11. Morphisms between varieties I

We adopt the following working definition of a morphism between projective varieties:

Definition 11.1. *A* ***morphism***

$$f\colon V \longrightarrow W,$$

between two projective varieties, where $V \subset \mathbb{P}^m$ and $W \subset \mathbb{P}^n$, is given by picking a collection of $n+1$ homogenous polynomials $F_0, F_1, \dots, F_n \in K[X_0, X_1, \dots, X_m]$ of the same degree such that

$$f(x) = [F_0(x) : F_1(x) : \dots : F_n(x)] \in W,$$

for every $x \in V$, ***where for every $x \in V$ there is an i such that*** *$F_i(x) \neq 0$.*

Note that this gives us the category of projective varieties, with maps given by projective morphisms.

Observe that if $w = \lambda v$, and each F_i has degree d, then

$$F_i(w) = \lambda^d F_i(v),$$

so that

$$\begin{aligned}[F_0(w) : F_1(w) : \dots : F_n(w)] &= [\lambda^d F_0(v) : \lambda^d F_1(v) : \dots : \lambda^d F_n(v)] \\ &= [F_0(v) : F_1(v) : \dots : F_n(v)].\end{aligned}$$

Example 11.2. *The map*

$$f\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^2,$$

given by

$$[S : T] \longrightarrow [S^2 : ST : T^2],$$

is a morphism. Indeed we only need to check that S^2, ST and T^2 cannot be simultaneously zero, which is clear.

Consider the image. Suppose that we pick coordinates $[X : Y : Z]$ on $\mathbb{P}^2$. On the image we have

$$\begin{aligned}Y^2 &= (ST)^2 \\ &= S^2T^2 \\ &= XZ.\end{aligned}$$

Thus the image lies in the locus $Y^2 - XZ = 0$. On the other hand suppose we have a point $[X : Y : Z] \in \mathbb{P}^2$. If $X \neq 0$, set $S = X$ and

$T = Y$. Then

$$\begin{aligned}[S^2 : ST : T^2] &= [X^2 : XY : Y^2] \\ &= [X^2 : XY : XZ] \\ &= [X : Y : Z],\end{aligned}$$

as $X \neq 0$. We need to worry about the case $X = 0$. One way to proceed is to observe that then $Y = 0$ so that we have the point $[0 : 0 : 1]$, the image of $[0 : 1]$. Or observe that X and Z cannot simultaneously be zero, and if $Z \neq 0$, we set $S = Y$ and $T = Z$ and argue as before (using the obvious symmetry). Either way we have established that the image of $\mathbb{P}^1$ is a conic in $\mathbb{P}^2$.

This example has many interesting generalisations.

Example 11.3. *Consider the morphism*

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^3,$$

given as

$$[S : T] \longrightarrow [S^3 : S^2T : ST^2 : T^3].$$

The image C is known as the ***twisted cubic****.*

Consider the image. We have $[X : Y : Z : W] = [S^3 : S^2T : ST^2 : T^3]$. Thus certainly $Y^2 = XZ$, $XW = YZ$ and $Z^2 = YW$. It is an interesting exercise to check that these equations define the image.

Example 11.4. *More generally still, we can look at the morphism*

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^d,$$

given as

$$[S : T] \longrightarrow [S^d : S^{d-1}T : \cdots : ST^{d-1} : T^d].$$

The image is called a ***rational normal curve of degree*** *d.*

Example 11.5. *More generally still, there is a morphism*

$$\mathbb{P}^n \longrightarrow \mathbb{P}^N,$$

given as

$$[X_0 : X_1 : \cdots : X_n] \longrightarrow [X^I],$$

where given an n-tuple $I = (i_0, i_1, \dots, i_n)$, X^I denotes the monomial $X_0^{i_0} X_1^{i_1} \dots X_n^{i_n}$. Here we choose coordinates Z_I, where I ranges over all n-tuples of positive integers, whose sum is d. N is equal to the number of such n-tuples, minus one. Note that not every X^I can be zero. Indeed if $X_0^d = X_1^d = \dots X_n^d = 0$, then $X_0 = X_1 = \cdots = X_n = 0$. This morphism is called the ***d-uple embedding****.*

Note that for every I, J, I' and J' such that $I + J = I' + J'$, the image lies in the hypersurface

$$Z_I Z_J = Z_{I'} Z_{J'},$$

since

$$X^I X^J = X^{I+J} = X^{I'+J'} = X^{I'} X^{J'}.$$

Once again, in fact the image is cut out by these equations.

Perhaps the most interesting example is to take $n = d = 2$.

Example 11.6. *In this case we get a morphism*

$$\mathbb{P}^2 \longrightarrow \mathbb{P}^5,$$

given as

$$[X : Y : Z] \longrightarrow [X^2 : Y^2 : Z^2 : YZ : XZ : XY] = \\ [Z_{(2,0,0)} : Z_{(0,2,0)} : Z_{(0,0,2)} : Z_{(0,1,1)} : Z_{(1,0,1)} : Z_{(1,1,0)}].$$

This morphism is called the **Veronese morphism** and the image is called the **Veronese surface**. It turns out that the Veronese surface is an exception to practically every (otherwise) general statement about projective varieties.

Finally it seems worthwhile to point out that there are other ways to construct rational normal curves.

Definition 11.7. *Let k be a positive integer. A subset X of projective space is a **determinental variety** if X is the locus where a matrix $M = (F_{ij})$ of homogeneous polynomials F_{ij} has rank at most k.*

For example, consider the matrix

$$\begin{pmatrix} X_0 & X_1 & X_2 & \dots & X_n - 1 \\ X_1 & X_2 & \dots & X_{n-1} & X_n \end{pmatrix}$$

The locus where this matrix has rank one is precisely a rational normal curve. Indeed if

$$[X_0 : X_1 : \dots : X_n] = [S^n : S^{n-1}T : \dots : T^n],$$

then clearly the second row is nothing more than the first row times T/S. Conversely if the given matrix has rank 1, then the second row is a scalar multiple of the first, and it is easy to get the result.

On the other hand, the locus where a matrix has rank at most one, is exactly the locus where the 2×2 minors are all zero. For example, for $n = 3$, we recover the three quadrics containing a twisted cubic.

We can do a similar thing for the Veronese. In this case, we look at the locus where the matrix

$$\begin{pmatrix} Z_{2,0,0} & Z_{1,1,0} & Z_{1,0,1} \\ Z_{1,1,0} & Z_{0,2,0} & Z_{0,1,1} \\ Z_{1,0,1} & Z_{0,1,1} & Z_{0,0,2} \end{pmatrix},$$

has rank one.