Birational geometry and moduli spaces of varieties of general type

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UCSD
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  - \( K_X \) positive: curves \( g \geq 2 \), large degree in \( \mathbb{P}^n \).
  - Form continuous families. Try to construct moduli spaces and investigate their geometry.
  - \( K_X \) is a natural polarisation on a curve of \( g \geq 2 \). In general classification is a birational problem.
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- $K_{\overline{M}_g} + D$ is log canonical and ample, where $D$ is the sum of the boundary divisors, $\partial\overline{M}_g = \overline{M}_g - M_g$. 

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- Moduli of varieties of general type of higher dimension. We will focus on this moduli space:
  - projective $X$, $K_X$ is ample (canonically polarised).
  - more generally log canonical pairs $(X, \Delta)$ such that $K_X + \Delta$ is ample (from $\overline{M}_g$ to $\overline{M}_{g,n}$).
Gieseker showed that the set of smooth projective surfaces $S$ of general type with fixed Hilbert polynomial $h(t) = \chi(S, \mathcal{O}_S(tK_S))$ is the set of closed point of a quasi-projective variety $\mathcal{M}(h)$. 
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- Call this the smooth case.
Murphy’s law

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**Moral**: we should expect many complications in higher dimensions.
Construction of $\mathcal{M}_g$

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(Mumford) Realise $\mathcal{M}_g$ as a GIT quotient by the action of $\text{PGL}(5g - 5)$.
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  - Equivalently: $C$ is nodal and $\text{Aut}(C)$ is finite.
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- This implies uniqueness of semi-stable reduction and highlights the significance of general type.
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- Focus on the latter problem in these lectures.
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- **Keel, Hacking, Tevelev:** cubic surfaces $S$.
- Look at the component of the moduli space of log pairs containing $(S, L_1 + L_2 + \cdots + L_{27})$, where $L_1, L_2, \ldots, L_{27}$ are the 27 lines on the cubic surface.
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- **Note**: $K_S + L_1 + L_2 + \cdots + L_{27} = -8K_S$ is ample.
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Note $K_S + L_1 + L_2 + \cdots + L_{27} = -8K_S$ is ample.

Sekiguchi: A cubic surface is determined by the $j$-invariants of all intersections of any line with any other four lines.
Construction of smooth moduli space

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As a consequence every variety $Y$ of general type is birational $\phi_{mK_Y} : Y \dashrightarrow X \subset \mathbb{P}^r$ to a variety $X$ naturally embedded in $\mathbb{P}^r$, where $K_X$ is ample and $X$ has canonical singularities (Reid).
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For curves $m_0 = 3$ works, surfaces $m_0 = 5$. 
Suppose that we fix the volume of $K_Y$,

$$d = \text{vol}(Y, K_Y) = \limsup_{m \to \infty} \frac{n! H^0(Y, \mathcal{O}_Y(mK_Y))}{m^n}.$$
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Fix $m \geq m_0$. The degree of $X$ in $\mathbb{P}^r$ is at most $m^n d$. 
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Degree versus Hilbert polynomial

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For this we use the Chow variety, not the Hilbert scheme.

Just need to fix the degree $d$ and the dimension $n$. 
Birational boundedness

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- Note that in the definition of boundedness, we do not a priori require that every fibre of $\mathcal{Y}$ is a variety of general type of volume $d$.
- Use the moduli space of canonically polarised varieties $X$ (the smooth case) to give a birational classification of smooth projective varieties $Y$ of general type.
Semi-stable reduction

- An **alteration** is a proper generically finite surjective morphism $V \longrightarrow U$. 
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- **Theorem:** (Abramovich-Karu) Given a morphism $\mathcal{Y} \rightarrow U$ whose generic fibre is geometrically irreducible then there is an alteration $V \rightarrow U$ and an alteration $\mathcal{X} \rightarrow \mathcal{Y}'$ of the main component $\mathcal{Y}'$ of $\mathcal{Y} \times_U V$ such that $\mathcal{X} \rightarrow V$ has reduced fibres and $\mathcal{X}$ is canonical.
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- So we may assume we have a semi-stable family of projective varieties which includes every variety of dimension $n$ and volume $d$. 
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- So we may assume that the Hilbert polynomial is constant.
Take the relative canonical model of $\pi : Y \longrightarrow U$

\[ \mathcal{X} = \text{Proj}_U \left( \bigoplus_{m \in \mathbb{N}} \pi_* \mathcal{O}_Y(mK_Y) \right) \longrightarrow U. \]
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Kovács & Patakfalvi generalised to log pairs.
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- Then we get a reasonable deformation theory.
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- Construct new Campedelli surfaces $S$, smooth projective surfaces, $K_S$ ample, with (Y. Lee, J. Park): $K_S^2 = 2$ and $\pi_1(S) \in \{0, \mathbb{Z}_2, \mathbb{Z}_4\}$. 
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- Non-trivial deformation theory, $\mathbb{Q}$-Gorenstein deformation (Wahl).
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- \(X\) has nodal singularities in codimension one and the normalisation \((X^\nu, D + \Delta^\nu)\) is log canonical, where \(D\) is the double locus.
- the fibres of the relative canonical model are semi log canonical—part of the statement of adjunction.
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- \(a = \inf_{i,Y} a_i\) is the log discrepancy of \((X, \Delta)\).
- \((X, \Delta)\) is **log canonical** if \(a \geq 0\).
Properties of log canonical pairs

If \((X, \Delta)\) is log canonical then

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H^0(X, \mathcal{O}_X(m(K_X + \Delta))) = H^0(Y, \mathcal{O}_Y(m(K_Y + \Gamma))).
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- We must work with log canonical pairs since the double locus always comes with coefficient one.