A tour of algebraic geometry

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UCSD

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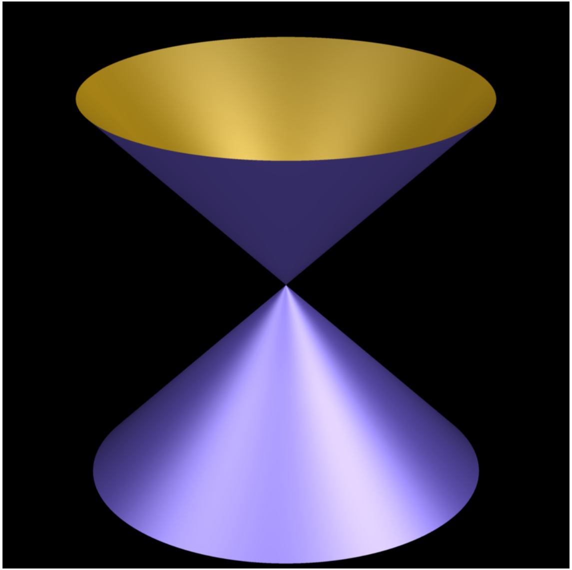
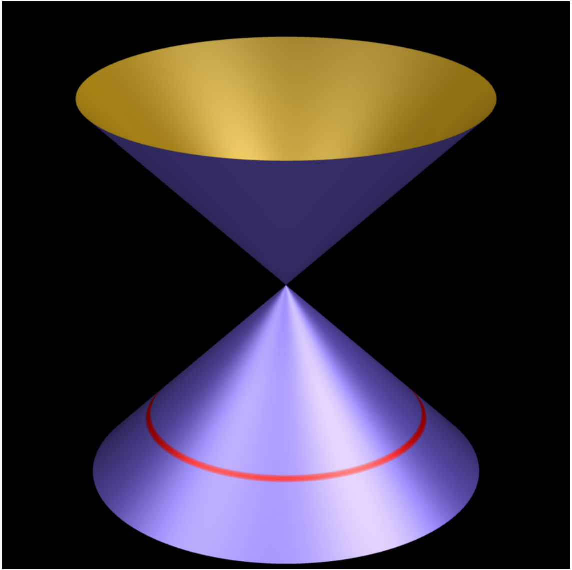
Classical geometry: conic sections



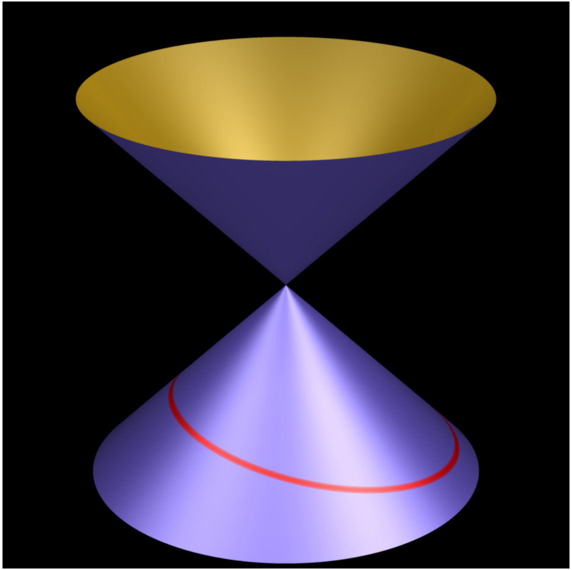
Menaechmus studied conic sections in the 3rd century BC.

Conic sections: circle



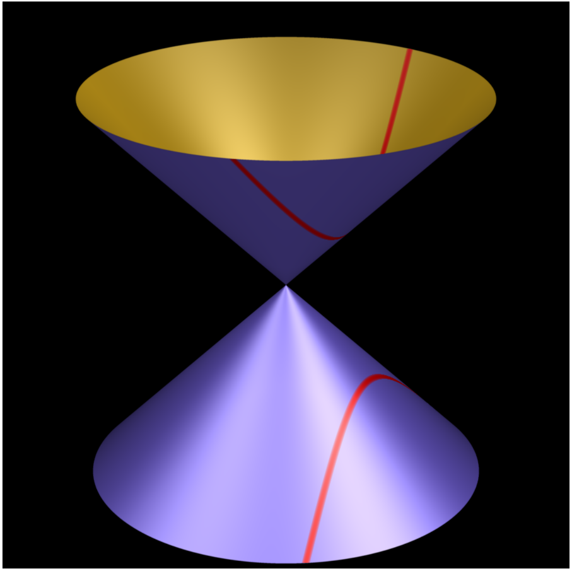
A horizontal slice.

Conic sections: ellipse



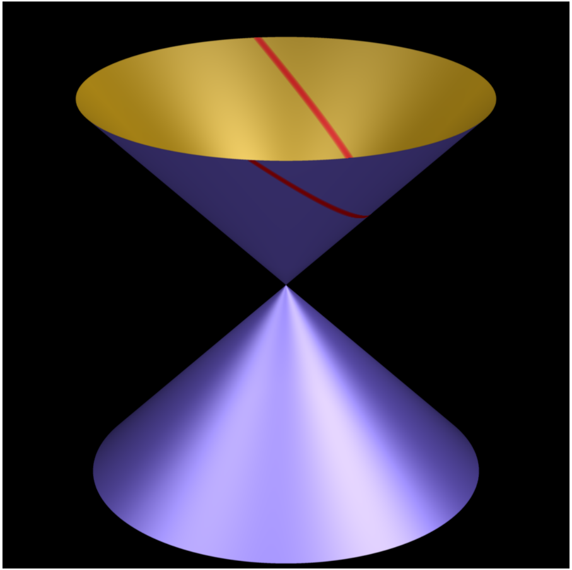
Small perturbation of a horizontal slice.

Conic sections: hyperbola



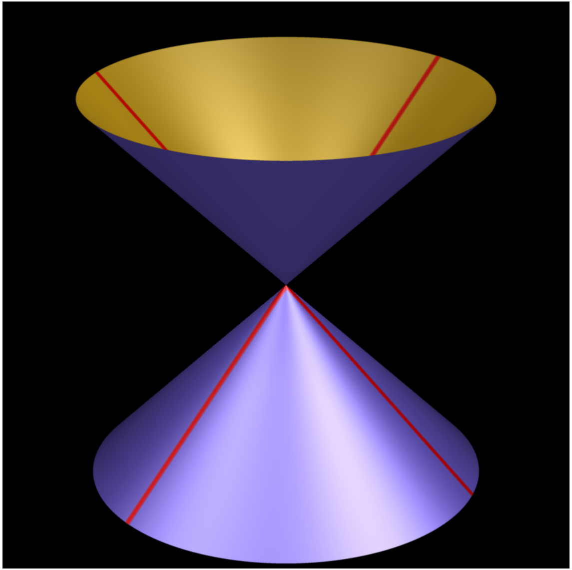
A vertical slice.

Conic sections: parabola



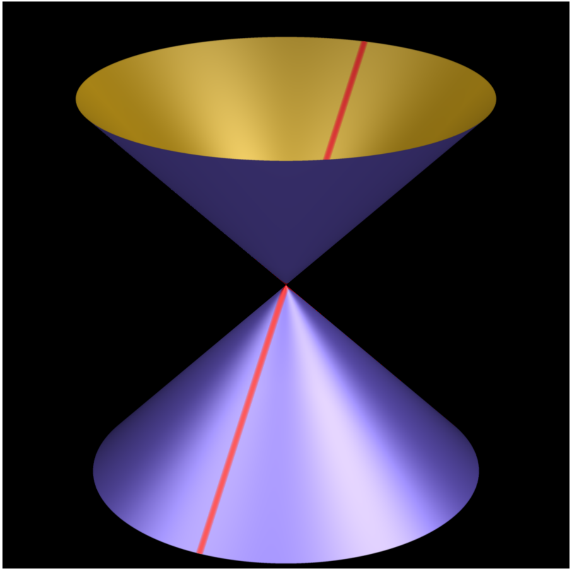
Diagonal slice.

Conic sections: pair of lines



Vertical slice through the vertex.

Conic sections: double line



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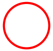

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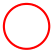

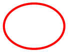

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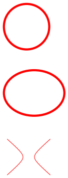

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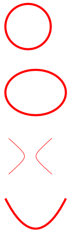

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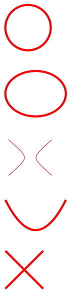

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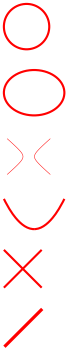

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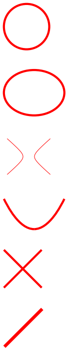
- The algebraic perspective both unifies and offers the opportunity to consider more complicated examples:



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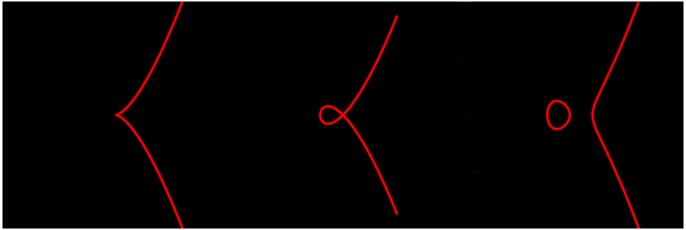
$$y^2 = x^3 \qquad\qquad y^2 = x^3 + x^2 \quad \text{or} \quad y^2 = x^3 - x :$$



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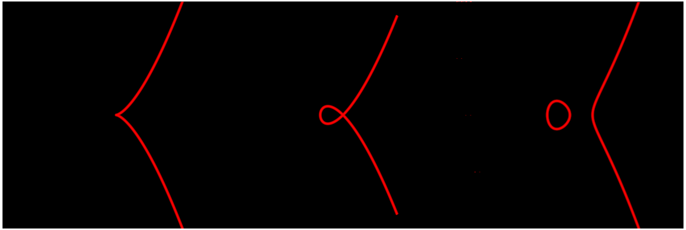

Newton, circa 1700, looked at cubics and found 72 different (topological) types of cubics.

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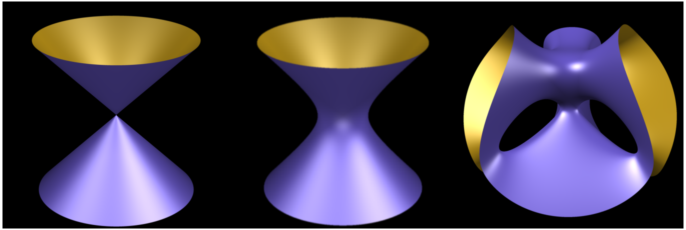

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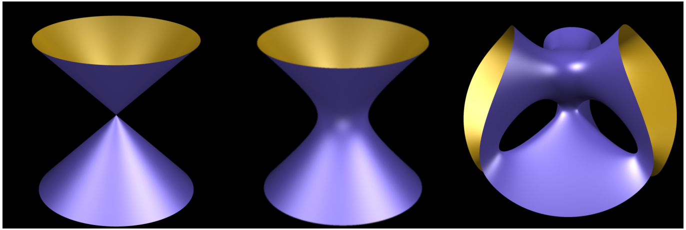
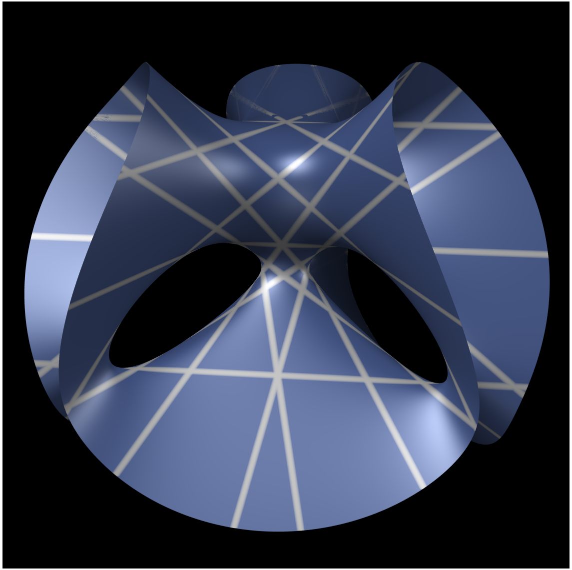


- Cayley communicated to the Royal society the discovery of the 27 lines on a cubic surface in 1869.

27 lines on the Clebsch cubic



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- Future challenge: One interesting scientific challenge of the twenty first century will be to understand both theoretically and practically how to solve large systems of polynomial equations in lots of variables.
- Many of the interesting practical applications of algebraic geometry involve solving such systems.

Painting perspectives

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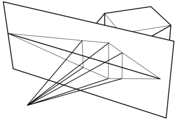

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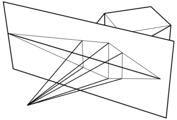

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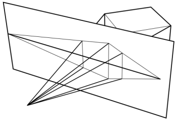

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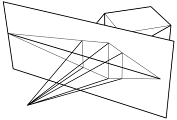

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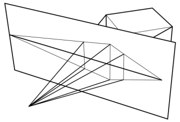

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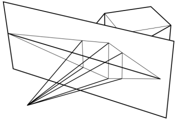

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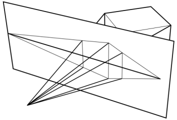
- To better understand what is happening, mathematically it is convenient to drop the dimension.



A mathematical perspective

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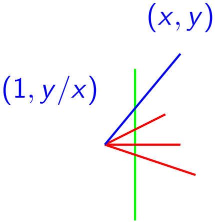

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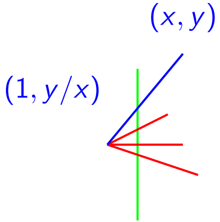

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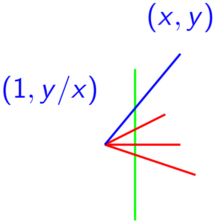

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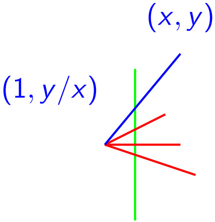

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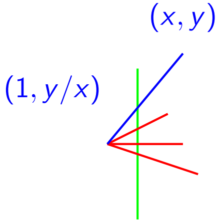

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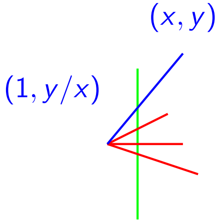

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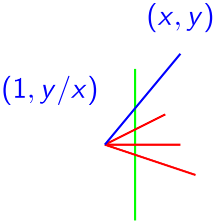

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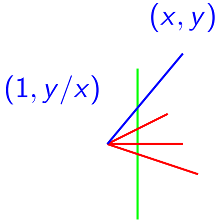
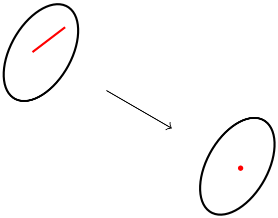
- But where do we send the origin, the point $p = (0, 0)$?
- Zen-like question: what is the slope of the line connecting the origin to the origin? What is the ratio $0/0$?
- Mathematical fix: We construct a new surface, with the origin replaced with a line representing all slopes. We blow up the origin.





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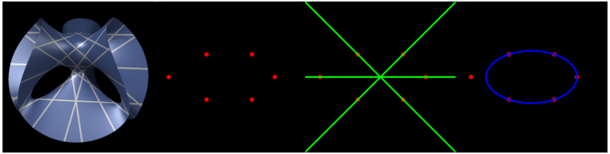
- If we pick six skew lines on the cubic surface, we can replace them by six points, to get the usual plane (we blow down the six lines).
- The 27 lines on the cubic are: the six skew lines, the fifteen lines connecting the six points we blow up and the six conics which pass through five of the six points.



$$27 = 6 + 15 + 6.$$

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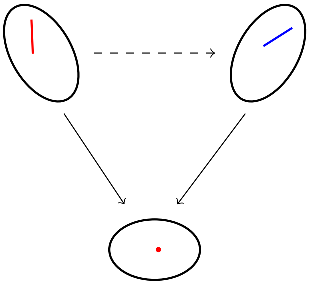
- Closely related to flops are flips.

Flips and flops



In this schematic picture of a flip the red line is flipped to the blue line.

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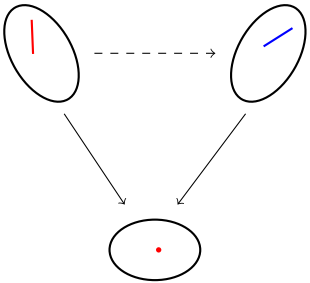

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Fundamental question: Blow ups first appear in dimension two, flips and flops in dimension three. Is there a similar feature in higher dimensions?

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- Summary: Use the MMP to break a variety into simpler pieces.

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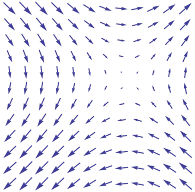

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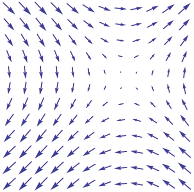
- and foliations, which study vector fields on varieties. 
- This work is exciting as it is quite unexpected that one can use the MMP in these contexts.

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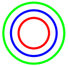
- Varieties are often parametrised by continuous geometric invariants, radius of a circle, length, area, volume. 

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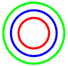
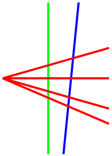
- Varieties are often parametrised by continuous geometric invariants, radius of a circle, length, area, volume.  
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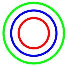
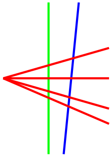
- Varieties are often parametrised by continuous geometric invariants, radius of a circle, length, area, volume.  
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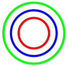
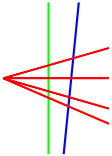
- Varieties are often parametrised by continuous geometric invariants, radius of a circle, length, area, volume.  
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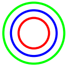
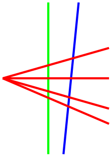
- Varieties are often parametrised by continuous geometric invariants, radius of a circle, length, area, volume.  
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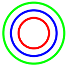
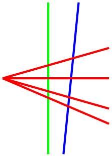
- Varieties are often parametrised by continuous geometric invariants, radius of a circle, length, area, volume.  
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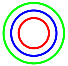
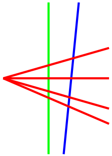
- Varieties are often parametrised by continuous geometric invariants, radius of a circle, length, area, volume.  
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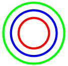
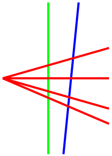
- Varieties are often parametrised by continuous geometric invariants, radius of a circle, length, area, volume.  
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- These slides were produced using LaTeX (beamer and TikZ) and surf (a program to draw curves and surfaces).