mathcal{M}(h)$.

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- Call this the smooth case.

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  - The moduli space is non-reduced: $Z = \operatorname{Spec}\mathbb{Z}[x]/\langle x^n\rangle$.
  - The moduli space has arbitrarily bad singularities.
- Moral: we should expect many complications in higher dimensions.

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- The family of all smooth curves embedded by $|3K_C|$ is a locally closed subset.
- (Mumford) Realise $\mathcal{M}_g$ as a GIT quotient by the action of $\mathrm{PGL}(5g-5)$.

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- Equivalently: C is nodal and $\mathrm{Aut}(C)$ is finite.

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- This implies uniqueness of semi-stable reduction and highlights the significance of general type.

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  - Try to generalise these results to all dimensions.
- Focus on the latter problem in these lectures.

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- Note $K_S + L_1 + L_2 + \cdots + L_{27} = -8K_S$ is ample.
- Sekiguchi: A cubic surface is determined by the j-invariants of all intersections of any line with any other four lines.

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- For curves $m_0 = 3$ works, surfaces $m_0 = 5$.

Degree versus Hilbert polynomial

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- Therefore $r \leq m^n d + n$.

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- Fix $m \geq m_0$. The degree of X in $\mathbb{P}^r$ is at most $m^n d$.
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- Just need to fix the degree d and the dimension n.

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- Note that in the definition of boundedness, we do not a priori require that every fibre of $\mathcal{Y}$ is a variety of general type of volume d.
- Use the moduli space of canonically polarised varieties X (the smooth case) to give a birational classification of smooth projective varieties Y of general type.

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- Choose V smooth and $\mathcal{X}$ with quotient singularities.
- So we may assume we have a semi-stable family of projective varieties which includes every variety of dimension n and volume d.

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- So we may assume that the Hilbert polynomial is constant.

Relative canonical model

- Take the relative canonical model of $\pi\colon \mathcal{Y} \longrightarrow U$

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- Kovács & Patakfalvi generalised to log pairs.

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- Then we get a reasonable deformation theory.

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- Non-trivial deformation theory, $\mathbb{Q}$-Gorenstein deformation (Wahl).

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- the fibres of the relative canonical model are semi log canonical—part of the statement of adjunction.

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- $a = \inf_{i,Y} a_i$ is the log discrepancy of (X, Δ). (X, Δ) is log canonical if $a \geq 0$.

Properties of log canonical pairs

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- Some of the basic results about log canonical pairs fail and we are ignorant of many of the rest.
- We must work with log canonical pairs since the double locus always comes with coefficient one.